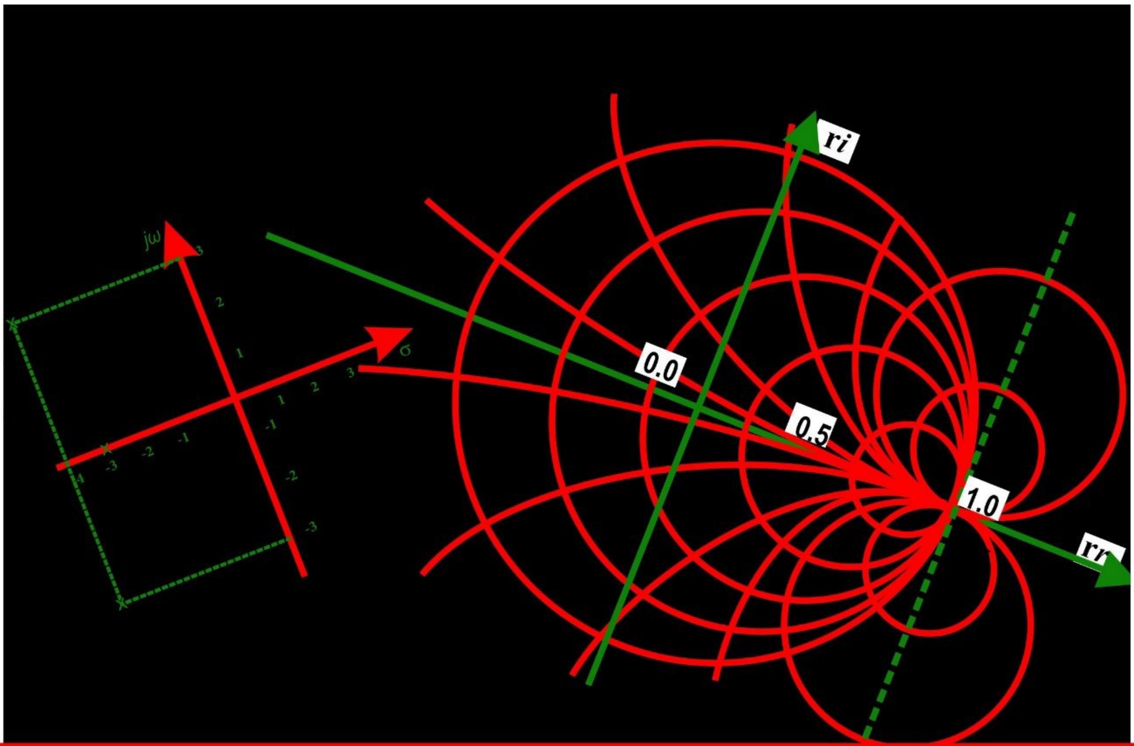
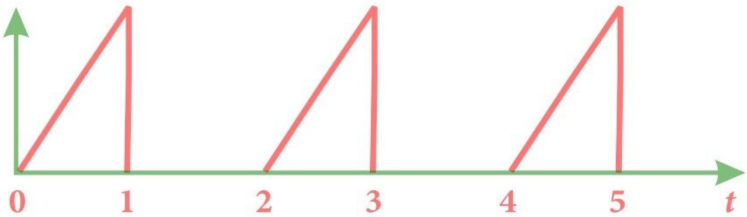


CIRCUIT ANALYSIS TEXTBOOK

Using Laplace Transform with Application



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CIRCUIT ANALYSIS TEXTBOOK

Using Laplace Transform with Application

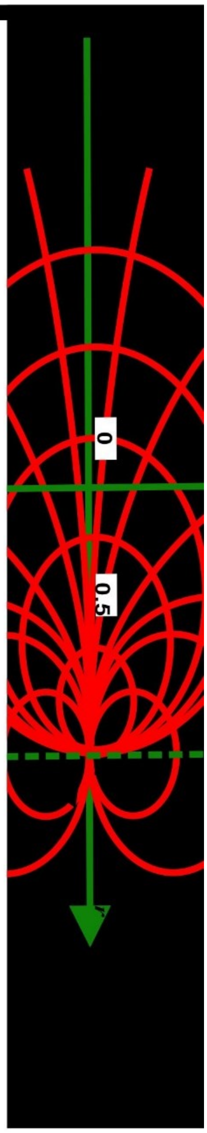
$$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

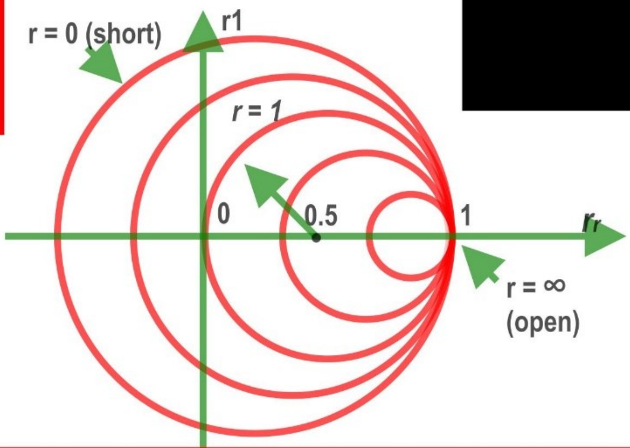
$$F(s) = \frac{F_1(s)}{1 - e^{-Ts}}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)^n}\right] = \frac{t^{n-1}e^{-at}}{(n-1)!}u(t)$$

$$f(\infty) = \lim_{s \to 0} sF(s)$$

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}$$





CIRCUIT ANALYSIS TEXTBOOK

Using Laplace Transform with Application

By

Kenneth Ugo Udeze

Circuit Analysis Textbook (*Using Laplace Transform with Application*)

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Preface

This book originates from notes used in teaching Electrical Circuit Theory courses at the third-year level of Electrical/Electronic Engineering Department, Federal Polytechnic, Oko, Anambra State, Nigeria. Along with other materials gathered by the author during his degree and post-degree years of academic pursuit, and over fifteen (15) years of teaching experience in accordance with course curriculum guidelines from the National Board for Technical Education (NBTE), this text, “CIRCUIT ANALYSIS Using Laplace Transform with Application”, was written.

The content of each chapter was designed to accommodate Higher National Diploma (HND) and Bachelor of Science/Engineering (B.Sc./B.Eng.) undergraduate students as the materials presented were made comprehensive enough to cover both classes of programs at their mid-course levels.

Chapter 1 covers the basic knowledge of Laplace Transform with its fundamental formula; chapter 2, partial fractions with different methods of resolving same.

Chapter 3 discusses Laplace transforms in the reverse direction, i.e., Inverse-Laplace Transform.

Chapter 4 covers solutions of differential equations by Laplace transform method, and how one can use it in solving time-domain equations by transforming to the frequency or s-domain.

Chapters 5 and 6 cover electrical circuits using Laplace transform, and the various applications thereof.

Chapter 7 is about Small Signal Transmission Lines, with primary and secondary constants. Chapter 8 covers Ultra High Frequency (UHF) transmission lines, along with the use of the Smith chart in solving small-signal problems.

At the end of the chapters are enough review problems designed to help the students exercise their level of comprehension of the treated matters, and by so doing internalize the underlying principles of the lessons taught.

About the Author

Udeze Kenneth Ugo hails from Onicha Ugbo in Aniocha North Local Government Area in Delta State, Nigeria. He attended his primary school at Aniemeke Primary School Onicha Ugbo, Delta State, Nigeria and attended his secondary education at Model Secondary School Maitama, Abuja, FCT, Nigeria where he obtained his Senior School Certificate in 2003.

Between 2005 and 2010, he obtained his National Diploma (ND) and Higher National Diploma (HND) in Electrical and Electronics Engineering (Telecommunication and Electronics Options) with a CGPA of 3.68/4 i.e., Distinction Honors from Federal Polytechnic Oko, Anambra State, Nigeria. He also obtained his first degree in Electrical and Electronics Engineering (Power, Telecommunication and Electronics Options), in 2013 from the prestigious University of Ibadan, Oyo State, Nigeria with a CGPA of 5.8/7 i.e., a Second-Class Upper Division (2.1).

After his one year mandatory National Youth Service Corp (NYSC) in Electrical and Electronics Engineering Department in Federal Polytechnic Oko, Anambra State, Nigeria in 2015, he proceeded to obtain his Masters degree in Offshore Engineering in 2016, majored in Offshore Design and installations, Subsea umbilical cables designed, Installation and maintenance of offshore facilities, Submarine power cable design and maintenance, Subsea instrumentation and control system (E&I) from Offshore Technology Institute, School of Advance Engineering, University of Port-Harcourt, Rivers State, Nigeria. Then a second Masters degree in Electrical and Electronics Engineering and majored in Power System Engineering, from University of Lagos, Lagos State, Nigeria in 2023. He graduated with a CGPA of 4.7/5 i.e., Distinction Honors.

He is currently a staff of Federal Polytechnic Oko, Anambra State, Nigeria attached to Electrical and Electronics Engineering Department. He teaches Mathematics and Electrical Engineering courses.

He is presently prospecting for PhD admission overseas for researches in Renewable Energy.

CHAPTER 1

LAPLACE TRANSFORM

1.0 The Concept of Complex Frequency

Laplace transform is a tool or device conveniently utilized to transform time-domain functions (s-domain) for the purpose of circuit analysis. Experience has shown that while dealing with transient analysis, it was found to be rather tedious and cumbersome dealing in time domain with several steps, intermediate or otherwise, involved in order to determine the initial conditions etc. Laplace transform makes it possible to solving time-domain integer differential equations in s-domain by working algebraically. Here differentiation in time domain corresponds to multiplying by s in frequency domain, whereas integration in time domain is equivalent to division by same $\boldsymbol{s}$.

The identity of the transformed function has to be presented, and the exponential function comes in handy! This is because it is the only function in all of mathematics that has the unique character of retaining its identity upon differentiating or integrating $\left(\frac{de^{at}}{dt} = ae^{at} \Leftrightarrow \int e^{at}\, dt = \frac{e^{at}}{a}\right)$, [note how the function is reproduced (reappears) in the foregoing two operations in parenthesis].

For a given function $f(t)$ in time domain, its Laplace transform is

$$\mathcal{L}f(t) = F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st} dt \qquad 1.0$$

On the frequency domain (s domain).

This is the article of faith, our working tool which is strictly a definition, so do not ask for a ‘’proof” since none is required for a defined relation.

A given function, when Laplace transform, has a unique value, so the use of Laplace transform table is a perfectly valid method of getting the answer (response) back into the original time domain which by reason of habit is presumed to be more familiar. An analogous situation is the process of multiplication or division whereby the logarithm table (found as part of the four-figure table in use up until a few decades back when it got to be supplemented by the laziness-inducing handheld pocket “calculator”) is used to convert inconvenient numbers to simpler numbers equivalent to raising **10** to other numbers. Law of indices then allows the resulting index numbers to be added (or subtracted), and the result then determined by employing an antilog table.

Since a given function has a unique value when Laplace transformed, the use of an inverse transform table is a perfectly valid method of getting the transformed result (s domain) back into the original time domain.

For a given function $f(t)$, its two-sided Laplace transform is defined to be.

$$\mathcal{L}f(t) = \int_{-\infty}^{\infty} f(t)e^{-st}\,dt = F(s) \qquad 1.1$$

Where the upper-case letter is used to designate the transformed function. Note once again, this purely a definition, so no "proof" is required. However, its validity would be established if we step back a bit and show how to traverse from time domain to frequency (s) domain by subsuming or suppressing an e^{st} (exponential st) factor.

A general damped sinusoidal (i.e. time varying) function, say a voltage signal can be written as: $v(t) = V_m\, e^{\sigma t}\cos(\omega t + \theta)$ (the cosine is the conventional trigonometric function employed in the analysis, not the sine), where V_m is the maximum value (amplitude), $e^{\sigma t}$ is the damping function (σ the damping factor necessarily taken to be negative for positive time since any given signal must of necessity converge (ultimately to zero value) in the absence of reinforcing factor], ω (Greek alphabet omega, not "double-u") is the radian frequency in rad/s $[\omega = 2\pi f]$, f being the cyclic frequency in cycles per second (hertz); and θ is the phase angle of the given voltage signal with respect to the current assumed to have zero phase (angle). It can be written in degrees or radians, although the letter is more acceptable.

By Euler's identity:

$$e^{j\beta} = \cos\beta + j\sin\beta \qquad 1.2$$

The sinusoidal function $\cos(\omega t + \theta)$ can be written as:

$$\cos(\omega t + \theta) = Re\,\{\cos(\omega t + \theta)\ + j\sin(\omega t + \theta)\}$$

$$= Re\left\{e^{j(\omega t + \theta)}\right\}$$

So,

$$v(t) = V_m e^{\sigma t}\cos(\omega t + \theta) = Re\left\{V_m e^{\sigma t}\, e^{j(\omega t + \theta)}\right\}$$

$$= Re\left\{V_m\, e^{j\theta}\, e^{(\sigma + j\omega t)}\right\} \qquad 1.3$$

$s = \sigma + j\omega$ is known as complex frequency, where σ is the neper frequency, and ω (omega, not "w") is radian frequency as previously mentioned σ is the real part, and ω (not $j\omega$) the imaginary part.

$v(t)$ can then be expressed as:

$$v(t) = Re\left\{V_m\, e^{j\theta}e^{st}\right\}$$

For the general expression of $v(t)$ as a damped sinusoidal, if $s = \sigma + j\omega = 0$, so that both σ and ω are zero, then we have simply:

$$v(t) = V_m\, e^{(0)t} \cos(0 + \theta) = V_m \cos\theta = V_0,$$

A constant which means a d.c. (direct current) signal.

With $s = 0 + j\omega\ (\sigma - 0)$

$$v(t) = V_m \cos(\omega t + \theta)$$

A purely sinusoidal signal (with no damping factor).

For $s = \sigma + j0\ (\omega = 0)$,

$v(t) = V_m\, e^{\sigma t} \cos\theta = V_0 e^{\sigma t}$, just a damped exponential signal with no sinusoidal factor.

Example 1.1: Given that circuit in Fig. 1.1 takes us on a journey from time domain to frequency domain to get to a forced current response with a given input voltage.

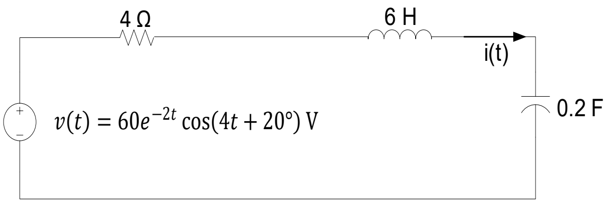


Figure 1.1

The voltage source in Fig. 1.1 is a fully damped (exponential) sinusoidal signal and we desire to determine the series current as the forced response.

We already know that in circuit analysis in time domain, that the forced response of a given signal assumes the same character as the input signal, it would have the same radian frequency and sinusoidal nature, differing only, perhaps in the amplitude and possibly the phase angle. Therefore, one needs only determine the amplitude and phase to be able to readily but down the expression for the forced response:

$i(t) = I_m\, e^{-2t} \cos(4t + \theta)\ and\ I_m$, the maximum current (amplitude) and phase (θ) are the only factors we need to determine from the input (nature).

$$v(t) = Re\left\{60\, e^{-2t} e^{j}(4t + 10^{\circ})\right\} = Re\left\{60\, e^{j10^{\circ}} e^{(-2+j4)t}\right\} \qquad 1.4$$

$$= Re\left\{\vec{V}\, e^{st}\right\}$$

where $\vec{v} = 60\angle 20^{\circ}$ is the polar representation of the voltage vector, and $s = -2 + j4$

$Re \implies 60\angle 20^{\circ} e^{st}$ merely for convenience.

Similarly, $i(t)$ "equals" $\vec{I}e^{st}$, where $\vec{I} = I_m \angle \emptyset$,

KVL:

$$v(t) = 4i + 6\frac{di}{dt} + 5\int i\,dt \qquad \left(Remark\colon \frac{1}{0.2} = 5\right)$$

Substituting,

$$60\angle 20^{\circ} e^{st} = 4Ie^{st} + 6s\,Ie^{st} + \frac{5}{s}Ie^{st}$$

$$60\angle 20^{\circ} = 4I + 6sI + \frac{5}{s}I$$

$$I = \frac{60\angle 20^{\circ}}{4 + 6s + \frac{5}{s}} = \frac{60\angle 20^{\circ}}{4 + 6(-2 + j4) + \frac{5}{(-2 + j4)}}$$

$$= \frac{60\angle 20^{\circ}}{4 - 12 + j24 + 5 \times \frac{(-2 - j4)}{20}} = \frac{60\angle 20^{\circ}}{-8.5 + j23} = \frac{120\angle 20^{\circ}}{17 + j46}$$

$$= \frac{120\angle 20^{\circ}}{49.04\angle 110.28^{\circ}} = 2.45\angle -90.28^{\circ}$$

So, the two qualities that we set out to determine, namely, I_m and $\emptyset$ are 2.45 A and -90.28°, respectively, and the time domain expression for $i(t)$ is:

$$i\,(t) = 2.45e^{-2t}\cos(4t - 90.28^{\circ})\ \mathrm{A}$$

In the foregoing **Example 1.1**, the extreme usefulness of Laplace transform was demonstrated as we shall soon see. Searchlight was also beamed on the exponential function as the pivotal factor in the definition of Laplace transform, albeit the negative factor.

For first order and some second order differential equations in time domain, given the initial conditions one in the case of 1st-order equation, and two for 2nd-order the response can be determined in relatively straightforward manner working in time domain, but for higher order differential equations, the process can be rather tedious and/or cumbersome. So, here Laplace transform comes in handy, and possesses a further advantage of delivering the complete response in one fell swoop, considering the initial conditions, and the forced response. This contrast with the time domain analysis where the component parts of the complete response had to be delivered piecemeal!

1.1 Definition of Laplace Transform

Because Laplace transform is valid only for positive values of time, its defining equations is as seen in Eq. 1.1:

$$\mathcal{L}f(t) = \int_{0}^{\infty} f(t)\, e^{-st} dt,$$

The lower limit of integration being 0^{-}rather than just 0, in order to consider any discontinuities and higher-order singularities that might occur at (exactly) time zero. Also, taking the lower limit of integration to be precisely zero might work for certain functions, but might get us in trouble with some peculiar or improper function

For the purpose of comparing and contrasting let's do one example before learning two sided

Example 1.2: Determine the two-sided Laplace transform of the function

$$f(t) = -2e^{-3t}[u\,(t+3) - u\,(t-2)]$$

Solution:

$$\mathcal{L}f(t) = -2\int_{-\infty}^{\infty} e^{-3t}\,[u\,(t+3) - u(t-2)]e^{-st}dt$$

$$= -2\,\int_{-3}^{\infty} e^{-3t}\;\; e^{-st}\, dt - \int_{2}^{\infty} e^{-3t}\;\; e^{-st}\, dt$$

$$u(t+3) = \begin{cases} 0, t < -3 \\ 1, t \geq -3 \end{cases}$$

$$u\,(t-2) = \begin{Bmatrix} 0, t < 2 \\ 1, t \;\; \geq 2 \end{Bmatrix}$$

$$\mathcal{L}f(t) = -2\int_{-3}^{\infty} e^{-(s+3)t}\;\, dt - \int_{2}^{\infty} e^{-(s+3)t}\;\, dt$$

$$= -2\left(\frac{e^{-(s+3)t}}{-(s+3)}\bigg|_{-3}^{\infty} - \frac{e^{-(s+3)t}}{-(s+3)}\bigg|_{2}^{\infty}\right)$$

$$= -2\left(0 - \frac{e^{-(s+3)(-3)}}{-(s+3)} + 0 - \frac{e^{-(s+3)(2)}}{(s+3)}\right)$$

$$= \frac{-2}{s+3} \times (e^{3s+9} - e^{-2s-6})$$

$$= \frac{-2}{s+3} \times \left(e^{-2s-6} - e^{3s+9}\right)$$

Example 1.3: Determine the one-sided Laplace Transform of **Example 1.2**.

Solution:

$$\mathcal{L}f(t) = -2\left(\int_0^{\infty} e^{-(s+3)t}\, dt - \int_2^{\infty} e^{-(s+3)t} dt\right)$$

$$-2\left(\int_{0-}^{\infty} e^{-(s+3)t} dt - \int_2^{\infty} e^{-(s+3)t}\, dt\right)$$

$$= -2\left[\left.\frac{e^{-(s+3)t}}{-(s+3)}\right|_0^{\infty} - \left.\frac{e^{-(s+3)t}}{-(s+3)}\right|_2^{\infty}\right]$$

$$= -2\left[\left(0 - \frac{1}{-(s+3)}\right) + \left(0 - \frac{e^{-(s+3)(2)}}{s+3}\right)\right] = \frac{2}{s+3}\left[e^{-2s-6} - 1\right]$$

Note: the first (exponential) expressions are identical for both the two sided and one sided but the second terms differ because the limits of integration for the first integral expression are taken from 0^- to ∞ for the one-sided ignoring from -3 to 0^-, resulting in $\frac{1}{(s+3)}$, whereas these limits are included for the two-sided.

1.2 Comparing Time Domain and S-Domain Analysis

Example 1.4: Determine the response $i(t)$ for the circuit of Fig. 1.2.

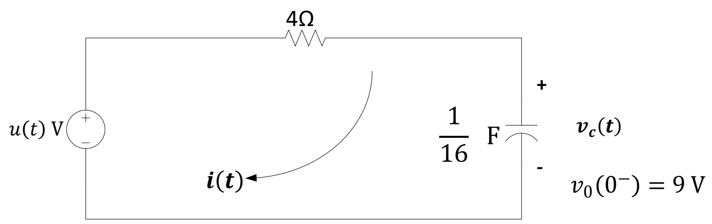


Figure 1.2

Solution: $u(t) = 14i + 16\int_{-\infty}^{t} i\,(t)dt$ (initial energy storage in the capacitor is accounted for, from time $-\infty$ to 0)

Differentiate across: $\delta(t) = 4\frac{di(t)}{dt} + 4i(t)$

$$\Longrightarrow \qquad 0 = \frac{di(t)}{dt} + 4i(t),$$

and zero input here means that complete response is the same as the natural response since the forcing function is implicitly zero.

$$\delta(t) = 0, \qquad\qquad t \neq 0 \text{ (i.e} \quad t = 0^{+})$$

$$i(t) = Ke^{-4t}$$

$$i(0) = \frac{(1-9)V}{4\Omega} = -2A = Ke^{o} = K$$

$$\Longrightarrow \qquad i(t) = 2e^{-4t}\,u(t) \text{ A}$$

By Laplace transform

$$u(t) = i(t) + 16\int_{-\infty}^{t} i\,(t)dt$$

The limits of integration here have to be properly readjusted to apply the transform:

$$u(t) = 4i(t) + 16\left[\int_{-\infty}^{0-} i(t)dt \; + \int_{0-}^{t} i\,(t)dt\right]$$

The second term on the right represent the initial capacitor voltage, 9 V

$$u(t) = 4i(t) + v\,(0^{-}) + 16\int_{0-}^{t} i\,(t)dt$$

Laplace transforming:

$$\frac{1}{s} = 4I(s) + \frac{9}{s} + 16\frac{I(s)}{s}$$

(Note: these Laplace transform will be derived later)

$$4I(s) + 16I(s)\frac{1}{s} = \frac{1}{s} - \frac{9}{s} = -\frac{8}{s}$$

$$I(s) + 4I(s)\frac{1}{s} = -\frac{2}{s}$$

$$\Longrightarrow I(s) = \left(-\frac{2}{s}\right) \times \left(\frac{s}{s+4}\right)$$

$$= -\frac{2}{(s+4)}$$

$$I(s) = -\frac{2}{(s+4)} \Longrightarrow -2e^{-4t}u(t)\text{ A} = i(t), \qquad \text{as before}$$

1.3 Laplace Transforms of Common Functions

1. The Heaviside unit step function $u(t)$, already encountered earlier in examples:

$$\mathcal{L}\,u(t) = \int_{0-}^{\infty} e^{-st}\,u(t)dt = \int_{0}^{\infty} e^{-st}\,(1)dt \qquad 1.5$$

(Note: $u(t)$ takes on the value of unity from $t = 0$)

$$\int_{0-}^{\infty} e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right|_{0}^{\infty} = 0 - -\frac{1}{s} = \frac{1}{s} = \mathcal{L}\,u(t)$$

2. The ramp function $t\,u(t)$

$$\int_{0-}^{\infty} e^{-st}\,tu(t)dt = \int_{0}^{\infty} te^{-st}\,dt$$

Employing integration by parts: $[\int udv = uv - \int v\,du]$

$$u = t; dv = e^{-st}\,dt$$

$$du = dt, v = -\frac{e^{-st}}{s}$$

$$\int udv = uv - \int vdu$$

$$\int_{0}^{\infty} te^{-st}dt = \left.\frac{-te^{-s}}{s}\right|_{0}^{\infty} - \int_{0}^{\infty}\left(-\frac{e^{-st}}{s}\right)dt$$

$$= 0 - 0 + \frac{1}{s}\int_{0}^{\infty} e^{-st}\,dt = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s}$$

Addendum: differentiation in the frequency domain

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_{0^-}^{\infty} e^{-st}\,f(t)dt = \int_{0^-}^{\infty} -te^{-st}\,f(t)dt$$

$$= \int_{0^-}^{\infty} e^{-st}\,[-tf(t)]dt = \mathcal{L}[-tf(t)]$$

This shows that differentiating with respect to s in the frequency domain, is equivalent to multiplying by $(-t)$ in the time domain, or

$\Longrightarrow \qquad \mathcal{L}tf(t) = -\frac{d}{ds}F(s) = -F'(s)$

Example 1.5: Find the $\mathcal{L}tu\ (t)$

Solution

If $\mathcal{L}tu(t) = \frac{1}{s^2}$, then $\mathcal{L}[t^2u(t)] = -\frac{d}{ds}\left(\frac{1}{s^2}\right)$,

Where $f\ (t) = t^2u(t)$

$$-\frac{d}{ds}\left(\frac{1}{s^2}\right) = -\frac{d}{ds}(s^2) = - - 2s^{-3} = \frac{2}{s^3}$$

Check: $\mathcal{L}t[tu(t)] = \mathcal{L}t^2u\ (t) = \left.\frac{n^1}{s^{n+1}}\right|_{n=2} = \frac{2}{s^3}$

$$\mathcal{L}t^3u(t) = \mathcal{L}t[t^2u(t)] = -\frac{d}{ds}\left(\frac{2}{s^3}\right) = -\frac{d}{ds}2(-3)s^{-3-1} = \frac{6}{s^4}$$

Check: $\mathcal{L}t^3 = \frac{3}{s^{3+1}} = \frac{6}{s^4}$

$$\mathcal{L}t^4u(t) = \mathcal{L}t[t^3u(t)] = -\frac{d}{ds}\left(\frac{6}{s^4}\right) = -\frac{d}{ds}(6(-4)s^{-4-1}) = \frac{24}{s^5}$$

Check:

$$\mathcal{L}\ t^4u(t) = \frac{4}{s^4+1} = \frac{24}{s^5}$$

$$\Longrightarrow \qquad \mathcal{L}t^nu(t) = \frac{n!}{s^{n+1}} \text{ earlier proven}$$

By replacing n by $n-1$ (i.e, $n+1$ by n),

$$\mathcal{L}t^nu(t) = \frac{(n-1)!}{s^n} \qquad 1.6$$

Example 1.6: Find the LT of $t^2u(t-3)$.

Solution

$$t^2u\ (t-3) = [(t-3)^2 + 6(t-3) + 9]u\ (t-3)$$

$$= (t-3)^2.u(t-3) + 6(t-3)u\ (t-3) + 9\ u\ (t-3)$$

$$\mathcal{L}\ t^2u\ (t-3) = \mathcal{L}(t-3)^2.u(t-3) + 6\mathcal{L}(t-3).u(t-3) + 9\mathcal{L}u(t-3)$$

$$= e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$$

Alliteratively $\mathcal{L}\ t^2u\ (t-3) = e^{-3s}\mathcal{L}\ (t-3)^2 = e^{-3s}\mathcal{L}[t^2 + 6t + 9]$

$$= e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$$

Example 1.7: Find the Laplace transform of cos 6t

Solution

$$\mathcal{L}\cos 6t \; = \frac{s}{s^2 + 36}$$

Example 1.8: Find the Laplace transform of $\cos^2$t.

Solution

$$\cos 2t = 2\, cos^2 t - 1$$

$$\therefore \qquad cos^2\, t = \tfrac{1}{2}\,[\cos 2t + 1\,]$$

$$\mathcal{L}(cos^2 t) = \mathcal{L}\left[\frac{1}{2}(\cos 2t + 1)\right] = \frac{1}{2}\,[\mathcal{L}(\cos 2t) + \mathcal{L}(1)]$$

$$= \frac{1}{2}\left[\frac{2}{s^2 \; + \; (2)^2} + \frac{1}{s}\right] \; = \frac{1}{2}\left[\frac{2}{s^2 \; + \; 4} + \frac{1}{s}\right]$$

3. The exponential function e^{at} with a constant:

$$\int_{0^-}^{\infty} e^{-st}\, e^{at}\, dt = \int_{0^-}^{\infty} e^{-\frac{s-a}{t}}\, dt = \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_{0^-}^{\infty}$$

$$= 0 - -\frac{1}{s-a} = \frac{1}{s-a} = \mathcal{L}e^{at}$$

$$\text{The Implicitly}, \mathcal{L}e^{-at} \; = \frac{1}{s+a}$$

As an exercise break $\cos at$ above into its "ruler parts" $\frac{(e^{j\,at} + e^{-j\,at})}{2}$ and then integrate procedure, to get the same answer as in (**3**) above. (It might necessitate complex numbers!)

4. Laplace transform of a derivative time differentiating theorem

$$\mathcal{L}\,\frac{df\,(t)}{dt} = \mathcal{L}\,f'(t) = \int_{0-}^{\infty} e^{-st}\, f'(t)dt$$

$$u = e^{-st}, du = -se^{-st}\,dt, dv = f'(t)dt, v = f(t)$$

$$\int udv = uv - \int vdu = e^{-st}f(t)|_{0-}^{\infty} + s\int_{0-}^{\infty} e^{-st}\ f\ (t)$$

The integral term on the right is simply the Laplace transform of $f(t)$, which is $F(s)$

So, $\mathcal{L}\, f'(t) = 0 - e^0 f(0^-) + sF(s)$

$$= sF(s) - f(0^-)$$

The transform of the second derivative of $f(t)$, that is $f'(t) = \frac{d^2 f(t)}{dt^2}$, may be solved either by interpolation or by direct application of the defining formula for Laplace transform using integration by parts:

By interpolation,

$$\mathcal{L}\, f''(t) = s[sF(s) - f(0^-)] - f'(0^-)$$

$$\mathcal{L}\, f''(t) = s^2F(s) - sf\ (0^-) - f'(0)$$

$$\mathcal{L}\, f'''(t) = s[s^2F(s) - sf(0^-) - f'(0^-)] - f''(0^-)$$

$$\mathcal{L} f'''(t) = s^3F(s) - s^2 f\ (0^-) - s\,f'(0^-) - f''(0^-)$$

For with derivative

$$\mathcal{L} f^n(t) = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - s^{n-3}\, f''(0^-) \ldots$$
$$-sf^{n-2}\,(0^-) - f^{n-1}(0^-)$$

Follow the pattern: from the second term on the indices of s and $f(0^-)$ are homogenous in add up to $n-1$, terminating where s takes up the zeroth index ($s^0 = 1$) and $f(0^-)$ therefore with $(n-1)$ index.

5. Laplace transform of an integral: Time-integration theorem

$$\mathcal{L} f^{-1}(t) = \mathcal{L}\left[\int_0^t f(x)dx\right] = \int_0^{\infty} e^{-st}\left[\int_0^t f(x)dx\right]dt$$

$$u = \int_0^t f(x)dx \Rightarrow du = f(t)dt$$

$$dv = e^{-st}$$

$$\Rightarrow v = -\frac{e^{-st}}{s}$$

$$\Longrightarrow \qquad \mathcal{L}f^{-1}(t) = \left[\int_0^t f\,(x)dx\right]\left(-\frac{e^{-st}}{s}\right) + \frac{1}{s}\int_{0^-}^{\infty} e^{-st}\ f(t)dt$$

Again, the integral expression on the far right in simply the Laplace transform of $f(t)$, the zeroth derivative, making that whole second term equal to $\frac{F(s)}{s}$. The first term is slightly tricky and must be evaluated carefully. Bear in mind that here, x is a dummy variable whereas t is the operating variable.

$$\left[-e^{-st}\int_0^t f(x)dx\right]_0^{\infty} = -0 - -\left[e^0\int_0^{0^-} f(x)dx = \int_0^{0^-} f(x)dx\right]$$

And thus, is simply the integral of $f(t)$ evaluated at 0^-

$$\mathcal{L}\int_0^t f(x)dx = \mathcal{L}f^{-1}(t) = \frac{F(s)}{s} + \frac{F^{-1}(0^-)}{s} \qquad 1.7$$

Table 1.1

Properties of the Laplace transform

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_2 f_2(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as}F(s)$
Frequency shift	$e^{\mp at}f(t)$	$F(s \pm a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2f}{dt^2}$	$s^2F(s) - sf'(0^-) - f(0^-)$
	$\frac{d^3f}{dt^3}$	$s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^{"}f}{dt^{"}}$	$s^nF(s) - s^{n-1}f(0^-)s^{n-2}f'(0^-) - \cdots - f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(t)dt$	$\frac{1}{s}F(s)$

Frequency differentiation	$tf(t)$	$-\frac{d}{ds}F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)\,ds$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
Initial value	$f(0)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$
Convolution	$f_1(t) * f_2(t)$	$F_1(s) \times F_2(s)$

Table 1.2

Laplace transform pairs

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$e^{\mp at}$	$\frac{1}{s \pm a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
$te^{\mp at}$	$\frac{1}{(s \pm a)^2}$
$t^n e^{\mp at}$	$\frac{n!}{(s \pm a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$

$\sin(\omega t + \theta)$	$\dfrac{s \sin\theta + \omega\cos\theta}{s^2 + \omega^2}$
$\cos(\omega t + \theta)$	$\dfrac{s \cos\theta - \omega\sin\theta}{s^2 + \omega^2}$
$e^{\mp at} \sin \omega t$	$\dfrac{\omega}{(s \pm a)^2 + \omega^2}$
$e^{\mp at} \cos \omega t$	$\dfrac{s + a}{(s \pm a)^2 + \omega^2}$

1.4 Laplace Transform Theorems

1. **Linearity Theorem**: states that the transform the sum of functions is simply the sum of the transforms of the individual's functions:
Proof:

$$\mathcal{L}\,[f_1(t) + f_2(t)] = \int_{0-}^{\infty} e^{-st}\,[f_1(t) + f_2(t)]dt \qquad 1.8$$

$$= \int_{0-}^{\infty} e^{-st}\, f_1(t)dt + \int_{0-}^{\infty} e^{-st} f_2(t)dt = f_1(s) + f_2(s)$$

2. **Homogeneity Property**: LT of a product of a constant and a function is simply the product of that constant and the Laplace transform of that function:

$$\mathcal{L}\,[cf\,(t)] = \int_{0^-}^{\infty} e^{-st}\, cf(t)dt = c\int_{0^-}^{\infty} e^{-st}\, f(t)dt = cF(s) \qquad 1.9$$

3. **First-Shift (frequency shift) theorem:** also called complex translation, states that multiplying a function by e^{at} in the time domain, results in subtracting a form s in the frequency domain after Laplacing the original unmultiplied function.
Proof:

$$\mathcal{L}[e^{at} f(t)] = \int_{0}^{\infty} e^{-st}\; e^{at} f(t)dt \qquad 1.10$$

$$= \int_{0^-}^{\infty} e^{-(s-a)t} f\,(t)dt$$

But,

$$\int_{0^-}^{\infty} e^{-st} f(t)dt = F(s), \text{so logically,}$$

$$\int_{0^-}^{\infty} e^{-(s-a)t}\; f(t)dt = F(s-a),$$

Where $(s - a)$ has replaced s in the exponential term.

4. Unit Step Function

Laplace Transform of units' step functions

$$\mathcal{L}\,[u(t-a)] = \frac{e^{-a\,s}}{s}$$

Proof.

$$\mathcal{L}\,[u(t-a)] = \int_0^\infty e^{-st}\,u(t-a)\,dt$$

$$= \int_0^\infty e^{-st}\,.0\ \ dt\ +\ \int_a^\infty e^{-st}\,.\ 1\ \ dt\ = 0 + \left[\frac{e^{-st}}{s}\right]_a^\infty$$

$$\mathcal{L}\,[u(t-a)] = \frac{e^{-as}}{s}$$

With the help of unit step functions, we can find the

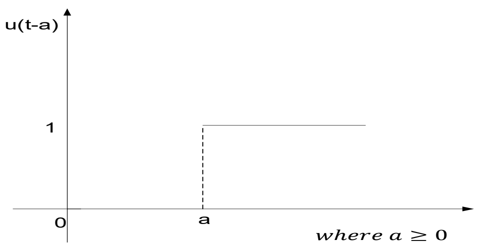


Figure 1.3

Inverse transform of functions, which cannot be determined

with previous methods.

The unit step functions $u\ (t-a)$ is define as follows:

$$u(t-a) = \begin{Bmatrix} o & when\ \ t\ \le a \\ 1 & when\ \ \ t \ge a \end{Bmatrix} \qquad\qquad where\ \ a\ \ge 0$$

Example 1.9: Express the following in terms of unit step function and find its Laplace transform:

$$f\ (t) = \begin{bmatrix} 8, & t\ \le 2 \\ 6, & t \ge 2 \end{bmatrix}$$

Solution:

$$f(t) = \begin{bmatrix} 8+0 & t\ \le 2 \\ 8-2 & t \ge 2 \end{bmatrix}$$

$$= 8 + \begin{bmatrix} 0 & t\ \le 2 \\ -2 & t\ \ge 2 \end{bmatrix}$$

$$= 8 - 2\,u\,(t-2)$$

$$\mathcal{L}f\ (t) = 8\mathcal{L}(1) - 2\mathcal{L}(t-2) = \frac{8}{s} - \ 2\ \frac{e^{-2s}}{s}$$

Example 1.10: Draw the graph of $u(t-a) - u\ (t-b)$

Solution:

The graph of $u(t-a)$ is a straight line from A to ∞. Similarly, the graph of $u(t-b)$ a straight line from B to ∞.

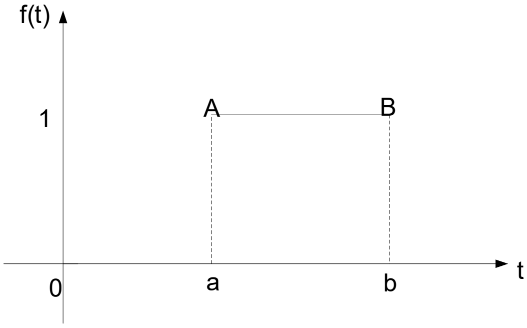


Figure 1.4

Hence, the graph of $u[t-a] - u[t-b]$ is AB.

Example 1.11: Express the following functions in terms of unit step function and find its Laplace transform:

$$f\ (t) = \begin{bmatrix} E & a \le t \le b \\ 0 & t > b \end{bmatrix}$$

Solution. $$f\ (t) = \begin{bmatrix} E & a \le t \le b \\ 0 & t > b \end{bmatrix} = E\ [u(t-a) - u(t-b)]$$

$$\mathcal{L}f\ (t) = E\left[\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right]$$

Example 1.12: Express the following function in terms step function:

$$f\ (t) = \begin{bmatrix} t-1 & 1 \le t \le 2 \\ 3-t & 2 \le t \le 3 \\ 0 & t > 3 \end{bmatrix}$$

and find its Laplace transform

Solution:

$$f(t) = \begin{bmatrix} t-1 & 1 \le t \le 2 \\ 3-t & 2 \le t \le 3 \\ 0 & t > 3 \end{bmatrix}$$

$$= (t-1)[u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)]$$

$$= (t-1)\,u(t-1) - (t-1)\,u(t-2) + \ (3-t)\,u(t-2) + (t-3)\,u(t-3)$$

$$= (t-1)\,u(t-1) - 2(t-2)\,u(t-2) + (t-3)\,u(t-3)$$

$$\mathcal{L}f(t) = \frac{e^{-s}}{s^2} - 2\frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$$

4. **Second Shift (Time-Shift) Theorem**: also called real translation, states that shifting a function by time c in the time domain and multiplying the result by an equally shifted unit step function, is equivalent to multiplying the Laplace transform of the original un-shifted function by the exponential of $(-cs)$:

Proof:

$$\mathcal{L}[f(t-c)u(t-c)] = \int_{0^-}^{\infty} e^{-st}\ f(t-c)u(t-c)dt \qquad 1.11$$

But $u(t-c)$" switches on" at $t = c$ to a value of simply unity resulting in the lower integral limit of c instead of 0^-

Let $t' = t - c$, where t is known as a "dummy" variable. So $dt' = dt - 0 = dt$,

$$\mathcal{L} = \int_{c}^{\infty} e^{-s(t'+c)}\ f(t')\ dt' \text{are what after the change of variables results}$$

$\mathcal{L}[\,f(t-c)u(t-c)] = e^{-sc}\int_{c^-}^{\infty} e^{-st}\ f(t)dt'$ not exactly in the "shape" of Laplace transform since the lower limit (of t, not t) is c instead if 0^-. This lower limit c, has to be properly adjusted to c^- to account for the fact that the lower limit for the transform expression is 0^- and not 0.

But $t = c^- \Longrightarrow t' = t - c = c^- - c = 0^-$ and so c^- for t variables correspond to 0^- for t variables we're finally able to express.

$$\mathcal{L}[f(t-c)u(t-c)] = e^{-sc}\int_{0^-}^{\infty} e^{-st}\ f(t')dt'$$

And the integral expression is still a Laplace transform regardless of the name given to the variable of the moment! (The variable t is not sacrosanct but only seemed so on the account of the fact that 't' is used to represent line a symbol!)

$$\mathcal{L}[f(t-c)u\,(t-c)] = e^{-sc}F(s) \qquad 1.12$$

Example 1.13: A function $f(t)$ is defined by $f(t) = \begin{cases} 4 & 0 \le t \le 2 \\ 2t-3 & 2 > t \end{cases}$

Sketch the graph of the function and determine its LT.

Solution: We see that for $t = 0$ to $t = 2, f(t) = 4$

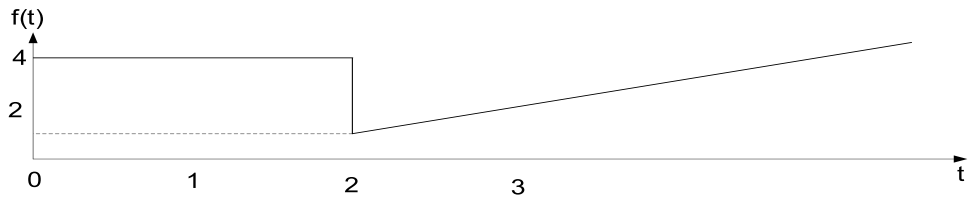


Figure 1.5

Notice the discontinuity at $t = 2$

Expressing the function in unit step form:

$$f(t) = 4u(t) - 4u(t-2) + u(t-2)\cdot(2t-3)$$

Note that the second term cancels $f(t) = 4$ at $t = 2$ and that the third switches on $f(t) = 2t - 3$ at $t = 2$

Before we can express this in Laplace transforms, $(2t-3)$ in the third term must be written as a function of $(t-2)$ to correspond to $u(t-2)$. therefore, we write $2t-3$ as $2(t-2)+1$.

Then

$$f(t) = 4u(t) - 4u(t-2) + u(t-2)\cdot\{2(t-2)+1\}$$

$$= 4u(t) - 4u(t-2) + u(t-2)\cdot 2(t-2) + u(t-2)$$

$$= 4u(t) - 3u(t-2) + u(t-2)\cdot 2(t-2)$$

$$\mathcal{L}\{f(t)\} = \frac{4}{s} - \frac{3e^{-2s}}{s} + \frac{2e^{-2s}}{s^2}$$

Example 1.14: A function is defined by $f(t) = \begin{cases} 6 & 0 \le t \le 1 \\ 8-2t & 1 \le t \le 3 \\ 4 & t > 3 \end{cases}$

Sketch the graph and find the Laplace transform of the function.

Solution:

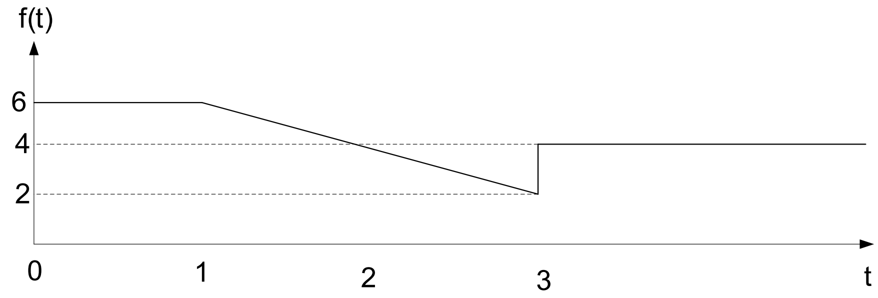


Figure 1.6

Expressing the graph in-unit step form we have:

$$f(t) = 6u(t) - 6u(t-1) + u(t-1)\cdot(8-2t)$$

$$-u(t-3)\cdot(8-2t) + 4.u(t-3)$$

Where the second term switches off the first function $f(t) = 6$ at $t = 1$ and the third term switches on the second function $f(t) = 8 - 2t,$ which in turn is switched off by the fourth term at $t = 3$ and replaced by $f(t) = 4$ in the fifth term.

Before we can write down the transform of the third and fourth terms, we must express $f(t) = 8 - 2t$ in terms of $(t-1)$ and $(t-3)$ respectively.

$$8 - 2t = 6 + 2 - 2t = 6 - 2(t-1)$$

$$8 - 2t = 2 + 6 - 2t = 2 - 2(t-3)$$

$$\therefore f(t) = 6u(t) - 6u(t-1) + u(t-1)\cdot\{6-2(t-1)\} - u(t-3)\cdot\{2-2(t-3)\} + 4u(t-3)$$

$$= 6u(t) - 6u(t-1) + 6u(t-1) - u(t-1)\cdot 2(t-1) - 2u(t-3) + u(t-3)\cdot 2(t-3) + 4u(t-3)$$

$$f(t) = 6u(t) - u(t-1)\cdot 2(t-1) + u(t-3)\cdot 2(t-3) + 2u(t-3)$$

$$\mathcal{L}\{f(t)\} = \frac{6}{s} - \frac{2e^{-s}}{s^2} + \frac{2e^{-3s}}{s^2} - \frac{2e^{-3s}}{s}$$

Alternatively, if

$$f(t) = 6u(t) - 6u(t-1) + u(t-1)\cdot(8-2t) - u(t-3)\cdot(8-2t) + u(t-3).4$$

$$= 6u(t) - 6u(t-1) + 8u(t-1) - 2tu(t-1) - 8u(t-3) + 2tu(t-3) + 4u(t-3)$$

$$= 6u(t) - 6u(t-1) + 8u(t-1) - 2(t+1)u(t-1) - 8u(t-3)$$

$$+2(t+3)u(t-3)+4u(t-3)$$

$$\mathcal{L}\{f(t)\}=\frac{6}{s}-\frac{6}{s}e^{-s}+\frac{8}{s}e^{-s}-2\left(\frac{1}{s^2}+\frac{1}{s}\right)e^{-s}-\frac{8}{s}\ e^{-3s}+2\left(\frac{1}{s^2}+\frac{1}{s}\right)e^{-3s}+\frac{4}{s}e^{-3s}$$

$$\mathcal{L}\{f(t)\}=\frac{6}{s}-\frac{6}{s}e^{-s}+\frac{8}{s}e^{-s}-\frac{2}{s^2}e^{-s}-\frac{2}{s}e^{-s}-\frac{8}{s}\ e^{-3s}+\frac{2}{s^2}e^{-3s}+\frac{2}{s}e^{-3s}+\frac{4}{s}e^{-3s}$$

Collecting like terms,

$$\mathcal{L}\{f(t)\}=\frac{6}{s}+\frac{1}{s}(-6+8-2)e^{-s}-\frac{2}{s^2}e^{-s}+\frac{1}{s}(-8+2+4)e^{-3s}+\frac{2}{s^2}e^{-3s}$$

$$F(s)=\frac{6}{s}-\frac{2e^{-s}}{s^2}-\frac{2e^{-3s}}{s}+\frac{2e^{-3s}}{s^2}$$

THEOREM $\mathcal{L}[f\ (t)\ u(t-a)=e^{-sa}\ \mathcal{L}\ [f\ (t+a)]$

Proof: $\mathcal{L}\ [f\ (t)u(t-a)]=\int_0^\infty e^{-st}\mathcal{L}[f\ (t)\ .\ u(t-a)]\ dt$

$$=0+\int_0^\infty e^{-st}\ [f(t).\ u(t-a)]\ dt$$

$$=0+\int_0^\infty e^{-st}.\ f\ (t)(1)dt$$

$$=0+\int_0^\infty e^{-s(y+a)}\ .\ f\ (y+a)\ dy=e^{-as}\int_0^\infty e^{-sy}\ .\ f\ (y+a)dy \quad but \quad (t-a=y)$$

$$\mathcal{L}[f\ (t)u(t-a)]=e^{-as}\int_0^\infty e^{-st}\ f(t+a)dt=e^{-as}\mathcal{L}f(t-a) \quad \textbf{Proved.}$$

Example 1.15: Find the Laplace transform of $t^2u(t-3)$.

Solution. $t^2.\ u\ (t-3)=[(t-3)^2+6(t-3)+9]u\ (t-3)$

$$=(t-3)^2\ .\ u(t-3)+\ 6(t-3)\ u(t-3)+9\ u(t-3)$$

$$\mathcal{L}t^2.\ u(t-3)=\mathcal{L}(t-3)^2.\ u(t-3)+6\mathcal{L}(t-3).\ u(t-3)\ +9\mathcal{L}u(t-3)$$

$$=e^{-3s}\left[\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right]$$

$$\mathcal{L}t^2u(t-3)=e^{-3s}\mathcal{L}\ (t-3)^2=e^{-3s}\mathcal{L}[t^2+6t+9]$$

$$=e^{-3s}\left[\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right]$$

Example 1.16: Represent $f(t) = sin\,2t,\ \ 2\,\pi < t < 4\,\pi$ and $f(t) = 0$ otherwise, in terms of unit step function and then find its Laplace transform.

Solution:

$$f(t) = \begin{bmatrix} \sin 2t & 2\pi \le t \le 4\,\pi \\ 0 & otherwise \end{bmatrix}$$

$$f(t) = \sin 2t\ \left[\,u(t - 2\,\pi) - u(t - 4\,\pi)\right]$$

$$\mathcal{L}f(t) = \mathcal{L}\sin 2\,t.u(t - 2\,\pi) - \mathcal{L}\sin 2t\,.u(t - 4\,\pi)$$

$$= \left(e^{-\,2\,\pi s}\ \frac{1}{s\ +\ 2} - e^{-\,4\,\pi s}\right)\frac{2}{s^3\ +\ 4}$$

$$= (e^{-\,2\,\pi s}\ - e^{-\,4\,\pi s})\ \frac{2}{s^3\ +\ 4}$$

5. Laplace of Polynomial Functions

$$\mathcal{L}\,t^n = \int_{0^-}^{\infty} t^n\ e^{-st}\,dt$$

$$v = t^n, dv = nt^{n-1}, dv = e^{-st}\,dt\,u = \frac{e^{-st}}{s}$$

$$\mathcal{L}t^n = \left.\frac{t^n\ e^{-st}}{-s}\right|_{0^-}^{\infty} + \frac{n}{s}\int_{0^-}^{\infty} t^{n-1}\ e^{-st}\,dt$$

$$= 0 - -0 + \frac{n}{s}\int_{0^-}^{\infty} t^{n-1}\ e^{-st}\,dt = \frac{n}{s}\ I_{n-1}$$

Where $\mathcal{L}t^n$ has been conveniently designated as I_n, so that $\int_{0^-}^{\infty} t^{n-1}\ e^{-st}$now becomes $I_n - 1$

Note: in evaluating the first term above, in taking the limits, the assumption has been made that the numerical value of s is sufficiently large to force the product $t^n\ e^{-st}$ to tend to zero for the upper limit of the integral expression, as the term e^{-st} diminishes faster than the term t^n is increasing, so that the product ultimately converges to zero. (The above assumption is justified on account of the law of energy conservation!)

Continuing and replacing n by $n - 1$ ("mathematricks!")

$$I_{n-1} = \frac{n-1}{s}\ I_{n-2} \Longrightarrow I_n = \left(\frac{n}{3}\right)\left(\frac{n-1}{s}\right)I_{n-2}$$

Repeating the process, and replacing n by $n - 1$,

$$I_{n-2} = \frac{n-2}{s} I_{n-3} \Longrightarrow I_n = \frac{n}{3}\left(\frac{n-1}{s}\right)\left(\frac{n-2}{s}\right) I_{n-3}$$

And so on. …… until, eventually,

$$I_n = \frac{n}{s}\left(\frac{n-1}{s}\right)\left(\frac{n-2}{s}\right)\left(\frac{n-3}{s}\right)\ldots.\left(\frac{i}{s}\right) I_0$$

But (do not break the rule!) $I_0 = \int_{0^-}^{\infty} t^0 \; e^{-st} \; dt = \int_0^{\infty} e^{-st} \; dt = \frac{1}{s}$

Which is the Laplace transform of unity earlier derived

$$\therefore \quad I_n = \frac{n}{s}\left(\frac{n-1}{s}\right)\left(\frac{n-2}{s}\right)\left(\frac{n-3}{s}\right)\cdots\cdots\cdot\left(\frac{1}{s}\right)\left(\frac{1}{2}\right) = \left(\frac{n^1}{s^n}\right)\left(\frac{1}{s}\right)$$

Finally, (and mercifully),

$$\mathcal{L}t^n = \frac{n!}{s^{n+1}} \qquad 1.13$$

Example 1.17: Find the Laplace transform of $t^{-1/2}$

Solution:

We know that $\mathcal{L}\,(t^n) = \dfrac{\lceil n+1}{s^{n+1}}$

Put $n = -\dfrac{1}{2}$, $\quad \mathcal{L}\left(t^{-\frac{1}{2}}\right) = \dfrac{\lceil -\frac{1}{2} + 1}{s^{-\frac{1}{2}+1}} \quad = \dfrac{\lceil \frac{1}{2}}{\sqrt{s}} \quad = \dfrac{\sqrt{\pi}}{\sqrt{s}} \quad$ *where* $\lceil \frac{1}{2} = \sqrt{\pi}$

A very powerful tool, since functions such as constants, quadratic, parabola etc. are now ours for the taking as far as Laplace transforms are concerned!

6. Multiplying a function by 't':

$$\mathcal{L}t\{f\,(t)\} = \int_{0^-}^{\infty} e^{-st} \; tf(t)dt \qquad 1.14$$

But te^{-st} can be expressed as $-\frac{d}{ds}\; e^{-st}$, with the variable now becoming s instead of the more familiar t. (Take the derivative to ascertain this to be so)

$$\Longrightarrow \mathcal{L}t\,\{f(t)\} = \int_{0-}^{\infty}\left(-\frac{d}{ds}\; e^{-st}\right) f(t)dt$$

Because is once again the variable, the derivative $-\frac{d}{ds}$ can now leave the integral sign, resulting in,

$$\mathcal{L}tf(t) = -\frac{d}{ds}\int_{0-}^{\infty} e^{-st} f(t)dt.$$

We once again recognize the integral expression as simply our old friend the Laplace transform of $f(t)$.

$$\Rightarrow \boxed{\mathcal{L}tf(t) = -\frac{d}{ds}F(s)} = -F'(s) \qquad 1.15$$

By the same reasoning, interpolating,

$$\mathcal{L}t^2 f(t) = \frac{d^2}{ds^2}F(s) \qquad \mathcal{L}t^3 f(t) = -\frac{d^2}{ds^2}F(s),$$

$$\boxed{\mathcal{L}t^3 f(t) = \mathcal{L}t[t^2 f(t)] = -\frac{d}{ds}\left(\frac{d^2F(s)}{ds^2}\right) = \frac{-d^3F(s)}{ds^3}}$$

$$\mathcal{L}t^n f(t) = (-1)^n \frac{d^nF(s)}{ds^n} = (-1)^n F^n(s) \qquad 1.16$$

Example 1.18: Find the Laplace transform of $t^2 \cos at$.

Solution:

$$\mathcal{L}(\cos at) = \frac{s}{s^2 - a^2}$$

$$\mathcal{L}(t^2 \cos at) = (-1)^2 \frac{d^2}{d\,s^2}\left[\frac{s}{s^2 - a^2}\right] = \frac{d}{ds}\,\frac{(s^2 - a^2)1 - s\ (2\,s)}{(s^2 - a^2)^2} = \frac{d}{ds}\,\frac{a^2 - s^2}{(s^2 - a^2)^2}$$

$$= \frac{(s^2 - a^2)^2 \times (-2\,s) - (a^2 - s^2) \times 2(s^2 - a^2)(2\,s)}{(s^2 - a^2)^4}$$

$$= \frac{-2\,s^3 - 2\,a^2\,s - 4\,a^2\,s + 4\,s^3}{(s^2 - a^2)^3}$$

$$= \frac{2\,(s^2 - 3\,a^2)}{(s^2 - a^2)^3}$$

The sinusoidal function $\cos at$

$$\int_{0-}^{\infty} e^{-st}\cos at\ dt = \mathcal{L}\cos at$$

$$\left(u = e^{-st}, \qquad du = -se^{-st};\ dv = \cos at\,dt \Rightarrow v = \frac{\sin at}{a}\ i.e\ by\ integration\right)$$

$$\mathcal{L}\cos at = \frac{e^{-st}\sin at}{a}\Bigg|_{0-}^{\infty} - -\frac{s}{a}\int_{0-}^{\infty} e^{-st}\sin at\ dt$$

Applying integration by parts a second time:

$$\mathcal{L}\cos at = 0 - 0 + \frac{s}{a}\left[\frac{-e^{-st}\cos at}{a}\Bigg|_{0-}^{\infty} - -\frac{s}{a}\int_{0-}^{\infty}(-e^{-st}\cos at)dt\right]$$

$$\Rightarrow \qquad \mathcal{L}\cos at = \frac{s}{a}\left[\left(-0 - \frac{1}{a}\right) - \frac{s}{a}\mathcal{L}\cos at\right]$$

$$\Rightarrow \qquad \mathcal{L}\cos at\left(1 + \frac{s^2}{a^2}\right) = \frac{s}{a^2}$$

$$\Rightarrow \boxed{\mathcal{L}\cos at = \frac{s}{s^2 + a^2}}$$

Example 1.19: Obtain the Laplace transform of $t^2 e^t\, sin\, 4t$.

Solution:

$$\mathcal{L}(\sin 4t) = \frac{4}{s^2 - 16}$$

$$\mathcal{L}\,(te^t \sin 4t) = \frac{4}{(s - 1)^2 + 16}$$

$$\mathcal{L}(te^t \sin 4t) = -\frac{d}{ds}\left(\frac{4}{s^2 - 2s + 17}\right) = \frac{4(2s - 2)}{(s^2 - 2s + 17)^2}$$

$$\mathcal{L}(t^2 e^t \sin 4t) = -\frac{d}{ds}\left(\frac{2(s - 2)}{(s^2 - 2s + 17)^2}\right)$$

$$= -\frac{2(s^2 - 2s + 17)^2 - 2(s - 2)(s^2 - 2s + 17)(2s - 2)}{(s^2 - 2s + 17)^4}$$

$$= \frac{-4(2s^2 - 4s + 34 - 8s^2 + 16s - 8)}{(s^2 - 2s + 17)^3}$$

$$= \frac{-4(-6s^2 + 12s + 26)}{(s^2 - 2s + 17)^3} = \frac{8(3s^2 - 6s - 13)}{(s^2 - 2s + 17)^3}$$

7. Dividing a function by ‘t’:

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_{\sigma=s}^{\infty} F(\sigma)\,d\sigma, \text{in so far as } \lim_{t\to 0}\left[\frac{f(t)}{t}\right] exist$$

Starting from the right-hand side,

$$\int_{\sigma=s}^{\infty} F(\sigma)d\sigma = \int_{\sigma=s}^{\infty}\left[\int_{0-}^{\infty} f(t)e^{-\sigma t}\, dt\right] d\sigma$$

Where in the inner integral expression, the domain has been changed s to σ, in order to restore the final integral answer back to s after evaluation. Also, here the variable is t $[\int_{t=0^-}^{\infty}]$

Interchanging the integrals (allowed by law convolution),

$$\int_{t=0^-}^{\infty}\int_{\sigma=s}^{\infty} f(t)e^{-\sigma t}\; d\sigma\, dt$$

$$= \int_{t=0^-}^{\infty} f(t)\left[\int_{\sigma=s}^{\infty} e^{-\sigma t}\, d\sigma\right] dt = \int_{t=0^-}^{\infty}\left\{f(t)\left[\frac{e^{-\sigma t}}{-t}\right]_{\sigma=s}^{\infty}\right\} dt$$

$$= \int_{t=0^-}^{\infty} f(t)\left[\frac{0 - -e^{-st}}{t}\right] dt = \int_{t=0^-}^{\infty}\frac{f(t)e^{-st}}{t}\, dt,$$

And this is simply the Laplace transform of $\frac{f(t)}{t}$ So, by going backward we've been able to show:

$$\mathcal{L}f(t) = \int_{\sigma=s}^{\infty} F(\sigma)d\sigma \qquad 1.17$$

Example 1.20: Evaluate $\mathcal{L}\left(\frac{\sin at}{t}\right)$

Testing for the existence of limit: $\lim\limits_{t\to 0}\frac{\sin at}{t} = \frac{0}{0}$ (division by zero is prohibited!)

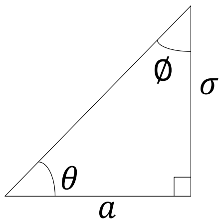
Applying L' H$\hat{o}$pital's rule, the derivatives of the numerator and denominator are taken:

$$\lim_{t\to 0}\frac{\frac{d}{dt}(\sin at)}{\left(\frac{dt}{dt}\right)} = \lim_{t\to 0}\frac{a\cos at}{1} = a$$

$$\mathcal{L}\sin at = \frac{a}{s^2 + a^2}$$

$$\mathcal{L}\,\frac{\sin at}{t} = \int_{\sigma=s}^{\infty}\left[\frac{a}{\sigma^2 +\, a^2}\right] d\sigma$$

To evaluate the integral, let's use some geometry:



$$\sigma = \mathrm{a}\tan\theta = \mathrm{a}\frac{\sin\theta}{\cos\theta}$$

$$\frac{d\sigma}{d\theta} = a\left(\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\right) = \frac{a}{\cos^2\theta}$$

$$\Longrightarrow \sigma^2 + a^2 = a^2\frac{\sin^2\theta}{\cos^2\theta} + a^2 = a^2\left(\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta}\right) = \frac{a^2}{\cos^2\theta}$$

$$\int_{\sigma=s}^{\infty}\frac{a}{\sigma^2 + a^2}d\sigma = \int_{\sigma=s}^{\infty}\frac{\left(\frac{a}{\cos^2\theta}\right)}{\left(\frac{a^2}{\cos^2\theta}\right)}d\theta = \int_{\sigma=s}^{\infty}d\theta$$

$$= \theta|_{\sigma=s}^{\infty} = arc\tan\left(\frac{\sigma}{a}\right)\Big|_{\sigma=s}^{\infty} = arc\tan\infty - arc\tan\frac{s}{a}$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \tan^{-1}\left(\frac{a}{s}\right),$$

Because $\frac{\pi}{2} - \arctan\left(\frac{s}{a}\right) = \frac{\pi}{2} - \theta = \emptyset = \arctan\left(\frac{a}{s}\right)$

So, finally (and mercifully too!), $\left[90^o - \tan^{-1}\left(\frac{s}{a}\right) = \tan^{-1}\left(\frac{a}{s}\right)\right]$

$$\boxed{\mathcal{L}\frac{sinat}{t} = \tan^{-1}\frac{a}{s}}$$

Example 1.21: Find the Laplace transform of the function: $\frac{\sin 2t}{t}$

Solution:

$$\mathcal{L}(\sin 2t) = \frac{2}{s^2 + 4}$$

$$\mathcal{L}\left(\frac{\sin 2t}{t}\right) = \int_s^{\infty}\frac{2}{s^2 + 4}ds = 2.\frac{1}{2}\left[tan^{-1}\frac{s}{2}\right]_s^{\infty}$$

$$= \left[tan^{-1}\infty - tan^{-1}\frac{s}{2}\right] = \frac{\pi}{2} - tan^{-1}\frac{s}{2}$$

$$= cot^{-1}\frac{s}{2}$$

Example 1.22: Find the Laplace transform of the equation $\int_0^t \frac{\sin t}{t}\, dt$.

Solution:

$$\mathcal{L}\sin t = \frac{2}{s^2 + 1}$$

$$\mathcal{L}\frac{\sin t}{t} = \int_s^\infty \frac{2}{s^2 + 1} \text{ds} = [tan^{-1} s]_s^\infty = \frac{\pi}{2} - tan^{-1} s$$

$$\mathcal{L}\int_0^t \frac{\sin t}{t}\, dt = \frac{1}{s}\left[\frac{\pi}{2} - tan^{-1} s\right]$$

Example 1.23: Find the Laplace transform of $\frac{1 - \cos t}{t^2}$

Solution:

$$\mathcal{L}(1 - \cos t) = \mathcal{L}(1) - \mathcal{L}(\cos t) = \frac{1}{s} - \frac{s}{s^2 + 1}$$

$$\mathcal{L}\frac{(1 - \cos t)}{t} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2 + 1}\right) ds = \left[\log s - \frac{1}{2}\log(s^2 + 1)\right]_s^\infty$$

$$= \frac{1}{2}\,[\log s^2 \log(s^2 + 1)]_s^\infty = \frac{1}{2}\left[\log\frac{s^2}{s^2 + 1}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2}{s^2\left(1 + \frac{1}{s^2}\right)}\right]_s^\infty = \frac{1}{2}\left[0 - \log\frac{s^2}{s^2 + 1}\right] = -\frac{1}{2}\log\frac{s^2}{s^2 + 1}$$

Again, $\mathcal{L}\left[\frac{(1 - \cos t)}{t^2}\right] = -\frac{1}{2}\int_s^\infty \log\frac{s^2}{s^2 + 1}\, ds = -\frac{1}{2}\int_s^\infty \left(\log\frac{s^2}{s^2 + 1}.1\right) ds$

Integrating by parts, we have

$$= -\frac{1}{2}\left[\log\frac{s^2}{s^2 + 1} \cdot s \int \frac{s^2 + 1}{s^2}\, \frac{(s^2 + 1)\, 2\, s - s^2\, (2\, s)}{(s^2 + 1)^2} \cdot s\; ds\right]_s^\infty$$

$$= -\frac{1}{2}\left[s\log\frac{s^2}{s^2 + 1} - 2\int \frac{1}{s^2 + 1}\, ds\right]_s^\infty = -\frac{1}{2}\left[s\log\frac{s^2}{s^2 + 1} - 2tan^{-1} s\right]_s^\infty$$

$$= -\frac{1}{2}\left[0-2\left(\frac{\pi}{2}\right)-s\log\frac{s^2}{s^2+1}+2tan^{-1}\ ds\right]_s^{\infty}$$

$$= -\frac{1}{2}\left[-\pi-s\log\frac{s^2}{s^2+1}+2tan^{-1}\ s\right]_s^{\infty}$$

$$= \frac{\pi}{2}+\frac{s}{2}\log\frac{s^2}{s^2+1}-\ tan^{-1}s$$

$$= \left(\frac{\pi}{2}-\ tan^{-1}s\right)+\frac{s}{2}\ \log\frac{s^2}{s^2+1} = \ cot^{-1}s+\frac{s}{2}\ \log\frac{s^2}{s^2+1}.$$

Example 1.24: Evaluate $\mathcal{L}\left[e^{-4t}\ \frac{\sin 3t}{t}\right]$.

Solution:

$$\mathcal{L}\sin 3t = \frac{3}{s^2+3^2} \Longrightarrow \mathcal{L}\frac{\sin 3t}{t} = \int_s^{\infty}\frac{3}{s^2+9}\quad ds = \left[\frac{3}{3}\ tan^{-1}\frac{s}{3}\right]_s^{\infty}$$

$$= \frac{\pi}{2}-tan^{-1}\frac{s}{3} = cot^{-1}\frac{s}{3}$$

$$\mathcal{L}\left[\ e^{-4t}\ \frac{\sin 3t}{t}\right] = cot^{-1}\ \frac{s\ +\ 4}{3}$$

8. Time-Scaling theorem

$$\boxed{\mathcal{L}f(at) = \frac{1}{a}F\left(\frac{s}{a}\right)} \qquad 1.18$$

Proof: Let at $= v \Longrightarrow t = \frac{v}{a} \Longrightarrow dt = \frac{1}{a}dv$

$$\Longrightarrow \qquad \mathcal{L}f\ (v) = \mathcal{L}f(at) = \int_{0^-}^{\infty} e^{-st}\, f(at)dt$$

$$= \frac{\int_{0^-}^{\infty} e^{-s\left(\frac{n}{a}\right)}\, f(n)1}{a}dv = \frac{1}{a}\int_{0^-}^{\infty} e^{-\left(\frac{s}{a}\right)v}\, f(v)dv,$$

Where the integral quantity if $F(\frac{s}{a})$, the Laplace transform of $f(t)$ with s replaced by s/a since a change of variables from t to v makes no difference to the result of the theorem. This is known as "dummy" variables theorems continued.

9. **Initial value theorem**:
   This states that if a function $f(t)$ and its first derivative $f'(t)$ are $F(s)$ exists, then:

$$\boxed{\lim_{s\to\infty} sF(s) = \lim_{t\to 0} f(t)} \qquad 1.19$$

The "initial" in the theorem is with respect to time domain($t \to 0$), and allows for the evaluation of the "final" value in the s (frequency) domain by evaluating the initial value in the time domain, (and vice versa) when the former is too complicated is to be carried out directly in the frequency domain. (Note the slight modification by the multiplier s. do not forget to do this in examinations!).

10. Final-value theorem:

If a function and its first derivative are Laplace transformable, and the poles of $sF(s)$ lie inside the left half of the s-plane, then:

$$\boxed{\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t)} \qquad 1.20$$

Proof:

Employing time differentiation theorem,

$$\mathcal{L}f'(t) = \int_{0^-}^{\infty} e^{-st} \frac{df(t)}{dt} dt = sF(s) - f(0^-)$$

As earlier derived.

Taking the limit $s \to 0$:

$$\lim_{s\to 0} \int_{0^-}^{\infty} e^{-st}\, df(t) = \lim_{s\to 0}[sF(s) - f(0^-)]$$

The limit of integral can be written as integral of limit, as long as the infinite integral on the left-hand side exists, and we have no reason to doubt its existence since we're dealing with a practical situation!

$$\Rightarrow \int_{0^-}^{\infty} \lim_{s\to 0} e^{-st}\ df(t) = \lim_{s\to 0}[sF(s) - f(0^-)]$$

$$= \lim_{s\to 0} sF(s) - f(0^-)$$

Because $f(0^-)$ is a constant term as it is, so its unaffected by the limit

$$\Rightarrow \quad f(t)|_{0^-}^{\infty} = \lim_{s\to 0} sF(s) - f(0^-)$$

$$f(\infty) - f(0^-) = \lim_{s\to 0} sF(s) - f(0^-)$$

$$\Longrightarrow \quad f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

NB: proof of earlier initial value theorem in left as an exercise.

Again, the name tag, “final” is with reference to the time domain $(t \to \infty)$ and allows for the computation of the “initial” value of the products of s and the Laplace transform of the zeroth derivative of $f(t)$, by evaluating the final value in the time domain, and vice versa. [so, one can always be rescued from the time domain by resorting to the frequency domain after the necessary multiplication by s (again, do not forget this) has been carried out].

Let $f(t) = e^{-at} \Longrightarrow \mathcal{L}f(t) = F(s) = \frac{1}{(s+a)}$ where a is necessarily assumed to be a positive quantity (as it should be since every realistic signal, if left unreinforced must eventually decay to zero).

$$\left[\lim_{t\to\infty} e^{-at} = 0\right]$$

The single pole of $sF(s) = \frac{s}{s+a}$, which is $-a$, lie inside the left half of the s-plane, therefore the final value theorem can be applied.

$$\operatorname*{Lim}_{s\to 0} sF(s) = \lim_{s\to 0} s\left(\frac{1}{s+a}\right) = \lim_{s\to 0}\frac{s}{s+a}$$

$$= \lim_{s\to 0}\frac{1}{1+\frac{a}{s}} = \lim_{s\to 0}\frac{s}{a} = 0,$$

After dividing throughout by s.

Note once again the single pole $(s = -a)$ lies inside the left half of the s-plane as required by the theorem to be valid.

By the same taken,

$$\lim_{s\to\infty} s\,F(s) = \lim_{s\to\infty}\frac{s}{s+a} = 1,$$

And $\qquad \lim_{t\to 0} f(t) = \lim_{t\to 0} e^{-at} = 1 = \lim_{s\to\infty} sF(s),$

Validating the initial-value theorem.

Note, once again, that in each case, both e^{-at} and its first derivative $(-ae^{-at})$ are Laplace transformable as required by the two theorems.

Observations:

The initial and final value theorems call to mind the duality principle encountered in an earlier course, and the two theorems could indeed be said to be the "duals" of each other (although not in the strictest sense of the principle). Similarly, the first and second shift theorems earlier treated resemble "duals" (not of each other, however), except that in the first shift theorem, multiplying a function by e^{at} results in adding "$-a$", not "a" to the s of the frequency domain. The second-shift theorem by itself, is a more fulfilled "dual" because shifting time by c (i.e. adding (-c) results in multiplying the Laplace transform of the original unshifted function by e^{-sc}

1.5 Laplace Transform of Periodic Functions

Recall that a periodic function reseat itself after every cycle that is, it looks like itself during every period, say T. for a function $f(t)$ with the period T, then for the first cycle the function is described by:

$$\bar{f}(t) = \begin{cases} f(t), & 0 \leq t < T \\ 0, & \text{elsewhere} \end{cases}$$

With the bar at the top indicating that the function $f(t)$ is periodic. The second cycle is identical to the first, except shifted by time T (the period), and can therefore be described by the Heaviside unit step function:

$$\bar{f}(t-T)u(t-T) = \begin{cases} f(t), & T \leq t\ 3T \\ 0, & \text{elsewhere} \end{cases}$$

For the third cycle:

$$\bar{f}(t-2T)u\ (t-2T) = \begin{cases} f(t), & (n-1)T \leq t < nT \\ 0, & elsewhere \end{cases}$$

Then, the nth cycle:

$$\bar{f}[t-(n-1)T]u[t-(n-1)T] = \begin{cases} f(t), & (n-1)T \leq t < nT \\ 0, & \text{elsewhere} \end{cases}$$

So, generally for a periodic function

$$f(t), f(t) = \bar{f}(t)u(t) + \bar{f}(t-T)u\ (t-T) + \bar{f}(t-2T)u(t-2T) + \cdots\cdots \\ + f[t-(n-1)T]u\ [t-(n-1)T]$$

Laplace transform of $\bar{f}(t)$ is then given by:

$$\mathcal{L}\ \bar{f}(t) = \int_{0^-}^{\infty} e^{-st}\ \bar{f}(t)u\ (t)dt = \int_{0^-}^{\infty} e^{-st}\ \bar{f}(t)dt = \bar{F}(s),$$

Since $\bar{f}(t) = 0^o$ for $t > T$ as stated above.

From second-shift (time-shift) theorem,

$$\mathcal{L}f(t) = \mathcal{L}\bar{f}(t)u\,(t) + \mathcal{L}\bar{f}(t-T)u\,(t-T) + \mathcal{L}\bar{f}(t-2T)u\,(t-2T) + \cdots + \mathcal{L}\bar{f}[t-(n-1)T]u\,[t-(n-1)T]$$

$$= \bar{F}(s) + e^{-sT}\,\bar{F}(s) + \,e^{-2sT}\,\bar{F}(s) \ldots + e^{-(n-1)sT}\bar{F}(s)$$

From the sum $1 + x + x^2 + x^3 + \cdots\cdots + x^n = \frac{1}{1-x}$, for a converging series (i.e $|x| < 1$, and $e^{-sT}$is obviously converging because T is necessarily positive),

$$\mathcal{L}f(t) = \frac{1}{1-e^{-sT}}\bar{F}(s) \qquad 1.21$$

$$\text{Where } \bar{F}(s) = \int_0^T e^{-st}\,f\,(t)dt$$

Example 1.25: For the periodic function defined by:

$$f(t) = \begin{matrix} 0, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{matrix}\Big\}\; f(t+2) = f(t) \qquad (\textit{indicating a period of } 2)$$

Solution:

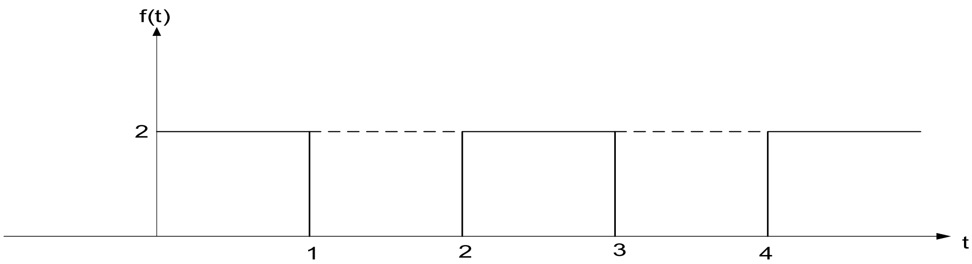


Figure 1.7

$$\mathcal{L}\,f(t) = \frac{1}{1-e^{-2s}}\int_0^\infty e^{-st}\,f(t)u(t)dt = \frac{1}{1-e^{-2s}}\int_0^1 e^{-st}\,f(t)dt$$

$$= \frac{1}{1-e^{-2s}}\int_0^1 e^{-st}\,2dt = \frac{2}{1-e^{-2s}}\int_0^1 e^{-st}\;dt$$

Since $f(t) = 0, 1 \le t < 2$

$$\mathcal{L}\,f(t) = \frac{2}{1-e^{-2s}}\,\frac{e^{-st}}{-s}\bigg|_0^1 = \frac{2}{1-e^{-2s}}\left(\frac{1-e^{-s}}{s}\right)\frac{2(1-e^{-s})}{s\,(1+e^{-s})(1-e^{-s})}$$

$$\mathcal{L}\,f(t) = \frac{2}{s(1+e^{-s})}$$

Example 1.26: Determine the Laplace transform of the half-wave rectifier output wave from defined by:

$$f(t) = \begin{Bmatrix} 4\sin t & 0 \le t < \pi \\ & \pi \le t < 2\pi \end{Bmatrix}, \qquad f(t+2) = f(t) \text{ (Indicating a period of } 2\pi)$$

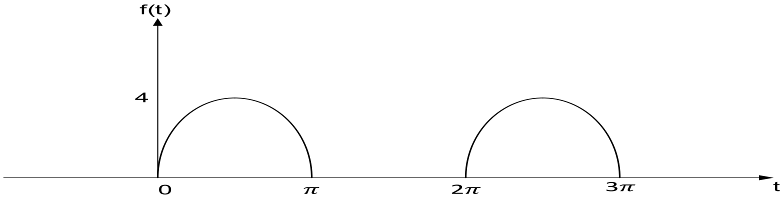


Figure 1.8

Solution:

$$\mathcal{L}\, f(t) = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t) dt = \frac{1}{1-e^{-2\pi s}} \int_0^{\pi} e^{-st}\, 4\sin t dt$$

Exponential function From Euler's identity, $e^{jt} = \cos t + j\sin t \Longrightarrow \sin t = I_m e^{jt}$, with I_m "imaginary" part of e^{jt}, not its maximum of current.

$$\Longrightarrow \qquad (1-e^{-2\pi s})\mathcal{L}f(t) = 4I_m \left[\int_0^{\pi} e^{-st}\, e^{jt}\, dt\right]$$

$$= 4I_m \int_0^{\pi} e^{-(s-j)t}\, dt = 4I_m \left[\frac{e^{-(s-j)t}}{-(s-j)}\right]_0^{\pi}$$

$$= 4I_m \frac{1}{s-j}\left(1-e^{-s\pi}e^{j\pi}\right) = 4I_m \frac{1}{s-j}[1-e^{-s\pi}(\cos\pi + j\sin\pi)]$$

$$= 4I_m \frac{1}{s-j}[1-e^{-s\pi}\,(-1)] = 4I_m \frac{1}{s-j}(1+e^{-s\pi})$$

$$= 4I_m \frac{(s+j)}{(s-j)\,(s+j)}\,(1+e^{-s\pi}) = \frac{4I_m\,(s+j)(1-e^{-s\pi})}{s^2+1}$$

$$= 4\frac{(1+e^{-s\pi})}{s^2+1}$$

$$\Longrightarrow \qquad \mathcal{L}f(t) = \frac{1}{1-e^{-2\pi s}} \times\; 4\left(\frac{1+e^{-s\pi}}{s^2+1}\right)$$

$$= \frac{4(1+e^{-s\pi})}{(1+e^{-s\pi})(1-e^{-s\pi})(s^2+1)}$$

$$= \frac{4}{(1 - e^{-s\pi})(s^2 + 1)}$$

The above result could have been obtained more directly (and easily) by noting that

$$\mathcal{L} \sin t = \frac{1}{(s^2 + 1)}$$

$$\mathcal{L}f(t) = \frac{1}{1 - e^{-2\pi s}} \int_0^{\pi} e^{-st}\, 4 \sin t \, dt = \frac{4}{1 - e^{-2\pi s}} \left(\frac{1 + e^{-s\pi}}{s^2 + 1} \right)$$

And so on, where the $e^{-s\pi}$ in the numerator is used to account for changing the upper limit of (transform) integration from ∞ to π.

1.6 The Dirac Delta-Unit Impulse "Function" ($\boldsymbol{\delta t}$)

The quotation marks above suggest that the Dirac delta is not a proper function in the strictest mathematical sense of the term "function". A function is typically characterized by its inputs and the corresponding output(s). That means that a function should be able to tell a "story". Implying a smooth transition from one point in space to the next (devoid of any abrupt or irregular behavior). The Dirac delta lacks this characteristic, the requirement above regarding functions are not obtainable with it. So, deprived of this tool, to treat the Dirac delta we have to resort to its effect on other functions *(reminds one of the Holy Spirit who's invisible to the naked eyes although this work is made manifest in our daily lives)! Please I am not preaching!!*

This is what the Dirac delta $\delta(t)$ does to a given function $f(t)$:

$$\int_{-\infty}^{\infty} f(t)\, \delta(t - a) dt = f(a) \qquad 1.22$$

$$\text{By itself, } \delta(\text{t} - \text{a}) = \begin{cases} 0 & for\ t \neq a \\ \text{undefined} & for\ t = a \end{cases}$$

Graphically,

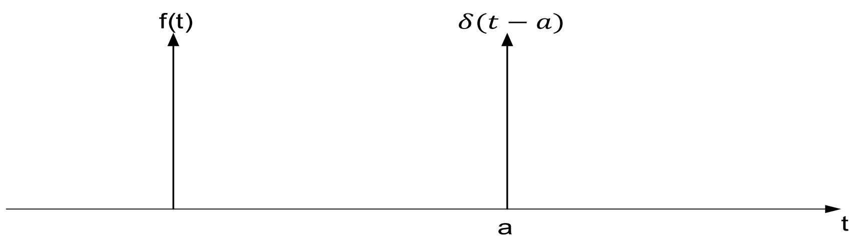


Figure 1.9

From the above integral expression relating the effect of the Dirac delta on other function, $(t) = 1 \Longrightarrow \int_{-\infty}^{\infty} 1\,\delta(t-a)dt = f(a) = 1$, because $f(t)$ is identically a constant, 1 (i.e. for all $t's$)

for $m < a < n$,

$$\int_{-\infty}^{\infty} \delta\,(t-a)dt = \int_{-\infty}^{m} \delta(t-a)\,dt + \int_{m}^{n} \delta(t-a)dt + \int_{n}^{\infty} \delta(t-a)dt$$

$$= 0 + \int_{m}^{n} \delta(t-a)dt + 0$$

The zeros result because the limits of integration do not include $t = a$ as required by the above specification for $\delta(t-a)$, since $\delta(t-a)$ is zero at t below m and above n.

$$\Longrightarrow \qquad \int_{-\infty}^{\infty} \delta\,(t-a)dt = 1 = \int_{m}^{n} \delta(t-a)dt, m < a < n$$

So, $\delta(t-a)$ is a horizontally axis with a vertical line of infinite length at $t = a$, that is, its an impulse at time $t = a$. From the integral expression involving the Dirac delta, it can be visualized as a "rectangle" of zero width but infinite length with an area of unity!

$$\int_{m}^{n} f(t)\delta\,(t-a)dt = f(a) \qquad 1.23$$

As long as 'a' lies between m and n. (This is an ultimately very important provisor)

Example 1.27: $\int_{1}^{6} (t^2+5)\,\delta(t-4)dt = f(4) = 4^2 + 5 = 21$

valid because $1 < 4 < 6$.

$$\int_{0}^{\pi} \sin 3t\,\delta\left(t-\frac{\pi}{6}\right)dt = f\left(\frac{\pi}{6}\right) = \sin\left(\frac{3\pi}{6}\right) = \frac{\sin\pi}{2} = 1$$

Valid because $0 < \frac{\pi}{6} < \pi$

$$\int_{1}^{4} e^{-3t}\,\delta(t-a)dt = f(2) = e^{-3t}t = 2 = e^{-6}$$

Valid because $1 < 2 < 4$

$$\int_{0}^{\pi} \cos 3t\delta\,\left(t-\frac{\pi}{3}\right)dt = f\left(\frac{\pi}{3}\right) = \cos 3\left(\frac{\pi}{3}\right) = \cos\pi = -1$$

Valid because $0 < \frac{\pi}{3} < \pi$

Laplace transform of $\delta(t-a)$

$$\mathcal{L}\,\delta(t-a) = \int_{0-}^{\infty} e^{-st}\,\delta(t-a)dt,$$

From the definition of Laplace transform.

Here, e^{-st} represent $f(t)$, per the effect of the Dirac delta on other function:

$$\Rightarrow \quad \int_{0^-}^{\infty} e^{-st}\,\delta(t-a)dt = \int_{0-}^{\infty} f(t)\delta\,(t-a)dt$$

$$\Rightarrow \quad \mathcal{L}\delta(t-a) = f(a) = e^{-st}|_{t=a} = e^{-sa} \qquad 1.24$$

[Note: we did a little manipulation, readjustment Eq. 1.23, albeit a legal one, by extending redundantly, the lower limit of the integral from 0^- (for Laplace transform) down to $-\infty$. As mentioned, this is indeed a redundant operation that has no effect whatsoever on the result but only allows us to have the expression in the form we want it in order to apply the foregoing effect of the Dirac delta on other functions]

At $t = 0$ (i.e, at the origin), $a = 0 \Rightarrow \mathcal{L}\delta(t)t = a = e^0 = 1$

Hurray! The Laplace transform of the impulse is a constant (1) where has the factor of s gone?!)

Laplace transform of the product of a given function and the unit impulse

$$\left[\mathcal{L}[f(t)\delta(t-a)] = \int_{0^-}^{\infty} e^{-st}\ f(t)\delta(t-a)dt\right]$$

At $\qquad t = a \Rightarrow e^{-st} = e^{-sa}$,

Then, $f(t) \Rightarrow f(a)$

$$\Rightarrow \qquad \mathcal{L}[f(t)\,\delta(t-a)] = f(a)e^{-as}\int_{0^-}^{\infty}\delta(t-a)dt$$

$$= f(a)e^{-as} \times 1 = f(a)e^{-as}$$

Example 1.28: $\mathcal{L}\,[4\delta(t-2)] = f(2)e^{-2s} = 4e^{-2s}$

Here because $f(t)$ is identically equal to 4

$$\mathcal{L}[t^2\delta(t-3)] = f(3)e^{-3s} = 3^2e^{-3s} = 9e^{-3s}$$

$$\mathcal{L}\left[\sin 3t\delta\left(t-\frac{\pi}{2}\right)\right] = f\left(\frac{\pi}{2}\right)e^{-\frac{\pi s}{2}} = \left(\sin\frac{3\pi}{2}\right)e^{-\frac{\pi s}{2}} = -1.e^{-\frac{\pi}{s}} = -e^{-\frac{\pi s}{2}}$$

Example 1.29: To find the Laplace transform of $f(t)$ in Fig. 1.10

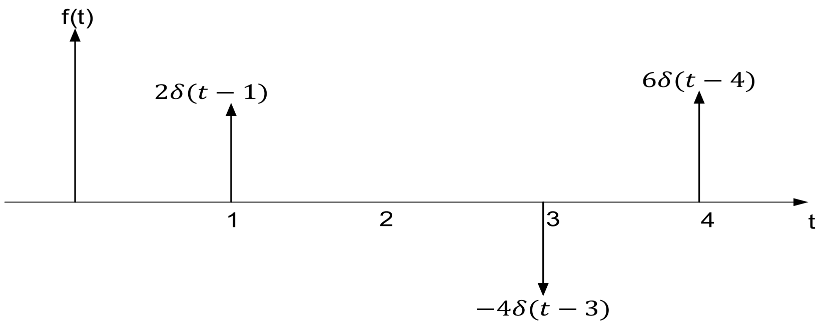


Figure 1.10

Solution $\mathcal{L}\,f(t) = \mathcal{L}[2\delta(t-1) - 4\delta(t-3) + 6\delta(t-4)] = 2e^{-s} - 4e^{-3s} + 6e^{-4s}$

Relating the unit step function and the unit impulse

Let $f(t) = 0$ for $t, < a$ and $t > b$

i. $$\Rightarrow \int_{-\infty}^{\infty} [u(t)f(t)]'\,dt = [u(t)f(t)]_{-\infty}^{\infty} = 0 - 0 = 0$$

because of the conditions imposed above with respect to the function $f(t)$, leading $f(\infty) = 0 = f(-\infty)$.

However, by the product rule, $[(uf)' = uf + uf']$ where both u and f are functions of time,

$$\Rightarrow \int_{-\infty}^{\infty} [u(t)f(t)]'\,dt$$

$$\Rightarrow \int_{-\infty}^{\infty} u'(t)f(t)dt + \int_{-\infty}^{\infty} u(t)f'(t)\delta t = 0 \qquad 1.25$$

Because of Eq (1.25).

ii. $\int_{-\infty}^{\infty} u'(t)f(t)dt = -\int_{-\infty}^{\infty} u(t)f'(t)dt = -\int_{0}^{\infty} f'(t)dt$

due to the fact that $u(t)$ factor forces the expression inside the integral sign to be zero up until time zero, from when $u(t)$ then takes on the value of unity.

$$-\int_0^\infty f'(t)dt = -f(\infty) + f(0) = f(0),$$

Since $f(t) = 0$ for $f(\infty)$ as originally imposed upon for $t > b$

iii. But $f(0) = \int_{-\infty}^{\infty} f(t)\delta\ (t\ 0)dt = \int_{-\infty}^{\infty} \delta(t)f(t)dt$, (according to the effect of $\delta(t-a)$ on the other function).

Comparison of equations (ii) and (iii), shows that $u'(t)$corresponds (indeed is equal) to $\delta(t)$! This we have through the back door, established a very important and interesting relationship, stating:

$$u'(t) = \frac{du(t)}{dt} = \delta(t) \qquad 1.26$$

i.e. the Dirac delta "function" is simply the derivative of the Heaviside unit step function.

Test: does this make any physical sense? (After all we're dealing with applied mathematics, to unit, engineering!) Recall that strictly speaking, whereas the unit step "function" $u(t-a)$ is identically zero up to $t = a^-$, and unit from $t = a^+$, its values at exactly $t = a$ is undefined. It makes sense, therefore, that vividly speaking, impulse $\delta(t-a)$ ought to be related to ir!

Recall also that:

$$\mathcal{L}\, u'(t) = \int_{0^-}^{\infty} e^{-st}\frac{du(t)}{dt}dt = sF(s) - f(0^-)$$

$$= s\left(\frac{1}{s}\right) - u(0^-) = 1 - 0 = 1,$$

And we have already determined that the Laplace transform of $\delta(t)$ is 1. So, since each of them is equal to 1, they must then be equal to each other! (This is by way of text of the fundamental proof performed above.)

Example 1.30: Evaluate $\int_1^3 (t^2+4)\cdot\delta(t-2)dt$

Solution:

The factor $\delta(t-2)$ shows that the impulse occurs at $t = 2$, i.e $a = 2$

$$f(t) = t^2 + 4$$

$$f(a) = f(2) = 4 + 4 = 8$$

$$\int_1^3 (t^2 + 4) \cdot \delta(t-2)dt = f(2) = 8$$

Example 1.31: To evaluate $\int_0^{\pi} \cos 6t \cdot \delta\left(t - \frac{\pi}{2}\right) dt$

$$\int_0^{\pi} \cos 6t \cdot \delta\left(t - \frac{\pi}{2}\right) dt = f\left(\frac{\pi}{2}\right) = \cos 3\pi = -1$$

And in the same way

(a) $\int_0^6 5 \cdot \delta(t-3)dt = 5 \times 1 = 5$

(b) $\int_2^5 e^{-2t} \cdot \delta(t-4)dt = f(4) = [e^{-2t}]_{t=4} = e^{-8}$

(c) $\int_0^\infty (3t^2 - 4t + 5) \, . \, \delta(t-2)dt = 12 - 8 + 5 = 9$

Nothing could be easier. It all rests on the fact that, provided $m < a < n$

Therefore, if $m = 0$ and $n = \infty$

$$\int_0^\infty f(t) \cdot \delta(t-a)dt = f(a)$$

Hence, if $f(t) = e^{-st}$, this becomes

$$\int_0^\infty e^{-st} \cdot \delta(t-a)dt = \mathcal{L}\{\delta(t-a)\}$$

$$= e^{-as}$$

i.e. the value of $f(t)$, i.e e^{-st}, at $t = a$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}$$

It follows from this that the Laplace transform of the impulse function at the origin is 1

Because, for $a = 0, \mathcal{L}\{\delta(t-a)\} = \mathcal{L}\{\delta(t)\} = e^0 = 1$

$$\mathcal{L}\{\delta(t)\} = 1$$

Finally, let us deal with the more general case of $\mathcal{L}\{f(t) \cdot \delta(t-a)\}$.

We have $\mathcal{L}\left\{f(t)\cdot\delta(t-a) = \int_0^\infty e^{-st}\cdot f(t)\cdot\delta(t-a)dt\right\}$. Now the integrand $e^{-st}\cdot f(t)\cdot\delta(t-a) = 0$ for all values of t except at $t = a$ which point $e^{-st} = e^{-as}$, and $f(t) = f(a)$

$$\mathcal{L}\{f(t)\cdot(t-a)\} = f(a)\cdot e^{-as}\int_0^\infty \delta(t-a)\,dt$$

$$= f(a)\cdot e^{-as}(1)$$

$$\mathcal{L}\{f(t)\cdot\delta(t-a)\} = f(a)e^{-as}$$

we have $\mathcal{L}\{f(t)\cdot\delta(t-a)\} = f(a)\cdot e^{-as}$

Therefore

a. $\mathcal{L}\{6.\delta(t-4)\}\quad a = 4$
$\mathcal{L}\{6\cdot\delta(t-4)\} = 6e^{-4s}$

b. $\mathcal{L}\{t^2\cdot\delta(t-2)\}\; a = 2$
$\mathcal{L}\{t^3\cdot\delta(t-2)\} = 8e^{-2s}$

Similarly

c. $\mathcal{L}\left\{\sin 3t\cdot\delta\left(t-\frac{\pi}{2}\right)\right\} = -e^{-\pi s/2}$

Because

$$\mathcal{L}\left\{\sin 3t\cdot\delta\left(t-\frac{\pi}{2}\right)\right\} = [\sin 3t]_{t=\pi/2}\cdot e^{-\frac{\pi s}{2}} = -e^{-\pi s/2}$$

And

d. $\mathcal{L}\{\cosh 2t\cdot\delta(t)\} = \cdots \qquad \boxed{1}$

Because

$$\mathcal{L}\{\cosh 2t\cdot\delta(t)\} = [\cosh 2t]_{t=0}\cdot e^0 = \cosh 0.(1) = 1$$

So, our main conclusion so far are as follows

1. $\int_n^m \delta(t-a)dt = 1$ Provided $m < a < n$

2. $\int_n^m f(t)\cdot\delta(t-a)dt = f(a)$ provided $m < a < n$

4. $\mathcal{L}\{\delta(t-a)\} = e^{-as}$

5. $\mathcal{L}\{f(t)\cdot\delta(t-a)\} = f(a)\cdot e^{-as}$

Example 1.32: Impulses of 1,4,7 units occur at $t = 1, t = 3$ and $t = 4$ respectively, in the directions shown. Write down an expression for $f(t)$ and determine its Laplace transform.

Solution: We have $f(t) = 1 \cdot \delta(t-1) - 4 \cdot \delta(t-3) + 7 \cdot \delta(t-4)$

$$\mathcal{L}\{f(t)\} = e^{-s} - 4e^{-3s} + 7e^{-4s}$$

1.7 Further Problems with Solutions

1. Given $\mathcal{L} \cos 4t = \frac{s}{(s^2+16)}$, determine the Laplace transform of $e^{3t} \cos 4t$.

Solution:
Since from $\mathcal{L}e^{-at} f(t) = F(s+a)$
$\Rightarrow \mathcal{L}(e^{3t} \cos 4t)$
$\Rightarrow \dfrac{s-3}{((s-3)^2+16)}$

2. Without first finding $f(t)$ determine $f(0^+)$, $f(\infty)$ for each of $F(s)$ equal to
$(i)\, \frac{4e^{3s}(2s+30)}{s}$ $(ii)\, \frac{(s^2-8)}{s(s^2+9)}$

Solution:

(i) $$f(0^+) = \lim_{s\to\infty} \frac{s4e^{-3s}}{(2s+30)s} = 0;$$

$$f(\infty) = \lim_{s\to 0} \frac{4e^{-3s}}{(2s+30)} = \frac{4}{30} = \frac{2}{15}$$

(ii) $$f(0^+) = \frac{\lim_{s\to\infty} s(s^2-8)}{(s^2+9)s} = \frac{s^2}{s^2} = 1;$$

$$f(\infty) = \frac{\lim_{s\to 0}(s^2-8)}{(s^2+9)} = \frac{-8}{9}$$

3. (a) Given a magnitude of 10 V, phase angle of 10^0, complex frequency of $s = -12 + j9$ put down the expression for the voltage in the time domain.

b) For the circuit of Fig.1.11, determine the forced response $i(t) = I_m e^{\sigma t} \cos(\omega t + \theta)$.

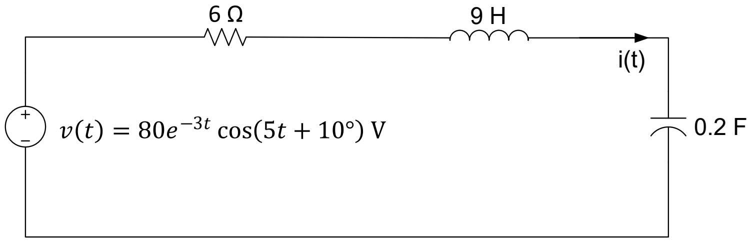


Figure 1.11

Solution:

(a) $v(t) = 10e^{-12t}\cos(9t + 10^o)\,\text{V}$

(b) $v(t) = 80e^{-3t}\cos(5t + 10^o)$

$$= 6i(t) + 9\frac{di(t)}{dt} + \frac{1}{0.2}\int_0^t i(t)dt$$

$$80\angle 10^o\, e^{st} = 6I(s)e^{st} + 9sI(s)e^{st} + \frac{5}{s}I(s)e^{st}$$

$$I(s) = \frac{80\angle 10^o}{\left[6 + 9s + \frac{5}{s}\right]}$$

$$= \frac{80\angle 10^o}{\left[6 + 9(-3 + j5) + \frac{5(-3 - j5)}{(9 + 25)}\right]}$$

$$= \frac{80\angle 10^o}{(6 - 27 + j45 - 0.44 - 0.74)}$$

$$= \frac{80\angle 10^o}{(-21.44 - j45.74)}$$

$$= \frac{80\angle 10^o}{50\angle - 115.4^o}$$

$$= 1.6\angle{-125.4^o}$$

$$i(t) = 1.6e^{-3t}\cos(5t - 125.4^o)\,\text{A}$$

4. Evaluate (i) $\int_{-6}^{-2}(2t^2 + 4)\,\delta(t + 5)dt$ (ii) $\int_1^{\pi}(\cos 2t)\,\delta(t - \pi/2)dt$, indicating time validity in each

Solution:

$$\text{(i)}\quad f(-5) = 2(-5)^2 + 4 = 54$$

$$\text{(ii)}\quad f\left(\frac{\pi}{2}\right) = \cos\left[2\left(\frac{\pi}{2}\right)\right] = \cos\pi = -1$$

$$\therefore\quad 1 < \frac{\pi}{2} < \pi$$

5. Obtain the Laplace transform of $f(t) = \delta(t) + 2u(t) - 3e^{-2t}u(t)$

Solution:

By the linearity property

$$F(s) = \mathcal{L}[\delta(t)] + 2\mathcal{L}[u(t)] - 3\mathcal{L}[e^{-2t}u(t)]$$

$$= 1 + 2\frac{1}{s} - 3\frac{1}{s+2} = \frac{s^2 + s + 4}{s(s+2)}$$

6. Determine the Laplace transform of $f(t) = t^2 \sin 2t\, u(t)$

Solution:

We know that

$$\mathcal{L}[\sin 2t] = \frac{2}{s^2 + 2^2}$$

Using frequency differentiation in Eq. (1.26)

$$F(s) = \mathcal{L}[t^2 \sin 2t] = (-1)^2 \frac{d^2}{ds^2}\left(\frac{2}{s^2 + 4}\right)$$

$$= \frac{d}{ds}\left(\frac{-4s}{(s^2+4)^2}\right) = \frac{12s^2 - 16}{(s^2+4)^3}$$

7. Find the Laplace transform of the gate function in Fig.1.12

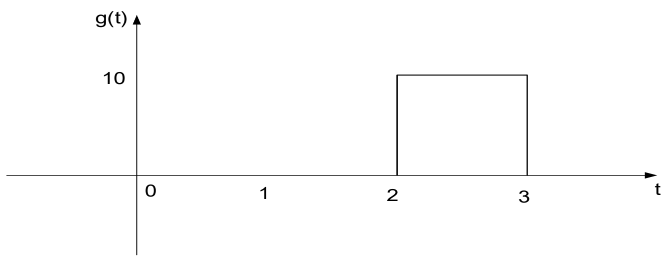


Figure 1.12 Gate function

Solution:

We can express the gate function in Fig.1.12 as

$$g(t) = 10[u(t-2) - u(t-3)]$$

Since we know the Laplace transform of $u(t)$, we apply the time-shift property and obtain

$$G(s) = 10\left(\frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}\right) = \frac{10}{s}(e^{-2s} - e^{-3s})$$

8. Calculate the Laplace transform of the periodic function in Fig. 1.13.

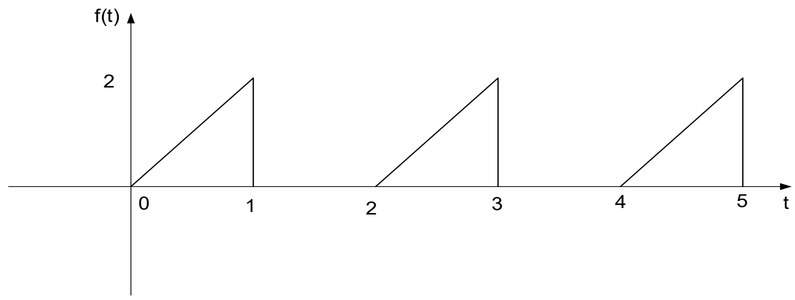


Figure 1.13

Solution

The period of the function is $T = 2$.

We first obtain the transform of the first period of the function.

$$f_1(t) = 2t[u(t) - u(t-1)] = 2u(t) - 2u(t-1)$$
$$= 2tu(t) - 2(t-1+1)u(t-1)$$
$$= 2tu(t) - 2(t-1)u(t-1) - 2u(t-1)$$

Using the time shift property

$$F_1(s) = \frac{2}{s^2} - 2\frac{e^{-s}}{s^2} - \frac{2}{s}e^{-s} = \frac{2}{s^2}(1 - e^{-s} - se^{-s})$$

$$F(s) = \frac{F_1(s)}{1 - e^{-Ts}} = \frac{2}{s^2(1 - e^{-2s})}(1 - e^{-s} - se^{-s})$$

9. Find the initial and final values of the function whose Laplace transform is

$$H(s) = \frac{20}{(s+3)(s^2+8s+25)}$$

Solution:

Applying the initial-value theorem

$$h(0) = \lim_{s\to 0} sH(s) = \lim_{s\to\infty} \frac{20s}{(s+3)(s^2+8s+25)}$$

$$= \lim_{s\to\infty} \frac{\frac{20}{s^2}}{\left(1+\frac{3}{s}\right)\left(1+\frac{8}{s}+\frac{25}{s^2}\right)} = \frac{0}{(1+0)(1+0+0)} = 0$$

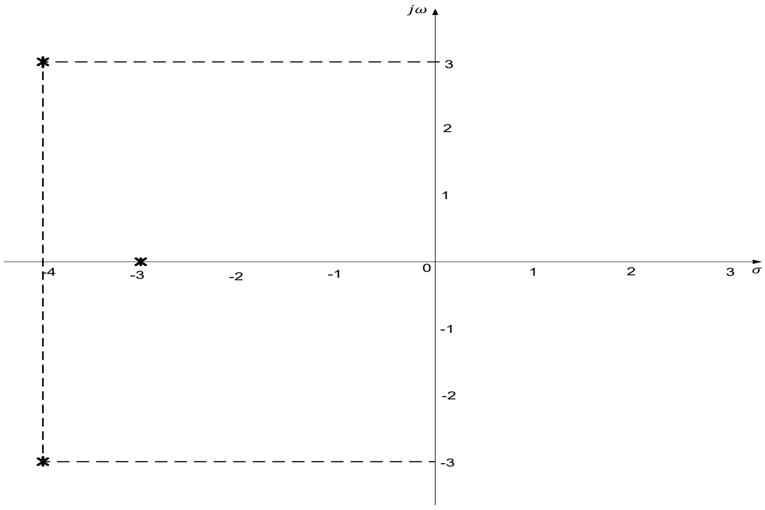


Figure 1.14

To be sure that the final-value theorem is applicable, we check where the poles of H(s) are located. The poles of $H(s)$ are $s = -3, -4 \pm j3$, which all have negative real parts: they are all located on the left half of the s plane (Fig.1.14). Hence, the final-value theorem applies and

$$h(\infty) = \lim_{s\to 0} sH(s) = \lim_{s\to 0} \frac{20s}{(s+3)(s^2+8s+25)}$$

$$= \frac{0}{(0+3)(0+0+25)} = 0$$

1.8 Exercise

1. a. (i) Given $\mathcal{L}\cos 6t = s(s^2+36)$, determine

(ii) the laplace transform of $e^{3t}\cos 6t$;

(ii) The Laplace transform of $\cos 4(t-3)u(t-3)$

b.(i). State and prove the final value theorem.

(ii) State and prove the initial value theorem

2. (i) Determine the initial value of $y(t)$, given $Y(s) = \frac{(3s^2+2)}{(s^3+7s^2+12s)}$

(ii) Evaluate $\int_{-1}^{0} (\sin^2)\, \delta(t+\frac{\pi}{4})dt$, stating the validity thereof

3. By Laplace transform, determine $y(t)$, given:

$$\frac{d^2y(t)}{dt^2} - \frac{3dy(t)}{dt} - 10y(t) = e^{-t} \qquad y(0^-) = 2 \qquad y(0^-) = 1$$

4. (a) Given a magnitude of 20 V, phase angle of 30^0, complex frequency of $s = -10 + 7j$ put down the expression for the voltage in the time domain.

(b) For the circuit of Fig. A, determine the forced response:

$$i(t) = I_m e^{\sigma t} \cos(\omega t + \theta).$$

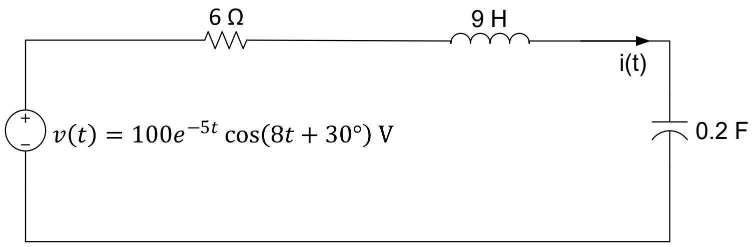


Figure A

c). Give the geometrical interpretation of the Heaviside (unit) impulse function

5. a) Sketch and determine the Laplace transform of the function given by:

$$i(t) = \begin{cases} I_0\left(\frac{t}{t_0} - 1\right), & t_0 \le t \le 2t_0 \\ I_0, & 2t_0 \le t \le 3t_0 \\ 0, & t > 3t_0 \end{cases}$$

b) Prove the time integration theorem

6. a (i). State the initial theorem. without first finding $f(t)$ determine $f(0^+)$, $f(\infty)$ for each of $F(s)$ equal to (1) $4e^{3s}(2s+60)/s$ (2) $(s^2-10)/(s^2+9)s$

7. Evaluate (i) $\int_{-6}^{-2}(2t^2+4)\,\delta(t+5)dt$ (ii) $\int_{1}^{\pi}(\cos 2t)\,\delta(t-\pi/2)dt$, indicating time validity in each

8. Determine the Laplace transform of the periodic signal in Fig. B

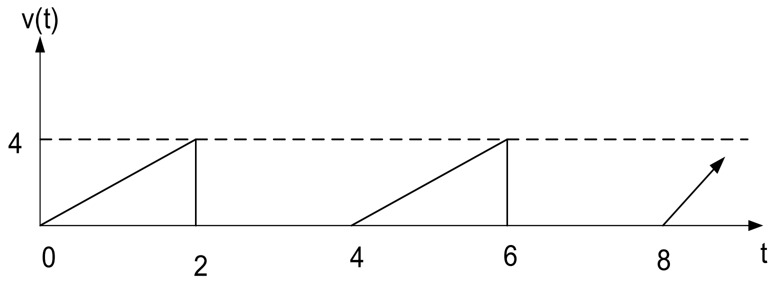


Figure B

9. Determine the Laplace transform of the wave form of Fig. C

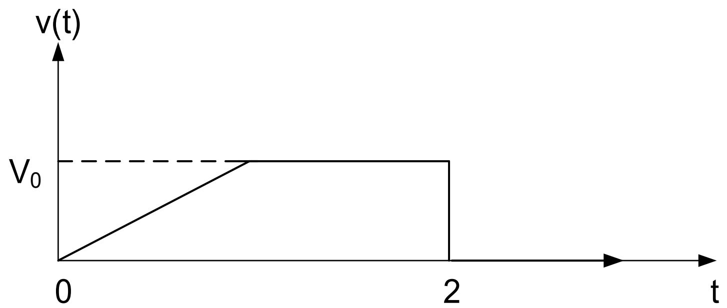


Figure C

10. (a) For the second-order differential equation:

$$\frac{d^2y(t)}{dt^2}-\frac{7dy(t)}{dt}+12y(t)=4e^{-2t}$$

$[y(0)=2, y(0)=5]$, determine the response $y(t)$

b) Evaluate (i) $\int_{-6}^{-2}(2t^2+4)\,\delta(t+5)dt$ (ii) $\int_{1}^{\pi}(\cos 2t)\,\delta(t-\pi/2)dt$, indicating time validity in each

11. (a) Given magnitude of 80 V, phase angle of 30^0, complex frequency of

$s=-6+j8$, put down the expression for the voltage in the time domain for the.

b) For the circuit of Fig. D, determine the forced response

$i(t) = I_m e^{\sigma t}\cos(\omega t + \theta)$. (Hint: just determine I_m, σ and θ)

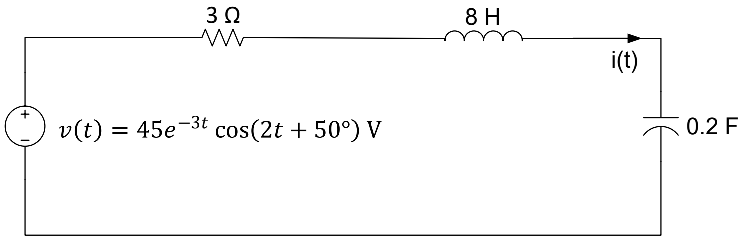


Figure D

(c) What is the rationale behind the lower limit of the one-sided Laplace transform. (State the lower limit).

12. (a) Find the two-side Laplace transformation of the function

$$f(t) = -3e^{-2t}\{u(t+3) - u(t-2)\}$$

(b) also determine its one-sided Laplace transformation.

13. State and prove the final-value and initial-value theorem.

14. Determine the step response of a system with unit impulse response of $-4e^{-t}6e^{-2t}, t \geq 0$

15. A first order linear system is initially relaxed for a unit step signal u(t). The response is $v_1(t) = (1 - e^{-3t})$ for $t > 0$. If a signal $3u(t) + \delta(t)$ is applied to the same initial relax system, what is the response?

16. The response of initially relaxed linear circuit to a signal v_s is $e^{-3t}u(t)$. Determine the response if the signal is changed to $(v_s + 2\frac{dv_s}{dt})$.

17. The response of a network for $t > 0$ is $v(t) = Kte^{-\alpha t}$, with α real and positive. What is the value of t that results in maximum value of $v(t)$.

18. By working backwards, determine the Laplace transform of $\frac{f(t)}{t}$

19. Sketch and determine the Laplace transform of the function given by:

$$f = \begin{cases} 6, & 0 \leq t \leq 1 \\ 8 - 2t, & 1 \leq t \leq 3 \\ 2, & t > 3 \end{cases}$$

20. The delta impulse function can be thought of as a rectangle with zero width and infinite length, with an area of unity. Use the (integral) effect of $\delta(t)$ on other functions to show this to be the case

21. The unit step response of a network is $(2 - 3e^{-\alpha t})$. What is the unit impulse response?

22. Determine the Laplace transform of the wave form of Fig. K

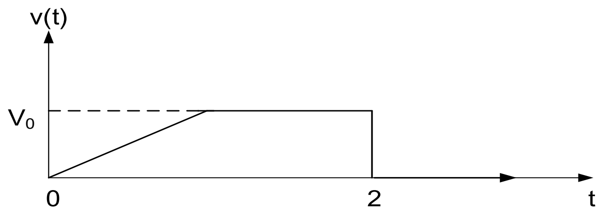


Figure K

23. What is the Laplace transform of $(t^2 - 2t)u(t-1)$

24. prove that $\frac{du(t)}{dt} = \delta(t)$. (Do not Laplace)

25. Find the Laplace of the following functions:

$(i)\ \sin 2t + \frac{t}{2}\cos 2t$

$(ii)\ \frac{1}{4}\sin 2t + \frac{t}{2}\cos 2t$

$(iii)\ \frac{1}{4}\sin 2t + \ t\cos 2t$

$(iv)\ \frac{1}{4}\sin 2t + \frac{t}{4}\cos 2t$

26. Evaluate the Laplace transform of the following functions:

(i)	$\int_0^\infty e^{-3t}\delta(t-4)\,dt$	**Ans.** e^{-12}
(ii)	$\int_{-\infty}^{\infty} \sin 2t\,\delta\left(t - \frac{\pi}{4}\right)$	**Ans.** 1
(iii)	$\int_{-\infty}^{\infty} e^{-3t}\delta(t-2)'$	**Ans.** $3e^{-6}$
(iv)	$\frac{\delta(t-4)}{t}$	**Ans.** $\frac{e^{-4s}}{4}$
(v)	Laplace transform of $\cos t \log t\,\delta(t-\pi)$	**Ans.** $-e^{-\pi s}\log\pi$
(vi)	$e^{-4t}\,\delta(t-3)$	**Ans.** $e^{-3\ (s+4)}$

27. Evaluate the Laplace transform of the following functions:

(a) $f(t) = \begin{bmatrix} t-1 & 1<t<2 \\ 0 & otherwise \end{bmatrix}$ **Ans.** $\frac{e^{-s}-e^{-2s}}{s^3} - \frac{e^{-2s}}{s}$

(b) $e^t\, u(t-1)$ **Ans.** $\frac{e^{-(s-1)}}{s-1)}$

28. Evaluate the Laplace transform of the following functions:

a. $t\, u_2(t)$ **Ans.** $\left(\frac{1}{s^2}+\frac{1}{s}\right) e^{-2x}$

b. $\frac{1-e^{2t}}{5} + tu(t) + \cosh t . \cos t$ **Ans.** $\frac{s-2}{s} + \frac{1}{s^2} + \frac{s^3}{s^4+4}$

c. $t^2 u(t-2)$ **Ans.** $\frac{e^{-4s}}{s^4+1}(4s^2+4s+2)$

d. $\sin t . u\,(t-4)$ **Ans.** $\frac{e^{-2s}}{s^3}[\cos 4 + s \sin 4]$

e. $f\,(t) = K(t-2)[u(t-2) - u(t-3)]$ **Ans.** $\frac{K}{s^2}\,\{e^{-2s} - (s+1)e^{-3s}\}$

$f\,(t) = \frac{K \sin \pi\, t}{T}\,[u(t-2T) - u(t-3T)]$ **Ans.** $\frac{K\,\pi\,T}{s^2\,T^2 + \pi^2}\,(e^{-2sT} - e^{-3sT})$

29. Express the following in terms of unit step functions and obtain Laplace transform

(i) $f\,(t) \begin{Bmatrix} t & 0<t<2 \\ 0 & \end{Bmatrix}$ **Ans.** $u(t) - u(t-2), \frac{1-2\,(2s+1)e^{-2s}}{s^2}$

(ii) $f\,(t) \begin{Bmatrix} t & 0<t<2 \\ 0 & \end{Bmatrix}$ **Ans.** $u(t) - u(t-2), \frac{1-2\,(2s+1)e^{-2s}}{s^2}$

(iii) $f\,(t) \begin{Bmatrix} 4 & 0<t<1 \\ -2 & 1<t<3 \\ 5 & t>3 \end{Bmatrix}$ **Ans.** $\frac{4-6e^{-s}+7e^{-3s}}{s}$

(iv) The Laplace transform $tu_2(t)$ is

$(a) \left(\frac{1}{s^2}+\frac{2}{s}\right) e^{-2s}$ $(b)\ \frac{1}{s^2}\ e^{-2s}$ $(c) \left(\frac{1}{s^2}-\frac{2}{s}\right) e^{-2s}$ $(d) \frac{e^{-2s}}{s^2}$ **Ans.** (a)

30. Find the Laplace Transform of the following:

i. $\frac{1}{t}(1-e^t)$ **Ans.** $\log\frac{s-1}{s}$

ii. $\frac{1}{t}(e^{-at} - e^{-bt})$ **Ans.** $\log\frac{s+b}{s+a}$

iii. $\frac{1}{t}(1-\cos at)$ **Ans.** $-\frac{1}{2}\,\log\frac{s^2}{s^2+a^2}$

iv. $\frac{1}{t}(\cos at - \cos bt)$ **Ans.** $-\frac{1}{2}\,\log\frac{s^2+a^2}{s^2+b^2}$

v. $\frac{1}{t}\ sin^2\,t$ **Ans.** $\frac{1}{4}\,\log\frac{s^2+4}{s^2}$

vi. $\frac{1}{t}\,\sinh t$ **Ans.** $-\frac{1}{2}\,\log\frac{s-1}{s+1}$

vii. $\frac{1}{t}(e^{-t}\sin t)$ **Ans.** $cot^{-1}\,(s+1)$

viii. $\frac{1}{t}(1-\cos t)$ **Ans.** $\frac{1}{2}\log(s^2+1)-\log s$

ix. $\int_0^\infty t\,e^{-2t}\sin t\ \ dt$ **Ans.** $\frac{4}{25}$

x. $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\ dt$ **Ans.** $\log 3$

31. Find the Laplace transform of the following:

a. $t\sin 2t$ **Ans.** $\frac{4\,s}{(s^2-\ 4)^2}$

b. $t\sin at$ **Ans.** $\frac{2\ as}{(s^2+a^2)^2}$

c. $t\cosh a\,t$ **Ans.** $\frac{s^2+a^2}{(s^2-a^2)^2}$

d. $t\cos t$ **Ans.** $\frac{s^2-1}{(s^2+1)^2}$

e. $t\cosh t$ **Ans.** $\frac{s^2+1}{(s^2-1)^2}$

f. $t^2\sin t$ **Ans.** $\frac{2\,(3\,s^2-1)}{(s^2+1)^3}$

g. $t^3t^{-3\,t}$ **Ans.** $\frac{6}{(s\ +\ 3)^4}$

h. $t sin^2\,3\,t$ **Ans.** $\frac{1}{2}\left[\frac{1}{s^2}-\frac{s^2-36}{(s^2+36)^2}\right]$

i. $tt^{at}\sin a\,t$ **Ans.** $\frac{2\,a\,(s-\ a)}{(s^2-2\,as+\ 2\,a^2\,)^2}$

j. $\int_0^t e^{-2t}\ t\ \ sin^3\ t\ dt.$ **Ans.** $\frac{3\ \ (s+\ 2)}{2\ \ s}\left[\frac{1}{[(s+2)^2+9]^2}-\frac{1}{[(s+2)^2+1]^2}\right]$

k. $t\ \ e^{-t}\cosh t$ **Ans.** $\frac{s^2+2s+\ 2}{(s^2+2s)^2}$

l. $t^2e^{-2t}\cos t$ **Ans.** $\frac{2(s^3+\ 10\,s^2+\ 25s+\ 2)}{(s^2+\ 4s+\ 5)^2}$

m. Laplace transform t^ne^{-at} is

$(i)\ \frac{\lceil n}{(s+a)^n}$ $(ii)\ \frac{\lceil n+1}{(s+a)^{n+1}}$ $(iii)\frac{\lceil n}{(s+a)^n}$ $(iv)\ \frac{\lceil n+1}{(s+a)^{n+1}}$

32. Find the Laplace transform of the following:

a. $t+t^2+\ t^3$ **Ans.** $\frac{1}{s^2}+\frac{2}{s^3}+\frac{6}{s^4}.$

b. $\sin t \cos t$ **Ans.** $\frac{1}{s^2+4}$.

c. $t^3\, e^{-2t}$ **Ans.** $\frac{6}{(s+2)^4}$

d. $sin^3\, 2\, t$ **Ans.** $\frac{48}{(s^2+4)\,(s^2+36)}$.

e. $e^{-t}\, cos^2$ **Ans.** $\frac{1}{2\,s+2} + \frac{s+1}{2\,s^2+4\,s+10}$

f. sin 2 t cos 3 t **Ans.** $\frac{2\,(s^2-5)}{(s^2+1)\,(s^2+25)}$.

g. sin 2 t cos 3 t **Ans.** $\frac{12\,s}{(s^2+1)\,(s^2+25)}$

i. cos at sinh at **Ans.** $\frac{1}{2}\,[\frac{s-a}{(s-a)^2+a^2} - \frac{s+a}{(s+a)^2+a^2}]$

j. $sinh^3\, t$ **Ans.** $\frac{6}{(s^2-1)\,(s^2-9)}$

k. cos t cos 2 t **Ans.** $\frac{2\,(s^2-5)}{(s^2+1)\,(s^2+9)}$

l. cosh at sin at **Ans.** $\frac{a\,(s^2-2a^2)}{s^2+4a^4}$

m. $f\,(t)\begin{Bmatrix} t^2 & 0<t<2 \\ t-1 & 2<t<3 \\ 7 & t>3 \end{Bmatrix}$ **Ans.** $\frac{2}{s^3} - \frac{e^{-2s}}{s^2}\,(2+3s+3s^2)\,\frac{e^{-3s}}{s^2}(5s-1)$

n. $f\,(t) = \begin{Bmatrix} \cos\left(t-\frac{2\pi}{3}\right) & t>\frac{2\pi}{3} \\ 0 & t<\frac{2\pi}{3} \end{Bmatrix}$ **Ans.** $\frac{e^{-2\pi s}}{3}.\frac{s}{s^2+1}$

33. Determine the Laplace transform of the ramp function in Fig. L.

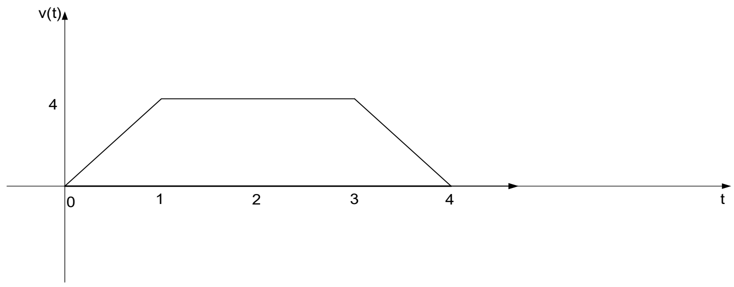


Figure L

33. The step response of a system is given by $f(t) = t^2 + t + 1$. Determine its impulse response.

34. Determine the Laplace transform of the ramp function in Fig. M.

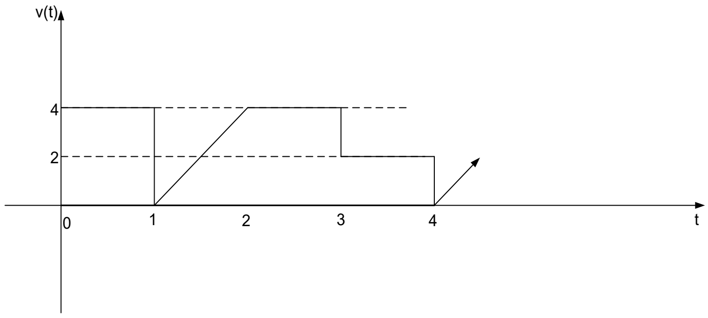


Figure M

35. Determine the Laplace transform of the function in Fig. N

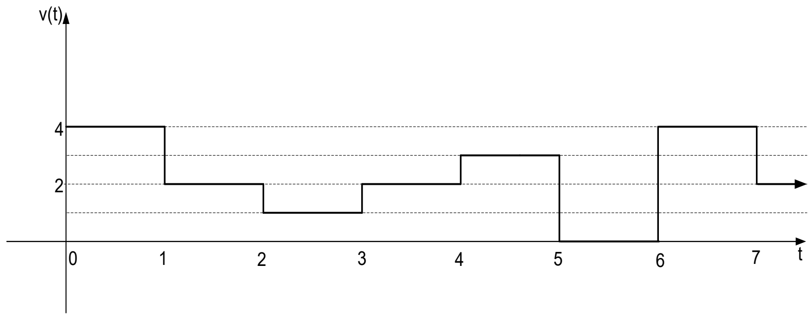


Figure N

36. Find the Laplace transform of $f(t) = (\cos(3) + e^{-5t})u(t)$ **Ans.** $\frac{2s^2+5s+9}{(s+5)(s^2+9)}$

37. Find the Laplace transform of $f(t) = t^2 \cos 3t\, u(t)$ **Ans.** $\frac{2s(s^2-27)}{(s^2+9)^3}$

38. Find the Laplace transform of the function $h(t)$ in Fig. O.

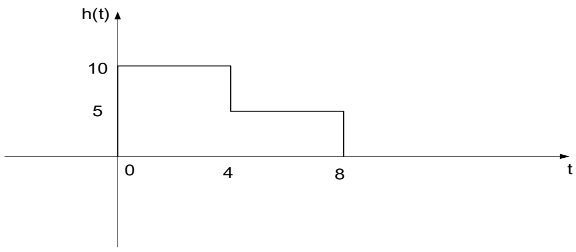


Figure O

Ans. $\frac{5}{s}(2 - e^{-4s} - e^{-8s})$

CHAPTER 2

PARTIAL FRACTIONS

2.0 Introduction

By the use of transform table 1.1 & 1.2, inverse transform of s-domain functions can be determined, to get the corresponding time domain functions. But oftentimes, the s-domain functions can be so complex that it would be necessary to break them into (simpler) partial fractions so that their corresponding time domain equivalent can be readily and easily put down.

Given a typical s-based quotient:

$$F(s) = \frac{{a_m}^s\, s^m + a_{m-1}s^{m-1} + a_{m-2}s^{m-2} + \cdots a_0}{a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots a_0} \qquad 2.0$$

From the above expression might take various forms, whether the numerator and/or the denominator can be factorized or not. The coefficients, i.e., the $a's$, both at the numerator and denominator, are constants with the a_0 (at top) and $a's$ (at bottom) generally unequal. For this quotient to be breakable into partial fractions, $F(s)$must be a proper function which means that the degree of the numerator, m, must be less than that of the denominator, n. (For instance $\frac{4}{3}$ is an improper fraction, whereas $\frac{3}{4}$ is a proper function!) if the two are either of the same order, or that of the numerator is greater than that of the denominator, then a long division must first be performed in order to break the original given polynomial into a divided ("whole number") plus a quotient that is now a proper fraction $\left(\frac{4}{3} = 1 + \frac{1}{3}\right)$, and the quotient can now be reduced into its partial fractions employing any of various methods:

2.1 Method of Comparing Coefficient 's'

$$\frac{2}{s+1} + \frac{3}{s+2} = \frac{2\,(s+2) + 3(s+1)}{(s+1)(s+2)}$$

$$= \frac{2s + 4 + 3s + 3}{s^2 + 3s + 2} = \frac{5s + 7}{s^2 + 3s + 2}$$

Example 2.1: But suppose that we initially were given $\frac{5s+7}{s^2+3s+12}$, and we've required to break it into partial fractions. We know that there would be two, since the denominator has two roots and therefore two factors:

Solution:

$$\frac{5s+7}{s^2+3s+12} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$\Rightarrow \quad \frac{A\,(s+2) + B(s+1)}{(s+1)(s+2)} \equiv \frac{5s+7}{(s+1)(s+2)}$$

This is an identity (hence the three stripes), so that this equality is valid for all values of $s, -\infty < s < \infty$, open-ended interval.

Gathering terms

$$As + 2A + Bs + B = (A+B)s + (2A + B = 5s + 7)$$

Comparing coefficients:

$$\left.\begin{matrix} A+B=5 \\ 2A+B=7 \end{matrix}\right\} \Rightarrow A = 2, B = 3$$

After solving the simultaneous equations. [this agrees with the original given values of A and B]

Generally, given the quotient $\frac{a}{(s-b)(s-c)}$, where the degree of the numerator is zero $(a = as^o)$ and that of the denominator is 2 $[(s-b)(s-c) = s^2 - (b+c)s + bc]$, $\frac{a}{(s-b)(s-c)}$ can be written as:

$\frac{A}{s-b} + \frac{B}{s-c}$, where A, B are constant (although they generally are complex numbers). Our task is to determine their values which would then enable us to write the corresponding time domain expression of the inverse transform.

$$\frac{A}{s-b} + \frac{B}{s-c} = \frac{A(s-c) + B(s-b)}{(s-b)(s-c)} \equiv \frac{a}{(s-b)(s-c)}$$

The two numerators are necessarily equal regardless of the values that s takes on, since this relationship is an identity.

$$\Rightarrow (A+B)s - (Ac + Bb) = a$$

$$A + B = 0 \Rightarrow A = -B$$

$$-(Ac + Bb) = a \Rightarrow -(Ac - Ab) = a$$

$$\Rightarrow A = \frac{a}{b-c}$$

We notice that, on comparing the coefficient, on the left the coefficient of s is $A + B$, but is zero on the right. Also, the constant term (sum of terms) on the left is $-(Ac + Bb)$ which must then equate to a on the right side. So, finally,

$$\frac{a}{(s-b)(s-c)} = \frac{\frac{a}{b-c}}{s-b} - \frac{\frac{a}{b-c}}{s-c}$$

And the inverse transform would give

$$\frac{a}{(s-b)(s-c)} \Leftrightarrow \frac{a}{b-c}e^{bt} - \frac{a}{b-c}e^{ct}$$

$$= \frac{a}{b-c}(e^{bt} - e^{ct})u(t)$$

With the requisite unit step attached depending on the type of signal under consideration.

2.2 Method 2: "Cover Up" (Method of Residues)

This works best when the denominator of the polynomial in question has non-repeats, linear factors. With repeated factors, the "cover-up" rule can be used to determine the coefficients of the non-repeated portions as well as that of the highest-index repeated one. Then the method of comparing coefficients or successive differentiation can be used to determine the lower order repeated portions.

Example 2.2: To repeat the above generic example using the rule:

$$\frac{5s+7}{s^2+3s+12} = \frac{5s+7}{(s+1)(s+2)}$$

$$= \frac{A}{s+1} + \frac{B}{s+2}$$

Solution:

To determine A, for instance, we proceed to isolate it by "disappearing" B whether it's actually A or B that's being "covered-up" depends on subjective judgment!).

Multiplying through by $(s+1)$, the factor lying below A:

$$\frac{(5s+7)(s+1)}{s^2+3s+2} = \frac{A\,(s+1)}{(s+1)} + \frac{B\,(s+1)}{(s+2\,)}$$

$$\Rightarrow \quad \frac{5s+7}{s+2} = A + \frac{B(s+1)}{(s+2)}$$

Now, to "disappear B" thereby leaving 'A' all alone, we simply let s assume the value of -1

$$\left.\frac{5s+7}{s+2}\right|_{s=-1} = \frac{-5+7}{-1+2} = 3$$

as previously determined!

[Note: any finite value other than -1 could have been chosen for s since the equation is an identity. But that would entail closing two different values thereby ending up with a simultaneous linear equation before aiming at the result. This is an entirely unnecessary rigmarole!]

By the same token

$$B = \left.\frac{5s+7}{s+1}\right|_{s=-2} = \frac{-10+7}{-2+1} = 3$$

Also, as previously gotten.

Is this method (which by the way, I highly recommend for most situations) not much more straightforward than the previous one, the method of comparing coefficient? Furthermore, it's less prone to error!

2.2.1 The Procedure:

1. Factorized the given polynomial if possible
2. To determine the unknown constant for instance A above remove the factor "under" A in the polynomial function.
3. Set s equal to the root (-1) in the polynomial to determine A

Respect the above procedure for B taking cognizance of the factor under it and the associated root (-2 now taking the place of the previous -1 in step **3** above). It should be noted that these steps are to be repeated for polynomials with more than two factors until all the unknown numerators are exhausted.

Caveat to my earlier caution and advice about choosing values other than the roots. It's no sin, and may aid in determining the two unknowns at once with simultaneous linear equations. But how about when three or more unknown are involved?)

So, for a generic polynomial

$$\frac{a}{(s-b)(s-c)} = \frac{A}{(s-b)} + \frac{B}{(s-c)} \qquad 2.1$$

$$A = \left.\frac{a}{(s-c)}\right|_{s=b}, B = \left.\frac{a}{(s-b)}\right|_{s=c}$$

2.2.2 The Fast Placed Using this Method is as Follows

1. Factorized the denominator of the given quotient (or polynomial)

2. From the denominator, remove the factor on top of which is the unknown constant (or by the way, complex number) to be determined.
3. For the remaining partial fractions, let's assume the value of the root (or pole) of the factor under consideration, after multiplying throughout by that factor.
4. Whatever value so realized (real or complex) is then the constant representing the numerator of the partial fractions under consideration.

2.3 Types of Partial Fractions

1. **Linear factor: non-repeated factors**: $\frac{A}{s-a}+\frac{B}{s-b}+\frac{C}{s-c}$
2. **Repeated factor** $(s-a)^2$ leads to $\frac{A}{s-a}+\frac{B}{(s-a)^2}$; $(s-a)^3$ leads to $\frac{A}{s-a}+\frac{B}{(s-a)^2}+\frac{C}{(s-a)^3}$; $(s-a)^4$ leads to $\frac{A}{s-a}+\frac{B}{(s-a)^2}+\frac{C}{(s-a)^3}+\frac{D}{(s-a)^4}$, etc
3. **A quadratic factor** (s^2+as+b) leads to $\frac{As+B}{s^2+as+b}$ and
4. **Repeated quadratic factor** $(s^2+as+b)^2$ leads to

$$\frac{As+B}{s^2+as+b}+\frac{Cs+D}{(s^2+as+b)^2}$$

Example 2.3: Resolve into partial fraction: $\frac{s^2-15s+41}{s^3-4s^2-3s+18}$

Solution:

Factorize the denominator of this proper quotient ("fraction") (numerator has degree 2, while and denominator, has degree 3) by noting that both -2 and +3 are its roots, by inspection:

$$[(-2)^3-4(-2)^2-3(-2)+18=0],$$

$$[3^2-4\,(3^2)-3(3)+18=0]$$

Dividing $s^3-4s^2-3s+18$ by the factor $(s+2)$ leads to s^2-6s+9 and further dividing s^2-6s+9 by $s-3$, leads to $s-3$

So, $$\frac{(s^2-15s+41)}{s^3-4s^2-3s+18}=\frac{s^2-15s+41}{(s+2)(s-3)^2}$$

$$=\frac{A}{s+2}+\frac{B}{(s-3)}+\frac{C}{(s-3)^2}$$

$$\Longrightarrow \quad A=\left.\frac{(s^2-15s+41)}{(s-3)^2}\right|_{s=-2}=\frac{[(-2)^2-15(-2)+41]}{(-5)^2}$$

$$= \frac{75}{25} = 3$$

$$C = \left.\frac{(s^2 - 15s + 41)}{(s+2)}\right|_{s=3} = \frac{[3^2 - 15(3) + 41]}{s} = \frac{5}{5} = 1$$

$$\frac{3}{s+2} + \frac{B}{s-3} + \frac{1}{(s-3)^2} = \frac{3(s-3)^2 + B(s+2)(s-3) + 1(s+2)}{(s+2)(s-3)^2}$$

$$= \frac{(s^2 - 15s + 41)}{(s^3 - 4s^2 - 3s + 18)}$$

$$3(s-3)^2 + B(s+2)(s-3) \equiv s^2 - 15s + 41$$

Setting $s = 3$ or $s = -2$ would “disappear” B that we’re trying to evaluate! So, we choose numbers other than those two:

$$s = 0 \Longrightarrow 3(-3)^2 + B(2)(-3) + 2 \equiv 41$$

$$\Longrightarrow B = -2.$$

Finally,
$$\frac{s^2 - 15s + 41}{s^3 - 4s^2 - 3s + 18} \equiv \frac{3}{s+2} - \frac{2}{s-3} + \frac{1}{(s-3)^2}$$

To get back to the time domain, the inverse transform $\left[\frac{3}{(s+2)} - \frac{2}{(s-3)} + \frac{1}{(s-3)^2}\right]$

$$= (3e^{-2t} - 2e^{3t} + te^{3t})u(t)$$

(Note: remember the first-shift theorem with respect to the third term above, or viewed from another dimension, the effect of multiplying a function by t:

$$\left[\frac{-d}{ds}\left(\frac{1}{s-3}\right) = \frac{1}{(s-3)^2}\right]$$

An alternative, more straight forward approach to determining B is to multiply the quotient by $(s-3)^2$, leading to $\frac{(s^2-15s+41)\,(s-3)^2}{(S+2)\,(s-3)^2} = \frac{s^2-15s+41}{s+2}$, then differentiating once with respect to s, and then evaluating the resulting quotient at $s = 3$. If the pole is repeated twice (that is, a total of 3 times), the third numerator term, a_{n-2} is evaluated by again multiplying the quotient under consideration by $(s-p)^n$, differentiating the result twice with repeat to s, and then dividing by 2 (or 2!).

In general, given a quotient (of polynomials in s):

$$F(s) = \frac{a_n}{(s-p)^n} + \frac{a_{n-1}}{(s-p)^{n-1}} + \frac{a_{n-2}}{(s-p)^{n-2}} + \cdots + \frac{a_1}{(s-p)} \qquad 2.2$$

With n repeated ratios (each root is p,) to evaluate the a_{n-k} term [that is, the $(K+1)$ the numerator term], we first multiply F(s) by $(s-p)^n$, differentiate the result k times with respect to s, and then divide by L! (K factorial) after (or before) evaluating the result at $s = P$.

$$a_{n-k} = \frac{1}{k!}\left\{\frac{d^k}{ds^k}[\{(s-p)^n\}F(s)]\right\}\Bigg|_{s=p} \qquad 2.3$$

For the last example, B represent the a_{n-1} term, that is the numerator of the second repeated root (written in descending order of indices).

$$a_{n-1} = \frac{1}{1!}\left\{\frac{d}{ds}\left[(s-3)^2\left\{\frac{s^2-15s+41}{(s+2)(s-3)^2}\right\}\right]\right\}_{s=3}$$

$$= \frac{d}{ds}\,\frac{s^2-15s+41}{s+2}\Bigg|_{s=3}$$

$$= \frac{(s+2)(2s-15)-(s^2-15s+41)}{(s+2)^2}\Bigg|_{s=3}$$

$$= \frac{(5)\,(-9)-(9-54+41)}{25} = \frac{-45-9+45-41}{25}$$

$$= \frac{-50}{25} = -2,$$

With practice, it could be seen that the above procedure is much quicker and less subject to errors, than the method of comparing coefficients.

Example 2.4: Find the inverse transform of $\dfrac{1}{s^2 - 5s + 6}$.

Solution: Let us convert the given function into partial fractions.

$$\mathcal{L}^{-1}\left[\frac{1}{s^2-5s+6}\right] = \mathcal{L}^{-1}\left[\frac{1}{s-3} - \frac{1}{s-2}\right]$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s-3}\right) - \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = e^{3t} - e^{2t}$$

Note: We shall discourse the inverse Laplace transform in details in the proceeding chapter.

Example 2.5: Find the inverse Laplace transform of

$$\frac{s-1}{s^2-6s+25}$$

Solution: $\mathcal{L}^{-1}\left[\frac{s-1}{s^2-6s+25}\right]=\mathcal{L}^{-1}\left[\frac{s-1}{(s-3)^2+(4)^2}\right]=\mathcal{L}^{-1}\left[\frac{s-3+2}{(s-3)^2+(4)^2}\right]$

$$=\mathcal{L}^{-1}\left[\frac{s-3}{(s-3)^2+(4)^2}\right]+\frac{1}{2}\mathcal{L}^{-1}\left[\frac{4}{(s-3)^2+(4)^2}\right]$$

$$=e^{3t}\cos 4t+\frac{1}{2}e^{3t}\sin 4t$$

Example 2.6: Find the inverse Laplace transform of

$$\frac{s+4}{s(s-1)+(s^2+4)}.$$

Solution: Let us first resolve $\frac{s+4}{s(s+1)+(s^2+4)}$ in partial fractions.

$$\frac{s+4}{s(s-1)+(s^2+4)}\equiv\frac{A}{s}+\frac{B}{s-1}+\frac{Cs+D}{s^2+4}$$

$$s+4\equiv A(s-1)(s^2+4)+Bs(s^2+4)+(Cs+D)s(s-1) \qquad (*)$$

Putting $s=0,\ we\ get\ 4=-4A\ \ or\ \ A=-1$

Putting $s=1,\ we\ get\ 5=B.1.(1+4)\ or\ B=1$

Equating the coefficients of s^3 on both sides of (*), we have

$$0=A+B+C \quad or \quad 0=-1+1+C \quad or \quad C=0.$$

Equating the coefficients of s on both sides of (1), we get

$$1=4A+4B-D \quad or \quad 1=-4+4-D \quad or \quad D=-1$$

On putting the values of A, B, C, D in (*), we get

$$\frac{s+4}{s(s-1)+(s^2+4)}=-\frac{1}{s}+\frac{1}{s-1}+\frac{1}{s^2+4}$$

$\therefore$ $$\mathcal{L}^{-1}\left[\frac{s+4}{s(s-1)+(s^2+4)}\right]=\mathcal{L}^{-1}\left[-\frac{1}{s}+\frac{1}{s-1}+\frac{1}{s^2+4}\right]$$

$$=-\mathcal{L}^{-1}\left(\frac{1}{s}\right)+\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)-\frac{1}{2}\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right)$$

$$=-1+e^{t}-\frac{1}{2}\sin 2t\,.$$

Example 2.7: Find the Laplace inverse of

$$\frac{s^2}{(s^2 + a^2) + (s^2 + b^2)}$$

Solution: Let us convert the given into partial fractions.

$$\mathcal{L}^{-1}\left[\frac{s^2}{(s^2 + a^2) + (s^2 + b^2)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{a^2}{a^2 - b^2} \cdot \frac{1}{s^2 + b^2} - \frac{b^2}{a^2 - b^2} \cdot \frac{1}{s^2 + b^2}\right]$$

$$= \frac{1}{a^2 - b^2} \mathcal{L}^{-1}\left[\frac{a^2}{s^2 + a^2} - \frac{b^2}{s^2 - b^2}\right]$$

$$= \frac{1}{a^2 - b^2}\left[a^2\left(\frac{1}{a}\sin at\right) - b^2\left(\frac{1}{b}\sin bt\right)\right]$$

$$= \frac{1}{a^2 - b^2}[a \sin at - b \sin bt].$$

2.4 Repeated Poles

Suppose $F(s)$ has n repeated poles at $s = -p$. Then we may represent $F(s)$ as

$$F(s) = \frac{k_n}{(s+p)^n} + \frac{k_{n-1}}{(s+p)^{n-1}} + \cdots + \frac{k_2}{(s+p)^2} + \frac{k_1}{s+p} + F_1(s) \qquad 2.4$$

where $F_1(s)$ is the remaining part of $F(s)$ that does not have a pole at $s = -p$. We determine the expansion coefficient k_n as

$$k_n = (s+p)^n + F(s)|_{s=-p} \qquad 2.5$$

as we did above. To determine k_{n-1}, we multiply each term in Eq. (2.4) by $(s+p)^n$ and differentiate to get rid of k_n, then evaluate the result at $s = -p$ to get rid of the other coefficients except k_{n-1} Thus, we obtain

$$k_{n-1} = \frac{d}{ds}[(s+p)^n F(s)]|_{s=-p} \qquad 2.6$$

Repeating this gives

$$k_{n-2} = \frac{1}{2!}\frac{d^2}{ds^2}[(s+p)^n F(s)]|_{s=-p} \qquad 2.7$$

The mth term becomes

$$k_{n-m} = \frac{1}{m!}\frac{d^m}{ds^m}[(s+p)^n F(s)]|_{s=-p} \qquad 2.8$$

where $m = 1,2,\ldots,n-1$. One can expect the differentiation to be difficult to handle as m increases. Once we obtain the values of $k_1, k_2 \ldots . k_n$ by partial fraction expansion, we apply the inverse transform

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)^n}\right] = \frac{t^{n-1}e^{-at}}{(n-1)!}u(t) \qquad 2.9$$

to each term on the right-hand side of Eq. (2.4) and obtain

$$f(t) = \left(k_1 e^{-pt} + k_2 t e^{-pt} + \frac{k_3}{2!}t^2 e^{-pt} + \cdots + \frac{k_n}{(n-1)!}t^{n-1}e^{-pt}\right)u(t) + f_1(t) \qquad 2.10$$

2.5 Complex Poles

A pair of complex poles is simple if it is not repeated; it is a double or multiple pole if repeated. Simple complex poles may be handled the same way as simple real poles, but because complex algebra is involved the result is always cumbersome. An easier approach is a method known as completing the square. The idea is to express each complex pole pair (or quadratic term) in $D(s)$ as a complete square such as $(s+\alpha)^2 + \beta^2$ and then use Table 1.2 to find the inverse of the term.

Since $N(s)$ and $D(s)$ always have real coefficients and we know that the complex roots of polynomials with real coefficients must occur in conjugate pairs, $F(s)$ may have the general form

$$F(s) = \frac{A_1 s + A_2}{s^2 + as + b} + F_1(s) \qquad 2.11$$

where $F_1(s)$ is the remaining part of $F(s)$ that does not have this pair of complex poles. If we complete the square by letting

$$s^2 + as + b = s^2 + 2\alpha s + \alpha^2 + \beta^2 = (s+\alpha)^2 + \beta^2 \qquad 2.12$$

we also let

$$A_1 s + A_2 = A_1(s+\alpha) + B_1\beta \qquad 2.13$$

then Eq. (2.12) becomes

$$F(s) = \frac{A_1(s+\alpha)}{(s+\alpha)^2+\beta^2} + \frac{B_1\beta}{(s+\alpha)^2+\beta^2} + F_1(s) \qquad 2.14$$

From Table 1.2, the inverse transform is

$$f(t) = (A_1 e^{-\alpha t}\cos\beta t + B_1 e^{-\alpha t}\sin\beta t)u(t) + f_1(t) \qquad 2.15$$

Whether the pole is simple, repeated, or complex, a general approach that can always be used in finding the expansion coefficients is the method of algebra. To apply the method, we first set $F(s)=\frac{N(s)}{D(s)}$ equal to an expansion containing unknown constants. We multiply the result through by a common denominator. Then we determine the unknown constants by (i.e., by algebraically solving a set of simultaneous equations for these coefficients at like powers of s).

Another general approach is to substitute specific, convenient values of s to obtain as many simultaneous equations as the number of unknown coefficients, and then solve for the unknown coefficients. We must make sure that each selected value of s is not one of the poles of F(s).

Example 2.8: Find the inverse Laplace transform of

$$F(s)=\frac{3}{s}-\frac{5}{s+1}+\frac{6}{s^2+4}$$

Solution:

The inverse transform is given by

$$f(t)=\mathcal{L}^{-1}[F(s)]=\mathcal{L}^{-1}\left(\frac{3}{s}\right)-\mathcal{L}^{-1}\left(\frac{5}{s+1}\right)+\mathcal{L}^{-1}\left(\frac{6}{s^2+4}\right)$$
$$=(3-5e^{-t}+3\sin 2t)u(t), \qquad t\geq 0$$

where Table 1.2 has been consulted for the inverse of each term.

Example 2.9: Find $f(t)$ given that

$$F(s)=\frac{s^2+12}{s(s+2)(s+3)}$$

Solution:

Unlike in the previous example where the partial fractions have been provided, we first need to determine the partial fractions. Since there are three poles, we let

$$\frac{s^2+12}{s(s+2)(s+3)}=\frac{A}{s}+\frac{B}{s+2}+\frac{C}{s+3} \qquad 2.16$$

where A, B, and C are the constants to be determined, we can find the constants using two approaches

Residue method:

$$A = sF(s)|_{s=0} = \left.\frac{s^2+12}{(s+2)(s+3)}\right|_{s=0} = \frac{12}{(2)(3)} = 2$$

$$B = (s+2)F(s)|_{s=-2} = \left.\frac{s^2+12}{s(s+3)}\right|_{s=-2} = \frac{4+12}{(-2)(1)} = -8$$

$$C = (s+3)F(s)|_{s=-3} = \left.\frac{s^2+12}{s(s+2)}\right|_{s=-3} = \frac{9+12}{(-3)(1)} = -7$$

Algebraic method:

multiplying both sides of Eq. (2.17) by $s(s+2)(s+3)$ gives

$$s^2 + 12 = A(s+2)(s+3) + Bs(s+3) + Cs(s+2)$$

Or

$$s^2 + 12 = A(s^2+5s+6) + B(s^2+3s) + C(s^2+2s)$$

Equating the coefficient of like powers of s gives

Constant: $12 = 6A \implies A = 2$

s: $0 = 5A + 3B + 2C \implies 3B + 2C = -10$

s^2: $1 = A + B + C \implies B + C = -1$

Thus, $A = 2, B = -8, C = 7$ and Eq. (2.17) becomes

$$F(s) = \frac{2}{s} - \frac{8}{s+2} + \frac{7}{s+3}$$

By finding the inverse transform of each term, we obtain

$$f(t) = (2 - 8e^{-2t} + 7e^{-3t})u(t)$$

Example 2.10: Calculate $v(t)$ given that

$$V(s) = \frac{10s^2+4}{s(s+1)(s+2)^2}$$

Solution:

While the previous example is on simple roots, this example is on repeated roots. Let

$$V(s) = \frac{10s^2 + 4}{s(s+1)(s+2)^2}$$

$$= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+2)^2} + \frac{D}{s+2} \qquad 2.17$$

Residue method:

$$A = sV(s)|_{s=0} = \left.\frac{10s^2 + 4}{(s+1)(s+2)^2}\right|_{s=0} = \frac{4}{(1)(2)^2} = 1$$

$$B = (s+1)V(s)|_{s=-1} = \left.\frac{10s^2 + 4}{s(s+2)^2}\right|_{s=0} = \frac{14}{(-1)(1)^2} = 14$$

$$C = (s+2)^2V(s)|_{s=-2} = \left.\frac{10s^2 + 4}{s(s+1)}\right|_{s=-2} = \frac{44}{(-2)(-1)} = 22$$

$$D = \frac{d}{ds}[(s+2)^2V(s)]\Big|_{s=-2} = \frac{d}{ds}\left(\frac{10s^2 + 4}{s^2 + s}\right)\Bigg|_{s=-2}$$

$$= \left.\frac{(s^2 + s)(20s) - (10s^2 + 4)(2s+1)}{(s^2 + s)^2}\right|_{s=-2} = \frac{52}{4} = 13$$

Algebraic method:

Multiplying Eq. (2.17) by $s(s+1)(s+2)^2$, we obtain

$$10s^2 + 4 = A(s+1)(s+2)^2 + Bs(s+2)^2$$

$$= Cs(s+1) + Ds(s+1)(s+2)$$

Or

$$10s^2 + 4 = A\,(s^3 + 5s^2 + 8s + 4) + B(4s^2 + 4s)$$

$$C(s^2 + s) + D(s^3 + 3s^2 + 2s)$$

Equating coefficients,

Constant: $4 = 4A \implies A = 1$

s: $0 = 8A + 4B + C + 2D \implies 4B + C + 2D = -8$

s^2: $10 = 5A + 4B + C + 3D \implies 4B + C + 3D = 5$

s^3: $0 = A + B + D \implies B + D = -1$

Solving these simultaneous equations gives $A = 1, B = -14, C = 22, D = 13$, so that

$$V(s) = \frac{1}{s} = \frac{14}{s+1} + \frac{13}{s+2} + \frac{22}{(s+2)^2}$$

Taking the inverse transform of each term, we get

$$v(t) = (1 - 14e^{-t} + 13e^{-2t} + 223^{-2t})u(t)$$

Example 2.11: Find the inverse transform of the frequency-domain function of:

$$H(s) = \frac{20}{(s+3)(s^2+8s+25)}$$

Solution:

In this example, $H(s)$ has a pair of complex poles at $s^2 + 8s + 25 = 0$ or $s = -4 \pm j3$. We let

$$H(s) = \frac{20}{(s+3)(s^2+8s+25)} = \frac{A}{s+3} + \frac{Bs+C}{(s^2+8s+25)} \qquad 2.18$$

We now determine the expansion coefficient in two ways

Combination of Methods:

we can obtain A using the method of residue method

$$A = (s+3)H(s)|_{s=-3} = \left.\frac{20}{s^2+8s+25}\right|_{s=-3} = \frac{20}{10} = 2$$

Although B and C can be obtained using the method of residue, we will not do so, to avoid complex algebra. Rather, we can substitute two specific values of s [say $s = 0,1$, which are not poles of $F(s)$] into Eq. (2.18). This will give us two simultaneous equations from which to find B and C. If we let $s = 0$ in Eq. (2.18), we obtain

$$\frac{20}{75} = \frac{A}{3} + \frac{C}{25}$$

Or

$$20 = 25A + 3C \qquad 2.18.1$$

Since $A = 2$, Eq. (2.18.1) gives $C = -10$. Substituting $s = 1$ into Eq. (2.18) gives

$$\frac{20}{(4)(34)} = \frac{A}{4} + \frac{B + C}{34}$$

Or

$$20 = 34A + 4B + 4C \qquad 2.18.2$$

But $A = 2, C = -10$, so that Eq. (2.28.2) gives $B = -2$

Algebraic method:

Multiplying both sides by Eq (2.18) by $(s + 3)(s^2 + 8s + 25)$ yields

$$20 = A(s^2 + 8s + 25) + (Bs + C)(s + 3)$$

$$= A(s^2 + 8s + 25) + B(s^2 + 3s) + C(s + 3) \qquad 2.18.3$$

Equating coefficient gives

s^2: $\quad 0 = A + B \quad \Longrightarrow \quad A = -B$

s: $\quad 0 = 8A + 3B + C = 5A + C \quad \Longrightarrow C = -5A$

Constant: $\quad 20 = 25A + 3C = 25A - 15A \quad \Longrightarrow \quad A = 2$

That is, $B = -2, C = -10$ thus,

$$H(s) = \frac{2}{s + 3} - \frac{2s + 10}{(s^2 + 8s + 25)} = \frac{2}{s + 3} - \frac{2(s + 4) + 2}{(s + 4)^2 + 9}$$

$$= \frac{2}{s + 3} - \frac{2(s + 4)}{(s + 4)^2 + 9} - \frac{2}{3}\frac{3}{(s + 4)^2 + 9}$$

Taking the inverse of each term, we obtain

$$h(t) = \left(2e^{-3t} - 2e^{-4t}\cos 3t - \tfrac{2}{3}e^{-4t}\sin 3t\right)u(t) \qquad 2.18.4$$

It is alright to leave the result this way. However, we can combine the cosine and sine terms as

$$h(t) = (2e^{-3t} - Re^{-4t}\cos(3t - \theta))u(t) \qquad 2.18.5$$

To obtain Eq. (2.18.5) from Eq. (2.18.4).

Next, we determine the coefficient R and the phase angle θ:

$$R = \sqrt{2^2 + \left(\tfrac{2}{3}\right)^2} = 2.108, \quad \theta = \tan^{-1}\frac{\frac{2}{3}}{2} = 14.43^o$$

Thus

$$h(t) = (2e^{-3t} - 2.108e^{-4t}\cos(3t - 18.43^{o}))u(t)$$

2.6 Exercise

Solve the following partial fraction

1. $\frac{s^2 + 2s + 6}{s^2}$
2. $\frac{1}{s^2 - 7s + 12}$
3. $\frac{s - 2}{s^2 - 4s + 13}$
4. $\frac{3s + 1}{(s - 1)(s^2 + 1)}$
5. $\frac{11s^2 - 2s + 5}{2s^3 - 3s^2 - 3s + 2}$
6. $\frac{2s^2 - 6s + 5}{(s-1)(s-2)(s-3)}$
7. $\frac{3s + 1}{(s - 4)^2 + 9}$
8. $\frac{16}{(s^2 + 2s + 5)^2}$
9. $\frac{1}{(s-3)(s^2 + 2s + 2)}$
10. $\frac{1}{(s-2)(s^2 +)}$
11. $\frac{s^2 - 6s + 7}{(s^3 - 4s + 3)^2}$
12. $\frac{2s+3}{s^2+5s+4}$

CHAPTER 3

INVERSE LAPLACE TRANSFORMS

3.0 Introduction

Now we obtain $f(t)$ *when* $F(s)$ is given, then we say that inverse Laplace transform of $F\ (s)\ is\ f(t)$.

If $\mathcal{L}\ [f(t)] = F(s),\ then\ \mathcal{L}^{-1}\ [F(s)] = f(t)$. 3.1

where $\mathcal{L}^{-1}$ is called the inverse Laplace transform operator.

From the application point of view, the inverse Laplace transform is very useful.

3.1 Important Formulas

(1) $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

(2) $\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^n - 1}{n - 1}$

(3) $\mathcal{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$

(4) $\mathcal{L}^{-1}\left\{\frac{1}{s^2 - k^2}\right\} = \cosh kt$

(5) $\mathcal{L}^{-1}\left\{\frac{1}{s^2 - k^2}\right\} = \frac{1}{k}\cos kt$

(6) $\mathcal{L}^{-1}\left\{\frac{1}{s^2 + k^2}\right\} = \frac{1}{k}\sin kt$

(7) $\mathcal{L}^{-1}\left\{\frac{s}{s^2 + k^2}\right\} = \cos kt$

(8) $\mathcal{L}^{-1}\{F\ (s-k)\} = e^{kt}\ f\ (t)$

(9) $\mathcal{L}^{-1}\left\{\frac{1}{(s-m^2)^2 + n^2}\right\} = \frac{1}{n}\ e^{mt}\sin nt$

(10) $\mathcal{L}^{-1}\left\{\frac{s-m}{(s+m^2)^2 + n^2}\right\} = e^{mt}\cosh nt$

(11) $\mathcal{L}^{-1}\left\{\frac{1}{(s-m^2)^2 - n^2}\right\} = \frac{1}{n}\ e^{mt}\sinh nt$

(12) $\mathcal{L}^{-1}\left\{\frac{s-m}{(s-m^2)^2 - n^2}\right\} = \frac{1}{n}\ e^{mt}\cosh nt$

(13) $\mathcal{L}^{-1}\left\{\frac{1}{(s^2 - m^2)^2}\right\} = \frac{1}{2m^3}\ (\sin mt - mt\cos mt)$

(14) $\mathcal{L}^{-1}\left\{\frac{s}{(s^2 + m^2)^2}\right\} = \frac{1}{2m}t\sin mt$

(15) $\mathcal{L}^{-1}\left\{\frac{s^2 - m^2}{(s^2 + m^2)^2}\right\} = t\cos mt$

(16) $\mathcal{L}^{-1}\ \{1\} = \delta\ (t)$

(17) $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2 + m^2)^2}\right\} = \frac{1}{2m}\ [\sin mt + mt\cos mt]$

Example 3.1: Find the inverse Laplace Transform of the following

(i) $\dfrac{1}{s - 2}$

(ii) $\dfrac{1}{s^2 - 9}$

(iii) $\dfrac{s}{s^2 - 16}$

(iv) $\dfrac{1}{s^2 - 25}$

(v) $\dfrac{s}{s^2 - 9}$

(vi) $\dfrac{1}{(s - 2)^2 + 1}$

(vii) $\dfrac{s - 1}{(s - 2)^2 + 4}$

(viii) $\dfrac{1}{(s - 2)^2 - 4}$

(ix) $\dfrac{s + 2}{(s - 2)^2 - 25}$

(x) $\dfrac{1}{2s - 7}$

Solutions:

(i) $\mathcal{L}^{-1}\dfrac{1}{s-2} = e^{2t}$

(ii) $\mathcal{L}^{-1}\dfrac{1}{s^2-9} = \mathcal{L}^{-1}\dfrac{1}{3} \cdot \dfrac{3}{s^2-(3)^2} = \dfrac{1}{3}\sinh 3t$

(iii) $\mathcal{L}^{-1}\dfrac{s}{s^2-16} = \mathcal{L}^{-1}\dfrac{s}{s^2-(4)^2} = \cosh 4t$

(iv) $\mathcal{L}^{-1}\dfrac{1}{s^2+25} = \dfrac{1}{5}\dfrac{5}{s^2-(5)^2} = \dfrac{1}{5}\sin st$

(v) $\mathcal{L}^{-1}\dfrac{1}{s^2+9} = \dfrac{s}{s^2-(3)^2} = \cos 3t$

(vi) $\mathcal{L}^{-1}\dfrac{2}{(s-2)^2+1} = e^{2t} sin\ t$

(vii) $\mathcal{L}^{-1}\dfrac{s-1}{(s-1)^2+4} = e^{t}\cos 2t$

(viii) $\mathcal{L}^{-1}\dfrac{1}{(s-3)^2-4} = \dfrac{1}{2}\dfrac{2}{(s-3)^2-(2)^2} = \dfrac{1}{2}e^{-3t}\sinh 2t$

(ix) $\dfrac{1}{2s-7} = \dfrac{1}{2}e^{\frac{7}{2}t}$

(x) $\left[\mathcal{L}^{-1}F(s) = \dfrac{1}{m}f\left(\dfrac{1}{m}\right)\right]$

Example 3.2: Find the inverse Laplace transform of

$$(i)\quad \frac{s^2+s+2}{s^{\frac{3}{2}}}$$

$$(ii)\quad \frac{2s-5}{9s^2-25}$$

$$(iii)\quad \frac{s-2}{6s^2+20}$$

Solution:

$$(i)\quad \mathcal{L}^{-1}\frac{s^2+s+2}{s^{\frac{3}{2}}}=\mathcal{L}^{-1}\,s^{\frac{1}{2}}+\mathcal{L}^{-1}s^{-\frac{1}{2}}+\mathcal{L}^{-1}\frac{2}{s^{\frac{3}{2}}}$$

$$\Rightarrow \mathcal{L}^{-1}\frac{1}{s^{-\frac{1}{2}}}+\mathcal{L}^{-1}\frac{1}{s^{\frac{1}{2}}}+\mathcal{L}^{-1}\frac{2}{s^{\frac{3}{2}}}$$

$$\Rightarrow \frac{t^{-\frac{1}{2}-1}}{\Gamma\left(-\frac{1}{2}\right)}+\frac{t^{\frac{1}{2}-1}}{\Gamma\left(\frac{1}{2}\right)}+\frac{2\,t^{\frac{3}{2}-1}}{\Gamma\left(\frac{3}{2}\right)}$$

$$\Rightarrow \frac{1}{\Gamma\left(-\frac{1}{2}\right)t^{\frac{3}{2}}}+\frac{1}{\sqrt{\pi}\,t}=\frac{4\sqrt{t}}{\sqrt{\pi}}$$

$$(ii)\quad \mathcal{L}^{-1}\left[\frac{2s-5}{9s^2-25}\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{2s}{9s^2-25}-\frac{5}{9s^2-25}\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{2s}{9\left[s^2-\left(\frac{5}{3}\right)^2\right]}-\frac{5}{9\left[s^2-\left(\frac{5}{3}\right)^2\right]}\right]$$

$$\Rightarrow \frac{2}{9}\cosh\frac{5}{3}\,t-\frac{1}{3}\ \mathcal{L}^{-1}\left[\frac{\frac{5}{3}}{s^2-\left(\frac{5}{3}\right)^2}\right]$$

$$\Rightarrow \frac{2}{9}\cosh\frac{5t}{3}-\frac{1}{3}\sin\frac{5t}{3}$$

$$(iii)\ \mathcal{L}^{-1}\left[\frac{s-2}{6s^2+20}\right]=\mathcal{L}^{-1}\left[\frac{s}{6s^2+20}\right]-\mathcal{L}^{-1}\left[\frac{2}{6s^2+20}\right]$$

$$=\frac{1}{6}\mathcal{L}^{-1}\left[\frac{s}{s^2+\left(\sqrt{\frac{10}{3}}\right)^2}\right]-\frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s^2+\left(\sqrt{\frac{10}{3}}\right)^2}\right]$$

$$\Rightarrow\frac{1}{6}\cos\sqrt{\frac{10}{3}}\ t-\frac{1}{3}\times\sqrt{\frac{3}{10}}\mathcal{L}^{-1}\left[\frac{\sqrt{\frac{10}{3}}}{s^2+\left(\sqrt{\frac{10}{3}}\right)^2}\right]$$

$$\Rightarrow\frac{1}{6}\cos\sqrt{\frac{10}{3}}\ t-\frac{1}{\sqrt{30}}\sin\sqrt{\frac{10}{3}}\ t$$

3.2 Multiplication by s

$$\mathcal{L}^{-1}[sF(s)]=\frac{d}{dt}f(t)+f(0)\delta(t)$$

Example 3.3: Find the inverse Laplace transform of

$$(i)\ \frac{s}{s^2+1}\qquad(ii)\ \frac{s}{4s^2-25}\qquad(iii)\ \frac{3s}{2s+9}$$

Solution:

(i) $$\mathcal{L}^{-1}\frac{1}{s^2+1}=\sin t$$

$$\mathcal{L}^{-1}\frac{1}{s^2+1}=\frac{d}{dt}(\sin t)+\sin(0)\,\delta(t)$$

$$=\cos t$$

(ii) $$\mathcal{L}^{-1}\frac{1}{4s^2-25}=\frac{1}{4}\ \mathcal{L}^{-1}\frac{1}{s^2-\frac{25}{4}}=\frac{1}{4}\cdot\frac{2}{5}\ \mathcal{L}^{-1}\frac{\frac{5}{2}}{s^2-\left(\frac{5}{2}\right)^2}=\frac{1}{10}\sinh\frac{5}{2}\ t$$

(iii) $$\mathcal{L}^{-1}\frac{3}{2s+9}=\frac{3}{2}\ \mathcal{L}^{-1}\frac{1}{s+\frac{9}{2}}=\frac{3}{2}e^{-\frac{9}{2}t}$$

$$\mathcal{L}^{-1}\frac{3s}{2s\ +\ 9} = \frac{3}{2}\times\frac{d}{dt}\left(e^{-\frac{9}{2}t}\right)+\frac{3}{2}\ e^{-\frac{9}{2}(0)} = \frac{3}{2}\left(-\frac{9}{2}\right)\ e^{-\frac{11}{2}\,t}+\frac{3}{2}$$

$$= -e^{-\frac{11}{2}\,t}+\frac{3}{2}$$

3.3 Division by s (multiplication by $\frac{1}{s}$)

$$\mathcal{L}^{-1}\left[\frac{F\ (s)}{s}\right] = \int_0^t \mathcal{L}^{-1}[F\ (s)]\ dt = \int_0^t f(t)\ dt$$

Example 3.4: Find the inverse Laplace transform of

$(i)\ \frac{1}{s\ (s+\ m)}$

$(ii)\ \frac{1}{s\ (s^2\ +\ 1)}$

$(iii)\ \frac{s^2\ +\ 3}{s\ (s^2\ +\ 9)}$

Solution:

$(i)\ \mathcal{L}^{-1}\left(\frac{1}{s\ +\ m}\right) = e^{-mt}$

$$\mathcal{L}^{-1}\left[\frac{1}{s\ (s\ +\ \ m)}\right] = \int_0^t \mathcal{L}^{-1}\left(\frac{1}{s\ +\ \ m}\right)\ dt = \int_0^t e^{-mt}\ dt = \left[\frac{e^{-mt}}{-m}\right]_o^t$$

$$= \frac{e^{-mt}}{-m}+\frac{1}{m} = \frac{1}{m}\ \ [1-e^{-mt}]$$

$(ii)\ \mathcal{L}^{-1}\ \frac{1}{s^2\ +\ 1} = \sin t$

$$\mathcal{L}^{-1}\frac{1}{s}\ \left(\frac{1}{s^2\ \ +\ \ 1}\right) = \int_0^t \mathcal{L}^{-1}\ \left(\frac{1}{s^2\ \ +\ \ 1}\right)\ dt = \int_0^t \sin t\ \ dt = [-\cos t]_o^t\ = -\cos t+1$$

$$(iii)\quad \mathcal{L}^{-1}\frac{s^2\ \ +\ \ 3}{s(s^2\ \ +\ \ 9)} = \mathcal{L}^{-1}\left[\frac{s^2\ \ +\ \ 9\ \ -\ \ 6}{s(s^2\ \ +\ \ 9)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s}-\frac{6}{s(s^2\ \ +\ \ 9)}\right]$$

$$\Rightarrow 1-\int_0^t \mathcal{L}^{-1}\frac{3}{s^2\ \ +\ \ 9}\ ds = 1-2\int_0^t \sin 3t\, dt = 1+2\ \text{x}\ \frac{1}{3}\ \ [\cos 3t]_0^t$$

$$= 1+\frac{2}{3}\cos 3t-\frac{2}{3}\ = \frac{2}{3}\ [\cos 3t-1]$$

3.4 First Shifting Property

$$\mathcal{L}^{-1}F\,(s) = \ f(t),\ \text{ then }\ \mathcal{L}^{-1}F\,(s+m) = e^{-mt}\,\mathcal{L}^{-1}[F\,(s)]$$

Example 3.5: Find the inverse transform of

$(i)\ \frac{1}{(s\,+\,2)^5}$

$(ii)\ \dfrac{s}{s^2\,+\,4s\,+\,13}$

$(iii)\ \dfrac{3}{9s^2\,+\,6s\,+\,1}$

Solution:

$(i)\ \ \mathcal{L}^{-1}\,\frac{1}{s^5} = \frac{t^4}{4}$

Then

$\mathcal{L}^{-1}\frac{1}{(s\,+\,2)^5} = e^{-2t}\,\frac{t^4}{4}$

$$(ii)\qquad \mathcal{L}^{-1}\left(\frac{s}{s^2\,+\,4s\,+\,13}\right) = \mathcal{L}^{-1}\frac{s\,+\,2}{(s\,+\,2)^2\,+\,(3)^2}$$

$$= \mathcal{L}^{-1}\,\frac{s\,+\,2}{(s\,+\,2)^2\,+\,(3)^2} - \mathcal{L}^{-1}\frac{s\,+\,2}{(s\,+\,2)^2\,+\,3^2}$$

$$\Rightarrow e^{-2t}\mathcal{L}^{-1}\frac{s}{s^2\,+\,3^2} - e^{-2t}\mathcal{L}^{-1}\,\frac{2}{3}\left(\frac{s}{s^2\,+\,3^2}\right)$$

$$\Rightarrow e^{-2t}\cos 3t - \frac{2}{3}\,e^{-2t}\sin 3t$$

$$(iii)\qquad \mathcal{L}^{-1}\frac{s}{9s^2\,+\,6s\,\,+\,1} = \mathcal{L}^{-1}\frac{1}{(3s\,\,+\,1\,)^2} = \frac{1}{9}\,\mathcal{L}^{-1}\frac{1}{\left(s\,+\,\frac{1}{3}\right)^2}$$

$$= \frac{1}{9}\,e^{-\frac{t}{3}}\,\mathcal{L}^{-1}\,\frac{1}{s^2} = \frac{1}{9}\,e^{-\frac{t}{3}} = \frac{t}{9}e^{-\frac{t}{3}}$$

3.5 Second Shifting Property

$$\mathcal{L}^{-1}\,\,[e^{-ms}\,F\,(s)] = f\,(t-m)\,u\,(t-m)$$

Example 3.6: Obtain inverse Laplace transform of

$(i)\ \dfrac{e^{-\pi s}}{(s\ +\ 3)}$

$(ii)\ \dfrac{e^{-s}}{(s\ +\ 1)^3}$

Solution:

$(i)\ \ \mathcal{L}^{-1}\ \frac{1}{s+3} = e^{-3t}$

$$\mathcal{L}^{-1}\ \frac{e^{-\pi s}}{s+3} = e^{-3(t-\pi)}\,u(t-\pi)$$

$(ii)\ \ \mathcal{L}^{-1}\ \dfrac{1}{s^3} = \dfrac{t^2}{2}$

$$\mathcal{L}^{-1}\ \frac{1}{(s+1)^3} = e^{-t}\ \frac{t^2}{2}$$

$$\mathcal{L}^{-1}\ \frac{e^{-s}}{(s+1)^3} = e^{-(t-1)}.\frac{(t-1)^2}{2}.u\ (t-1)$$

Example 3.7: Find the inverse Laplace transform of

$$\frac{se^{-\frac{s}{2}}\ +\ \pi e^{-s}}{s^2\ +\ \pi^2}$$

in terms of unit step functions.

Solution:

$$\mathcal{L}^{-1}\quad \frac{\pi}{s^2\ +\ \pi^2}\ \sin\pi(t-1)\,.\,u\ (t-1)$$

$$\mathcal{L}^{-1}\left[e^{-s}\quad \frac{\pi}{s^2+\pi^2}\right] =\ \sin(\pi\ t)\,.\,u\ (t-1) \qquad *$$

and

$$\mathcal{L}^{-1}\ \frac{\pi}{s^2\ +\ \pi^2} = \sin\pi t$$

$$\mathcal{L}^{-1}\left[e^{-s/2}\quad \frac{s}{s^2\ +\ \pi^2}\right] = \cos\pi\ \left(t-\frac{1}{2}\right).u\ \left(t-\frac{1}{2}\right)$$

$$= \sin\ \pi\ \ t\,.\,u\left(t-\tfrac{1}{2}\right) \qquad **$$

On adding (*) and (**), we get

$$= \sin \pi\ t\,.\,u\left(t-\tfrac{1}{2}\right) -= \sin(\pi\ t)\,.\,u\,(t-1)$$

$$= \sin \pi\, t\left[u\left(t-\tfrac{1}{2}\right) - u(t-1)\right]$$

3.6 Inverse Laplace Transform of Periodic Functions

Since the transforms are obtained from integration over one cycle, instead of from zero to infinity as required by the definition of Laplace transform, a simple inverse transform table is not obtainable as in the transform of non-periodic functions.

Example 3.8: Determine $f(t)$ for $\mathcal{L}f(t) = F(s) = \frac{2e^{-4s}}{s(1-e^{-4s})}$

Solution:

$$F(s) = \tfrac{2}{s}e^{-4s}(1-e^{-4s})^{1} = \tfrac{2}{s}e^{-4s}(1+e^{-4s}+e^{-8s}+e^{-12s}+\cdots)$$

(By binomial expansion)

$$= \left(\frac{2}{s}\right)e^{-4s} + \left(\frac{2}{s}\right)e^{-8s} + \left(\frac{2}{s}\right)e^{-12s} + \left(\frac{2}{s}\right)e^{-16s}$$

$\Longrightarrow$ $$f(t) = \mathcal{L}^{-1}F(s) = 2u(t-4) + 2u(t-8) + 2u(t-12) + \cdots$$

By second-shift (time shift) theorem.

3.7 Inverse Laplace Transforms of Derivatives

$$\mathcal{L}^{-1}\left[\frac{d}{ds}F(s)\right] = t\,\mathcal{L}^{-1}\left[\,F\,(s)\right] = -t\,f(t)$$

$$or\quad \mathcal{L}^{-1}[\,F(s)\,] = -\,\mathcal{L}^{-1}\left[\frac{d}{ds}F(s)\right]$$

Example 3.9: Find inverse Laplace transform of $tan^{-1}\frac{1}{s}$.

Solution:

$$\mathcal{L}^{-1}\left(tan^{-1}\tfrac{1}{s}\right) = -\tfrac{1}{t}\,\mathcal{L}^{-1}\left[\tfrac{d}{ds}tan^{-1}\tfrac{1}{s}\right]$$

$$= -\frac{1}{t}\,\mathcal{L}^{-1}\left[\frac{1}{1\ +\ \frac{1}{s^2}}\left(-\frac{1}{s^2}\right)\right] = -\frac{1}{t}\,\mathcal{L}^{-1}\left[\frac{1}{1\ +\ s^2}\right] = \frac{\sin t}{t}$$

Example 3.10: Obtain the inverse Laplace transform of $\log\frac{s^2-1}{s^2}$.

Solution:

$$\mathcal{L}^{-1}\left[\log\frac{s^2-1}{s^2}\right] = -\frac{1}{t}\,\mathcal{L}^{-1}\left[\frac{d}{ds}\,\log\frac{s^2-1}{s^2}\right]$$

$$= -\frac{1}{t}\,\mathcal{L}^{-1}\left[\frac{d}{ds}\,\{\log(s^2-2) - 2\log s\}\right] = -\frac{1}{t}\,\mathcal{L}^{-1}\left[\frac{2s^2}{s^2} - \frac{2}{s}\right] = -\frac{1}{t}[2\cosh t - 2]$$

$$= \frac{2}{t}\,[1-\cosh t]$$

Example 3.11: Find $\mathcal{L}^{-1}[\,cot^{-1}(1+s)]$.

Solution:

$$\mathcal{L}^{-1}\,[\,cot^{-1}(1+s)] = -\frac{1}{t}\,\mathcal{L}^{-1}\,\left[\frac{d}{ds}cot^{-1}(1+s)\right]$$

$$= -\frac{1}{t}\,\mathcal{L}^{-1}\left[\frac{-1}{1+(s+1)^2}\right] = -\frac{1}{t}\,\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= \frac{1}{t}\,e^{-t}\sin t$$

3.8 Inverse Laplace Transform of Integrals

$$\mathcal{L}^{-1}\left[\int_t^\infty F(s)ds\right] = \frac{f(x)}{t} = \frac{1}{t}\mathcal{L}^{-1}[F(s)] \quad or \quad \mathcal{L}^{-1}[F(s)] = t\mathcal{L}^{-1}\left[\int_t^\infty F(s)ds\right]$$

Example 3.12. Obtain $\mathcal{L}^{-1}\frac{2s}{(s^2+1)^2}$.

Solution:

$$\mathcal{L}^{-1}\frac{2s}{(s^2+1)^2} = t\mathcal{L}^{-1}\int_s^\infty \frac{2s\;\;ds}{(s^2+1)^2} = t\mathcal{L}^{-1}\left[-\frac{1}{s^2+1}\right]_s^\infty$$

$$= t\mathcal{L}^{-1}\left[-0+\frac{1}{s^2+1}\right]$$

$$= t\sin t$$

Example 3.13. Obtain $\mathcal{L}^{-1}\dfrac{1}{s(s^2 - m^2)}$.

Solution:

$$\mathcal{L}^{-1}\frac{1}{s} = 1 \quad and \quad \mathcal{L}^{-1}\frac{1}{s^2 - m^2} = \frac{\sin mt}{m}$$

Hence by the convolution theorem

$$\mathcal{L}\int_0^t \left\{1 . \frac{\sin m\,(t - x)}{m}\,dx\right\} = \left(\frac{1}{s}\right)\left(\frac{1}{s^2 + m^2}\right)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s\ (s^2 + m^2)}\right\} = \int_0^t \frac{\sin m\,(t - x)}{m}\,dx = \left[\frac{-\cos(mt - mx)}{-m^2}\right]_0^t$$

$$= \frac{1}{m^2}[1 - \cos mt]$$

3.9 Exercise

1. Find the inverse Laplace transform of the following:

(a). $\frac{3s - 8}{4s^2 + 25}$ **Ans.** $\frac{3}{4}\cos\frac{5t}{2} - \frac{4}{5}\sin\frac{5t}{2}$

(b). $\frac{3\,(s^2 - 2)^2}{2s^2}$ **Ans.** $\frac{3}{2} - 3t^2 + \frac{1}{2}\,t^4$

(c). $\frac{3s - 8}{4s^2 + 25} + \frac{4s - 18}{9 - s^2}$ **Ans.** $\frac{1}{2}\,\cos\frac{5t}{2} - \frac{1}{2}\sin\frac{5t}{2} - 4\cosh 3t + 6\sinh 3t$

(d). $\frac{5s - 10}{9s^2 - 16}$ **Ans.** $\frac{5}{9}\,\cosh\frac{4}{3}\,t - \frac{5}{6}\sinh\frac{4}{3}\,t$

(e). $\frac{1}{4s} + \frac{16}{1 - s^2}$ **Ans.** $\frac{1}{4}\, - 16\sinh t$

2. Find the inverse Laplace transform of the following:

(i). $\frac{s}{s + 5}$ **Ans.** $1 - 5\,e^{-5t}$

(ii). $\frac{2s}{3s + 6}$ **Ans.** $\frac{2}{3}\, - \frac{4}{3}e^{-2t}$

(iii). $\frac{s}{2s^2 - 1}$ **Ans.** $\frac{1}{2}\,\cosh\frac{t}{2}$

(iv). $\frac{s^2}{s^2 + a^2}$ **Ans.** $1 - \, a\sin at$

(v). $\frac{s^2}{s^2 + a^2}$ **Ans.** $1 - \,\frac{5}{3}\sin 3t$

(vi). $\frac{1}{(s-3)^2}$ **Ans.** te^{3t}

(vii). $\mathcal{L}^{-1}\frac{s^2}{(s^2+4)^2}$

3. Solve the following partial fraction

(i). $\frac{1}{2s(s-3)}$ **Ans.** $\frac{1}{2}\left[\frac{e^{3t}}{3}-1\right]$

(ii). $\frac{1}{s(s+2)}$ **Ans.** $\frac{1-e^{-2t}}{3}$

(iii). $\frac{1}{(s^2-16)}$ **Ans.** $\frac{1}{16}\ [\cosh 4t-1]$

(iv). $\frac{1}{s(s^2-a^2)}$ **Ans.** $\frac{1-cos\,at}{a^2}$

(v). $\frac{s^2+2}{s(s^2+4)}$ **Ans.** $cos^2\,t$

(vi). $\frac{1}{s^2(s+1)}$ **Ans.** $t-e^{-t}$

(vii). $\frac{1}{s^3(s^2+1)}$ **Ans.** $\frac{t^2}{2}+\cos t-1$

(viii). $\mathcal{L}^{-1}\frac{s^2}{s\,(s^2-1)}$ is

$(a)\ 1-\cos t$ $(b)\ 1+\cos t$ $(c)\ 1-\sin t$ $(d)\ 1+\sin t$ **Ans.** (a)

4. Obtain the inverse Laplace transform of the following:

a. $\frac{s+8}{s^3+4s+5}$ **Ans.** $e^{-2t}\ (\cos t+\ 6\sin t)$

b. $\frac{s+8}{s^3+4s+5}$ **Ans.** $e^{-3t}\ (\cos 2t-\ 1.5\sin 2t)$

c. $\frac{s}{(s+7)^2}$ **Ans.** $e^{-7t}\frac{t^2}{6}\ (3-7t)$

d. $\frac{s+2}{s^2-2s-8}$ **Ans.** $e^t(cosh\ 3t+sinh\ 3t)$

e. $\frac{s}{s^2+6s+25}$ **Ans.** $e^{-3t}\ \left[\cos 4t\frac{3}{4}\sin 4t\right]$

f. $\frac{1}{2\,(s-1)^2+32}$ **Ans.** $\frac{e^t}{8}\sin 4t$

g. $\frac{s-4}{4\,(s-3)^2+16}$ **Ans.** $\frac{1}{4}e^{3t}\cos 2t-\frac{1}{8}\ e^{3t}\sin 2t$

5. Obtain inverse Laplace transform of the following:

(a) $\dfrac{s\,e^{-s/2} + \pi\,e^{-s}}{s^2 + \pi^2}$ **Ans.** $\left[u\left(t-\frac{1}{2}\right) - u(t-1)\right] \sin \pi t$

(b) $\dfrac{e^{-s}}{(s+2)^3}$ **Ans.** $e^{-(t-2)}\dfrac{(t-2)^2}{2}u(t-2)$

(c) $\dfrac{e^{-2s}}{(s+1)(s^2+2s+2)}$ **Ans.** $e^{-(t-2)}\{1-\cos((t-2)\}u(t-2)$

(d) $\dfrac{e^{-s}}{\sqrt{s+1}}$ **Ans.** $\dfrac{e^{-(t-1)}}{\sqrt{\pi(t-1)}}u(t-1)$

(e) $\dfrac{e^{-\frac{\pi}{2}s} + e^{-\frac{3\pi}{2}s}}{s^2+1}$ **Ans.** $\cot t\left[u\left(t-\frac{3\pi}{2}\right) - u\left(t-\frac{\pi}{2}\right)\right]$

(f) $\dfrac{e^{-4s}(s+2)}{s^2+4s+5}$ **Ans.** $e^{-2(t-u)}\cos(t-u)\,u(t-4)$

(g) $\dfrac{e^{-as}}{s^2}$ **Ans.** $f(t) = \begin{cases} t-a & when\ \ t \geq a \\ 0 & when\ \ t \geq a \end{cases}$

(h) $\dfrac{e^{-\pi s}}{s^2+1}$ **Ans.** $-\sin t\,u\,(t-\pi)$

(i) The inverse Laplace transform of $(e^{-3s})/s^3$, is

$(A)\ \ (t-3)u_3\,(t)$ $\quad (B)\ \ (t-3)^2\,u_3\,(t)$ $\quad (C)\ \ (t-3)^2u_3\,(t)$

$(D)\ \ (t-3)u_3\,(t).$ **Ans.** (D)

(j) If Laplace transform of a function $f\ (t)$ equals $(e^{-2s} - e^{-s})/s$, then

$(A)\ f(t) = 1,\ \ t \geq 1;$

$(B)\ f(t) = 1,\ \ when\ \ 1 \leq t \leq 2,\ \ \ and\ 0\ otherwise\ ;$

$(C)\ f(t) = -1,\ \ when\ \ 1 \leq t \leq 3,\ \ \ and\ 0\ otherwise\ ;$

$(D)\ f(t) = -1,\ \ when\ \ 1 \leq t \leq 2,\ \ \ and\ 0\ otherwise\ .$

Ans. (D)

(k) The Laplace inverse $\mathcal{L}^{-1}\left[\frac{2}{s}\,(e^{-2s} + e^{-4s}\right]$ equal

$(A)\ 2,\ \ if\ 0 \leq t \leq 4\ ;\ \ 0\ \ \ otherwise$

$(B)\ \ 2,\ \ if\ t \geq 0$

$(C)\ \ 2,\ \ if\ 0 \leq t \leq 2\ ;\ \ \ 0\ \ \ otherwise\ ;$

$(D)\ \ 2,\ \ if\ 0 \leq t \leq 4\ ;\ \ \ \ 0\ \ otherwise$

Ans. (D)

(l) The Laplace transform of $t\,u_2\,(t)$

$(A)\ \ \left(\frac{1}{s^2}+\frac{2}{2}\right)e^{-2s}$ $\quad (B)\ \ \frac{1}{s^2}e^{-2s}$ $\quad (C)\ \ \left(\frac{1}{s^2}-\frac{2}{s}\right)e^{-2s}$ $\quad (D)\ \ \frac{1}{s^2}e^{-2s}.$

Ans. (A)

(m) The inverse Laplace transform of $\frac{ke^{-as}}{s^2 + k^2}$ is

(A) $\sin kt$ (B) $\cos kt$ (C) $u\,(t-a)\sin kt$ (D) $none\ of\ these$ **Ans.** (D)

(n) The inverse Laplace transform of 1 is:

$(A)\ 1$ $(B)\ \delta\,(t)$ $(C)\ \delta(t-1)$ $(D)\ u(t)$ **Ans.** (B)

6. Obtain inverse Laplace transform of the following:

a. $\log\left(1+\frac{\omega^2}{s^2}\right)$ **Ans.** $\frac{2}{t}-\frac{2}{t}\cos\omega\, t$

b. $\frac{s}{1+s^2+s^4}$ **Ans.** $\frac{2}{\sqrt{3}}\sin\frac{3}{\sqrt{2}}\, t\sinh\frac{t}{2}$

c. $\frac{s+1}{(s^2+6s+13)^2}$ **Ans.** $\frac{e^{-3t}}{8}\,[2t\sin 2t + 2t\cos 2t - \sin 2t]$

d. $\frac{s}{(s^2+a^2)^2}$ **Ans.** $\frac{t\sin at}{2a}$

e. $\frac{1}{2}\log\left\{\frac{s^2+b^2}{(s-a)^2}\right\}$ **Ans.** $\frac{e^{-at}-\cos bt}{t}$

f. $tan^{-1}\,(s+1)$ **Ans.** $-\frac{1}{t}e^{-t}\sin t$

g. $\log\left(1+\frac{1}{s^2}\right)$ **Ans.** $\frac{2}{t}\,[1-\cos\omega\, t]$

h. $s\log\frac{s}{\sqrt{s^2+1}} + cot^{-1}\, s$ **Ans.** $\frac{1-\cos t}{t^2}$

7. Find the inverse transform of:

i. $\frac{s^2+2s+6}{s^2}$ **Ans.** $1+2t+3t^2$

ii. $\frac{1}{s^2-7s+12}$ **Ans.** $e^{4t}-e^{3t}$

iii. $\frac{s-2}{s^2-4s+13}$ **Ans.** $e^{3t}\cos 3t + \frac{4}{3}\,e^{2t}\sin 3t$

iv. $\frac{3s+1}{(s-1)(s^2+1)}$ **Ans.** $e^t - 2\cos t + \sin t$

v. $\frac{11s^2-2s+5}{2s^3-3s^2-3s+2}$ **Ans.** $2e^{-t}+5e^{2t}-\frac{3}{2}\,e^{t/2}$

vi. $\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$ **Ans.** $\frac{1}{2}\,e^t - e^{2t} + \frac{5}{2}\,e^{3t}$

vii. $\frac{3s+1}{(s-4)^2+9}$ **Ans.** $e^{4t}\cos 3t$

viii. $\frac{16}{(s^2 + 2s + 5)^2}$ **Ans.** $e^{-t}(\sin 2t - 2t\cos 2t)$

CHAPTER 4

SOLUTION OF DIFFERENTIAL EQUATIONS BY LAPLACE TRANSFORMS

4.0 Solving Differentials Equation Using Laplace Method

Ordinary linear differential equations with constant coefficients can be easily solved by the Laplace Transform method, without finding the general solution and the arbitrary constants.

The method will be clear from the following examples:

Example 4.1: Using Laplace transforms find the solution of the initial value problem

$$y'' - 4y' + 4y = 64\sin 2t$$

$$y(0) = 0, y'(0) = 1.$$

Solution:

$$y'' - 4y' + 4y = 64\sin 2t \qquad 4.1$$

$$y(0) = 0, y'(0) = 1$$

Taking Laplace transform of both sides of (4.1), we have

$$\left[s^2\,\bar{y} - sy\,(0) - y^{'}(0)\right] - 4[s^2\,\bar{y} - y\,(0)] + 4\bar{y} = \frac{64 \times 2}{s^2 + 4} \qquad 4.2$$

On putting the values of y (0) and $y'(0)$ into Eq. (4.2), we get

$$s^2\bar{y} - 1 - 4s\bar{y} + 4\,\bar{y} = \frac{128}{s^2 + 4}$$

$$(s^2 - 4s + 4)\,\bar{y} = 1 + \frac{128}{s^2 + 4}, \quad or \quad (s-2)^2\,\bar{y} = 1 + \frac{128}{s^2 + 4}$$

$$\bar{y} = \frac{1}{(s-2)^2} + \frac{128}{(s-2)^2\,(s^2 + 4)} = \frac{1}{(s-2)^2} - \frac{8}{s-2} + \frac{16}{(s-2)^2} + \frac{8s}{s^2 + 4}$$

$$y = \mathcal{L}^{-1}\left[-\frac{8}{s-2} + \frac{17}{(s-2)^2} + \frac{8s}{s^2 + 4}\right]$$

$$y = -8\,e^{2t} + 17e^{2t} + 8\cos 2t$$

Example 4.2: Applying convolution, solve the following initial value problem

$$y'' + y = \sin 3\ t$$

$$y\ (0) = 0,\ \ y'(0) = 0.$$

Solution:

$$y'' + y = \sin 3\ t$$

Taking Laplace transform of both the sides, we have

$$[s^2\ \bar{y} - sy\ (0) - y'(0)] + \bar{y} = \frac{3}{s^2\ +\ 9} \qquad 4.3$$

On putting the values of $y\ (0), y'(0)$ into Eq. (4.3) we get

$$s^2\ \bar{y} +\ \bar{y} = \frac{3}{s^2\ +\ 9} \quad or \quad (s^2 + 1)\ \bar{y} = \frac{3}{s^2\ +\ 9}$$

$$\bar{y} = \frac{3}{(s^2\ +\ 1)\ (s^2\ +\ 9)} = \frac{3}{8}\left[\frac{1}{s^2\ +\ 1} - \frac{1}{s^2 + 9}\right]$$

Taking the inversion, we get

$$y = \frac{3}{8}\ \mathcal{L}^{-1}\frac{1}{s^2\ +\ 1} - \frac{3}{8}\ \mathcal{L}^{-1}\frac{1}{s^2\ +\ 9}$$

$$y = \frac{3}{8}\sin t - \frac{3}{8} \times \frac{1}{3}\ \sin 3\ t$$

$$y = \frac{3}{8}\sin t - \frac{1}{8}\sin 3\ t$$

Example 4.3: Solve $[tD^2 + (1 - 2t)D - 2]\ y = 0.$ $\quad where\ y(0) = 1,$

$$y'(0) = 2$$

Solution:

Here, $tD^2 + (1 - 2t)Dy - 2y = 0$

$$\Longrightarrow \quad ty'' + y' - 2ty' - 2\ y = 0$$

Taking Laplace transform of given differential equation, we get

$\mathcal{L}(ty'') + \mathcal{L}(y') - 2\mathcal{L}(ty') - 2\mathcal{L}(y) = 0$

$$\Rightarrow\ -\frac{d}{ds}\mathcal{L}\ \{y''\} + \{y'\} + 2\frac{d}{ds}\mathcal{L}(y') - 2\mathcal{L}(y) = 0$$

$$-\frac{d}{ds}\ [s^2\,\bar{y} - s\,y\,(0) - y'(0)] + \ [s\,\bar{y} - y\,(0)] + \ 2\frac{d}{ds}[s\,\bar{y} - y\,(0)] - 2\,\bar{y} = 0$$

Putting the values of $y\,(0)$ and $y^{'}(0)$, we get

$$-\frac{d}{ds}\ (s^2\,\bar{y} - s - 2) + (s^2\,\bar{y} - 1) + 2\frac{d}{ds}\ (s^2\,\bar{y} - 1) = 2\ \ \bar{y} = 0$$

$$[\because\ y\,(0) = 1, y'(0) = 2]$$

$$\Rightarrow -s^2\frac{d}{ds}\bar{y} - 2\,\bar{y} + 1 + \ \bar{y} - 1 + 2\left(s\ \ \frac{d\,\bar{y}}{d\,s} + \bar{y}\right) - 2\,\bar{y} = 0$$

$$\Rightarrow\ \ -(s^2 - 2\,s)\ \frac{d\,\bar{y}}{d\,s} - s\,\bar{y} = 0$$

$$\Rightarrow -\frac{d\,\bar{y}}{\bar{y}} + \frac{1}{s\ -\ 2}ds = 0$$

$$\Rightarrow \int\frac{d\,\bar{y}}{\bar{y}} + \int\frac{1}{s\ -\ 2} = 0\quad \Rightarrow \log\bar{y} + \log(s-2) = \log C$$

$$\Rightarrow\ \bar{y}\,(s-2) = C\ \Rightarrow\ \ \bar{y} = \frac{C}{s-2} \Rightarrow\ y = C\,\mathcal{L}^{-1}\left\{\frac{1}{s-\ 2}\right\} \Rightarrow y = C\,e^{2\,t}\quad 4.4$$

Putting $y\,(0) = 1$ into Eq. (4.4), we get $\quad 1 = C\,e^{0} \quad \Rightarrow \quad C = 1$

Putting $C = 1$ into Eq. (4.4), we get $y = \ e^{2\,t}$

This is the required solution.

Example 4.4. Using Laplace transform technique solve the following initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 5\sin t\,, \qquad\qquad where\ y\,(0) = y'(0) = 0 \qquad 4.5$$

Solution:

$$y'' + 2y' + 2y = 5\sin t$$

$$y\,(0) = y'(0) = 0$$

Take the Laplace Transform of both sides, we have

$$[s^2\bar{y} - sy\,(0) - y'(0)] + 2[s\bar{y} - y\,(0)] + 2\bar{y} = 5\times\frac{1}{s^2\ +\ 1} \qquad 4.6$$

On substituting the value $y\,(0),\ and\ y^{'}(0)$ into Eq. (4.5), we get

$$s^2\bar{y} + 2\,s\bar{y} + \ 2\bar{y} = \frac{5}{s^2\ +\ 1}\quad or\quad [s^2 + 2\,s + 2]\bar{y} = \frac{5}{s^2\ +\ 1}$$

$$\bar{y} = \frac{5}{(s^2 + 2s + 2)(s^2 + 1)}$$

Resolving into partial fractions, $y = \dfrac{2s + 3}{s^2 + 2s + 2} + \dfrac{-2s + 1}{s^2 + 1}$

Taking the inverse transform, we get

$$y = \mathcal{L}^{-1}\left(\frac{2s + 3}{s^2 + 2s + 2}\right) + \mathcal{L}^{-1}\left(\frac{-2s + 1}{s^2 + 1}\right)$$

$$= \mathcal{L}^{-1}\left[\frac{2(s + 1)}{(s + 1)^2 + 1}\right] + \mathcal{L}^{-1}\left(\frac{-2s + 1}{s^2 + 1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2 + 1}\right)$$

$$= \mathcal{L}^{-1}\left[\frac{2(s + 1)}{(s + 1)^2 + 1}\right] + \mathcal{L}^{-1}\left[\frac{1}{(s + 1)^2 + 1}\right] - 2\cos t + \sin t$$

$$= 2\,e^{-t}\cos t + e^{-t}\sin t - 2\cos t + \sin t$$

Example 4.5. Solve the initial value problem

$$2y'' + 5y' + 2\,y = e^{-2\,t}, \qquad y\,(0) = 1,\; y'(0) = 1$$

using the Laplace transforms.

Solution:

$$2\,y'' + 5y' + 2\,y = e^{-2\,t}, \quad y\,(0) = 1, \qquad y'(0) = 1$$

Taking the Laplace Transform of both the sides, we get

$$2[s^2\bar{y} - sy\,(0) - y'(0)] + 5[s\bar{y} - y\,(0)] + 2\bar{y} = \frac{1}{s + 2}$$

On putting the values of $y\,(0)$ *and* $y'(0)$ in (1), we get

$$2[s^2\bar{y} - s - 1] + 5[s\bar{y} - 1] + 2\bar{y} = \frac{1}{s + 2}$$

$$[2s^2 + 5\,s + 2]\bar{y} - 2s - 2 - 5 = \frac{1}{s + 2}$$

$$\bar{y} = \frac{1 + (s + 2)(2s + 7)}{(2\,s^2 + 5s + 2)(s + 2)} = \frac{2s^2 + 11s + 15}{(2s + 1)\,(s + 2)^2}$$

$$= \frac{\frac{4}{9}}{2s + 1} - \frac{\frac{11}{9}}{s + 2} - \frac{\frac{1}{3}}{(s + 2)^2} = \frac{4}{9}\frac{1}{2}\frac{1}{s + \frac{1}{2}} - \frac{11}{9}\frac{1}{s + 2} - \frac{1}{3}\frac{1}{(s + 2)^2}$$

$$y = \frac{2}{9}e^{-\frac{1}{2}t} - \frac{11}{9}e^{-2t} - \frac{1}{3}te^{-2t}$$

Example 4.6: Solve $\frac{d^2 y}{d x^2} + 2\frac{dy}{dx} + 5y = e^{-x}\sin x \;\; for \;\; y(0) = 0, \quad y'(0) = 1$

Solution:

$$\frac{d^2 y}{d x^2} + 2\frac{dy}{dx} + 5y = e^{-x}\sin x$$

Taking the Laplace Transform of both the sides, we get

$$[s^2\bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] + 5\bar{y} = \mathcal{L}(e^{-x}\sin x)$$

$$[s^2\bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] + 5\bar{y} = \frac{1}{(s + 2)^2 + 1} \qquad 4.7$$

On substituting the values of $y(0)$ and $y^{'}(0)$ into Eq. (4.7), we get

$$(s^2\bar{y} - 1) + 2(s\bar{y}) + 5\bar{y} = \frac{1}{s^2 + 2s + 2}$$

$$(s^2 + 2s + 5)\,\bar{y} = 1 + \frac{1}{s^2 + 2s + 2} = \frac{s^2 + 2s + 3}{s^2 + 2s + 2}$$

$$\bar{y} = \frac{s^2 + 2s + 3}{(s^2 + 2s + 5)(s^2 + 2s + 2)}$$

On resolving the R.H.S. into partial fractions, we get

$$\bar{y} = \frac{2}{3}\frac{1}{s^2 + 2s + 5} + \frac{1}{3}\frac{1}{s^2 + 2s + 2}$$

On inversion, we obtain

$$y = \frac{2}{3}\mathcal{L}^{-1}\frac{1}{s^2 + 2s + 5} + \frac{1}{3}\mathcal{L}^{-1}\frac{1}{s^2 + 2s + 2}$$

Or

$$y = \frac{1}{3}\mathcal{L}^{-1}\frac{2}{(s + 1)^2 + (2)^2} + \frac{1}{3}\mathcal{L}^{-1}\frac{1}{(s + 1)^2 + (2)^2}$$

$$y = \frac{1}{3}\, e^{-x} \sin 2\ x + \frac{1}{3}\, e^{-x} \sin x \quad or \quad y = \frac{1}{3}\, e^{-x}\,(\sin x + \sin 2\ x)$$

Example 4.7. Using Laplace transforms, find the solution of the initial value problem:

$$y'' + 9y = 9u(t-3),\ \ y\,(0) = y'(0) = 0$$

where $u\,(t-3)$ is the unit step function.

Solution:

$$y'' + 9y = 9u\,(t-3). \tag{4.8}$$

Taking Laplace transform of Eq. (4.8), we have

$$s^2\bar{y} - sy\,(0) - y'(0) + 9\,\bar{y} = 9\,\frac{e^{-3s}}{s} \tag{4.9}$$

Putting the values of $y\,(0)$ and $y'(0) = 0$ into Eq. (4.9), we get

$$s^2\bar{y} + 9\,\bar{y} = \frac{9\,e^{-3s}}{s}$$

$$(s^2 + 9)\bar{y} = \frac{9\,e^{-3s}}{s}$$

$$\bar{y} = \frac{9\,e^{-3s}}{s\,(s^2 + 9)} \implies y\ = \mathcal{L}^{-1}\,\frac{9\,e^{-3s}}{s\,(s^2 + 9)}$$

$$\mathcal{L}^{-1}\,\frac{3}{s^2 + 9} = \sin 3t$$

$$3\,\mathcal{L}^{-1}\,\frac{3}{s\,(s^2 + 9)} \Longrightarrow 3\int_0^t \sin 3t\ \ dt = -\,[\cos 3t]_0^t = 1 - \cos 3t$$

$$y = \mathcal{L}^{-1}\,\frac{9\,e^{-3s}}{s\,(s^2 + 9)}$$

$$y = [1 - \cos 3\,(t-3)]u(t-3)$$

4.1 Solution of Simultaneous Differential Equations by Laplace Transform

Simultaneous differential equations can also be solved by Laplace Transform method.

Example 4.8: Solve $\frac{dx}{dt} + y = 0$ $\ and\ $ $\frac{dy}{dt} - x = 0$ $\ under\ the\ condition$
$x(0) = 0,\ y\,(0) = 0$

Solution:

$$x' + y = 0 \quad 4.10$$
$$y' - x = 0 \quad 4.11$$

Taking the Laplace transform of Eqs. (4.10) and (4.11) we get

$$[s\,\bar{x} - x(0)] + \bar{y} = 0 \quad 4.12$$
$$[s\,\bar{y} - y(0)] - \bar{x} = 0 \quad 4.13$$

On substituting the values of $x(0)$ and $y(0)$ into Eqs. (4.12) and (4.11) we get

$$s\,\bar{x} - 1 + \bar{y} = 0 \quad 4.14$$
$$s\,\bar{y} - \bar{x} = 0 \quad 4.15$$

Solving Eqs. (4.14) and (4.15) for $\bar{x}$ and $\bar{y}$ we get

$$\bar{x} = \frac{s}{s^2 + 1}, \qquad \bar{y} = \frac{s}{s^2 + 1}$$

On inversion, we obtain $x = \mathcal{L}^{-1}\left(\frac{s}{s^2 + 1}\right), \quad y = \mathcal{L}^{-1}\left(\frac{s}{s^2 + 1}\right)$

$$x = \cos t, \; y = \sin t$$

Example 4.9: Solve $\frac{dx}{dt} - y = e^t, \quad \frac{dy}{dt} + x = \sin t$ given: $x(0) = 1, \; y(0) = 0$

Solution:

$$x' - y = e^t \quad 4.16$$
$$y' + x = \sin t \quad 4.17$$

Taking the Laplace Transform of Eqs. (4.17) and (4.17), we get

$$[s\,\bar{x} - x(0)] - \bar{y} = \frac{1}{s - 1} \quad 4.18$$
$$[s\,\bar{y} - y(0)] + \bar{x} = \frac{1}{s^2 + 1} \quad 4.19$$

On substituting the values of $x(0)$ and $y(0)$ into Eqs. (4.18) and (4.19) we get

$$s\,\bar{x} - 1 - \bar{y} = \frac{1}{s - 1} \quad 4.20$$
$$s\,\bar{y} + x = \frac{1}{s^2 + 1} \quad 4.21$$

On solving Eqs. (4.20) and (4.21), we get

$$\bar{x} = \frac{s^4 + s^2 + s - 1}{(s - 1)(s^2 + 1)^2} = \frac{1}{2}\frac{1}{s - 1} + \frac{1}{2}\frac{s + 1}{s^2 + 1} + \frac{1}{(s^2 + 1)^2} \quad 4.22$$

$$\bar{y} = \frac{-s^3 + s^2 - 2s}{(s - 1)(s^2 + 1)^2} = -\frac{1}{2}\frac{1}{s - 1} + \frac{1}{2}\frac{s + 1}{s^2 + 1} + \frac{s}{(s^2 + 1)^2} \quad 4.23$$

On inversion Eq. (4.23), we get

$$\gamma = \frac{1}{2}\ \mathcal{L}^{-1}\frac{1}{s\ -\ 1} + \frac{1}{2}\ \mathcal{L}^{-1}\frac{1}{s^2\ +\ 1} + \frac{1}{2}\ \mathcal{L}^{-1}\frac{1}{s^2\ +\ 1}\ \mathcal{L}^{-1}\frac{s}{(s^2\ +\ 1)^2}$$

$$= \frac{1}{2}\ e^t + \frac{1}{2}\cos t + \frac{1}{2}\sin t + \frac{1}{2}\ (\sin t - t\cos t) = \frac{1}{2}\ [e^t + \cos t + 2\sin t - t\cos t]$$

On inverse we get

$$y = -\frac{1}{2}\ \mathcal{L}^{-1}\frac{1}{s\ -\ 1} + \frac{1}{2}\ \mathcal{L}^{-1}\frac{s}{s^2\ +\ 1} - \frac{1}{2}\ \mathcal{L}^{-1}\frac{1}{s^2\ +\ 1} + \mathcal{L}^{-1}\frac{s}{(s^2\ +\ 1)^2}$$

$$y = -\frac{1}{2}\ e^t + \frac{1}{2}\cos t - \frac{1}{2}\sin t + \frac{1}{2}\ t\sin t$$

$$y = \frac{1}{2}[-e^t - \sin t + \cos t +\ t\sin t]$$

Example 4.10: Using the Laplace transform solve the initial value problem

$$y_1'' = y_1 + 3y_2$$

$$y_2'' = 4y_1 - 4e^t$$

$$y_1(0) = 2, \qquad y_1'\ (0) = 3, \qquad y_2\ (0) = 1, \qquad y_2'\ (0) = 2$$

Solution:

$$y_1'' = y_1 + 3y_2 \qquad 4.24$$

$$y_2'' = 4y_1 - 4e^t \qquad 4.25$$

Taking the Laplace transform of Eqs. (4.24) and (4.25) we get

$$s^2\bar{y}_1 = sy_1(0) -\ y_1'\ (0) = \bar{y}_1 + 3\ \bar{y}_2 \qquad 4.26$$

$$s^2\bar{y}_1 = sy_2(0) -\ y_2'\ (0) = 4\ \bar{y}_1 - \frac{4}{s\ +\ 1} \qquad 4.27$$

Putting the values of $y_1'\ (0),\ \ y_2(0), y_2'\ (0)$ into Eqs. (4.25) and (4.26), we get

$$s^2\ \bar{y}_1 -\ 2\ s - 3 = \bar{y}_1\ +\ 3\ \bar{y}_2 \quad or \quad (s^2 - 1)\bar{y}_1 -\ 3\ \bar{y}_2 = 2s + 3 \qquad 4.28$$

$$s^2\ \bar{y}_2 - s - 2 = 4\ \bar{y}_1 - \frac{4}{s -\ 1} \quad or \quad 4\bar{y}_1 - s\bar{y}_2 = \frac{4}{s -\ 1} - s - 2 \qquad 4.29$$

On solving Eqs. (4.28) and (4.29), we get

$$\bar{y}_1 = \frac{(2\ s\ -\ 3)\ (s^2\ +\ 3)\ (s\ +\ 2)}{(s\ -\ 1)\ (s^2\ +\ 3)\ (s\ +\ 4)} = \frac{2\ s\ -\ 3}{(s\ -\ 1)(s\ -\ 2)} - \frac{1}{s\ -\ 1} + \frac{1}{s\ -\ 2}$$

$$y_1 = e^t + e^{2t}$$

$$\bar{y}_2 = \frac{(s\ -\ 2)\,(s^2\ +\ 3)}{(s^2\ +\ 3)\,(s^2 +\ 4)} = \frac{1}{s\ -\ 2}\,, \qquad \Longrightarrow \ y_2 = e^{2t}$$

4.2 Application to Integro Differential Equations

The Laplace transform is useful in solving linear integro differential equations. Using the differentiation and integration properties of Laplace transforms, each term in the integrodifferential equation is transformed.

Initial conditions are automatically taken into account. We solve the resulting algebraic equation in the s-domain. We then convert the solution back to the time domain by using the inverse transform. The following examples illustrate the process.

Example 4.11: Use the Laplace transform to solve the differential equation

$$\frac{d^2v(t)}{dt^2} + 6\frac{dv(t)}{dt} + 8v(t) = 2u(t)$$

Subject to $v(0) = 1, v'(0) = -2$

Solution:

We take the Laplace transform of each term in the given differential equation and obtain

$$[s^2V(s) - sv(0) - v'(0)] + 6[sV(s) - v(0)] + 8V(s) = \frac{2}{s}$$

Substituting $v(0) = 1, v(0) = -2$

$$s^2V(s) - s + 2 + 6sV(s) - 6 + 8V(s) = \frac{2}{s}$$

Or

$$(s^2 + 6s + 8)V(s) = s + 4 + \frac{2}{s} = \frac{s^2 + 4s + 2}{s}$$

Hence,

$$V(s) = \frac{s^2 + 4s + 2}{s(s+2)(s+4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4}$$

Where

$$A = sV(s)|_{s=0} = \left.\frac{s^2 + 4s + 2}{(s + 2)(s + 4)}\right|_{s=0} = \frac{2}{(2)(4)} = \frac{1}{4}$$

$$B = (s + 2)V(s)|_{s=-2} = \left.\frac{s^2 + 4s + 2}{s(s + 4)}\right|_{s=2} = \frac{-2}{(-2)(2)} = \frac{1}{2}$$

$$C = (s + 4)V(s)|_{s=-4} = \left.\frac{s^2 + 4s + 2}{s(s + 2)}\right|_{s=-4} = \frac{2}{(-4)(-2)} = \frac{1}{4}$$

Hence,

$$V(s) = \frac{\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s} + \frac{\frac{1}{4}}{s + 4}$$

By the inverse Laplace transform

$$v(t) = \frac{1}{4}(1 + 2e^{-2t} + e^{-4t})u(t)$$

Example 4.12: Solve for the response $y(t)$ in the following integro differential equation.

$$\frac{dy}{dt} + 5y(t) + 6\int_0^t y(\tau) = u(t), \qquad y(0) = 2$$

Solution

Taking the Laplace transform of each term, we get

$$[sY(s) - y(0)] + 5Y(s) + \frac{6}{s}Y(s) = \frac{1}{s}$$

Substituting $y(0) = 2$ and multiplying through by s,

$$Y(s)(s^2 + 5s + 6) = 1 + 2s$$

Or

$$Y(s) = \frac{2s + 1}{(s + 2)(s + 3)} = \frac{A}{s + 2} + \frac{B}{s + 3}$$

Where

$$A = (s + 2)Y(s)|_{s=-2} = \left.\frac{s^2 + 1}{s + 3}\right|_{s=-2} = \frac{-3}{1} = -3$$

$$B = (s+3)Y(s)|_{s=-3} = \left.\frac{2s+1}{s+2}\right|_{s=-3} = \frac{-5}{-1} = 5$$

Thus

$$Y(s) = \frac{-3}{s+2} + \frac{5}{s+3}$$

Its inverse transform is

$$y(t) = (-3e^{-2t} + 5e^{-3t})u(t)$$

4.3 Exercise

Solve the following differential equations:

1. $\frac{d^2y}{dx^2} + y = 0,\ where\ \ y = 1\ \ and\ \ \frac{dy}{dx} = -1\ \ at\ \ x = 0$

Ans. $y = \cos x - \sin x$

2. $\frac{d^2y}{dx^2} - 4y = 0,\ where\ \ y = 0\ \ and\ \ \frac{dy}{dx} = -6\ \ at\ \ x = 0.$

Ans. $y = -\frac{3}{2}\ e^{2\,x} + \frac{3}{2}\ e^{-2\,x}$

3. $\frac{d^2y}{dx^2} + y = 0,\ where\ \ y = 1,\ \frac{dy}{dx} = 1\ \ at\ \ x = 0.$

Ans. $y = \sin x + \cos x$

4. $\frac{d^2y}{dx^2} + 2\ \frac{dy}{dx} + 5\,y = 0,\ where\ \ y = 2,\ \frac{dy}{dx} = -4\ \ at\ \ x = 0$

Ans. $y = e^{\,x}(2\ \cos 2\ \ x - \sin 2\ \ x)$

5. $\frac{d^3y}{dx^3} + 2\ \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2\,y = 0,\ given\ y = \frac{dy}{dx} = 0,\ \frac{d^2y}{dx^2} = 6\ at\ \ x = 0$

Ans. $y = e^{\,x} - 3e^{-\,x} + 2e^{-\,x}$

6. $\frac{d^2y}{dx^2} + y = 3\ \cos 2\,x\,,\ where\ \ y = \frac{dy}{dx} = 0\ at\ \ x = 0.$ **Ans.** $y = \cos x - \cos 2\,x.$

7. $\frac{d^3y}{dx^3} + \frac{dy}{dx} - 2\,y = 1 - 2\,x,\ given\ y = 0, \frac{dy}{dx} = 4\ at\ \ x = 0$

Ans. $y = e^{\,x} - e^{-\,2x} + x$

8. $\frac{d^3y}{dx^3} - 3\ \frac{dy}{dx} + 2\,y = 4e^{-2x}, given\ y = -3, and\ \frac{dy}{dx} = 5\ at\ x = 0.$

Ans. $y = -7e^{\,x} + 4e^{-2x} + 4\,xe^{2\,x}$

9. $\frac{d^3y}{dx^3} - 3\ \frac{dy}{dx} + 2\,y = 4x + e^{-2x}, where\ y = 1, \frac{dy}{dx} = -1\ at\ x = 0.$

Ans. $y = 3 + 2\,x + \frac{1}{2}e^{\,3x} - \frac{1}{2}e^{x}$

10. $\frac{d^3y}{dx^3} + 2\ \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2\,y = 0,\ where\ y = 1, \frac{dy}{dx} = 2,\ \frac{d^2y}{dx^2} = 2\ at\ \ x = 0.$

Ans. $y = \frac{5}{3}e^{\,x} - e^{-\,x} + \frac{1}{3}e^{-2x}$

11. $(D^2 - D - 2)x = 20\sin 2\,t\,,\ x_0 = -1, x_1 = 2$

Ans. $x = 2e^{2t} - 4e^{-t} + \cos 2t - 3\sin 2t$

12. $(D^3 + D^2)x = 6t^2 + 4. x(0) = 0, x'(0) = 2, x''(0) = 0$

Ans. $x = \frac{1}{2}t^4 - 2t^3 + 8t^2 - 16t - 16e^{-t}$

13. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + x = e^t, where\ x(0) = 2,\ \frac{dy}{dx} = -1\ at\ t = 0.$

Ans. $x = 2e^t - 3te^t + \frac{1}{2}t^2e^t$

14. $(D^2 - n^2)x = a\sin(nt + \alpha)\ where\ x = D \quad x = 0\ \ at\ \ t = 0$

Ans. $x = an\ cos\ \alpha\ (sin\ nt - nt\ cos\ nt) + \frac{a \sin 2\ \alpha}{2n}\ (t \sin nt)$

15. $y'' + 2y' + y = te^{-t}\ \ if\ y(0) = 1,\ y'(0) = -2$

Ans. $y = \left(1 - t + \frac{t^3}{6}\right)e^{-t}$

16. $\frac{d^2y}{dx^2} + y = x\cos 2x, where\ \ y = \frac{dy}{dx} = 0\ at\ t = 0.$

Ans. $y = \frac{4}{9}\ sin\ 2x - \frac{5}{9}\ sin\ x - \frac{x}{3}\ cos\ 2x$

17. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - y = x^2x^{2x},\ where\ y = 1, \frac{dy}{dx} = 0,\ \frac{d^2y}{dx^2} = -2\ at\ \ x = 0$

Ans. $y = e^{2x}(x^2 - 6x + 12) - e^x(15x^2 + 7x + 11)$

18. $y'' + 4y' + 3y = t, t > 0;\ given\ that\ y(0) = 0\ and\ \ y'(0) = 1$

Ans. $y = -\frac{4}{9} + \frac{t}{6} + e^{-t} - \frac{5}{9}e^{-3t}$

19. $y'' + 2y = r(t),\ y(0) = 0,\ \ y'(0) = 0\ where\ \ r(t) = \begin{Bmatrix} 0, & t \geq l \\ 1, & 0 \leq t < l \end{Bmatrix}$

Ans. $v = \frac{1}{2} - \frac{1}{2}\ cos\ \sqrt{2}\ t.$

20. $\frac{d^2y}{dx^2} + 4y = y(t - 2)$, where u is unit step function $\quad y(0) = 0\ and\ y'(0) = 1$

Ans. $y = \frac{1}{2}\sin 2t\ \ for\ t < 2$

21. $\frac{d^2y}{dx^2} + y - u(t - \pi) - u(t - 2\pi),\ \ y(0) = y'(0) = 0$

Ans. $y = (1 + \cos t)u(t - \pi) - (1 - \cos t)u(t - \pi)$

22. A condenser of capacity C is charged to potential E and discharged at t = 0 through an inductance L and resistance R. The charge q at time t is governed by the differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = q(t)$$

$$q(t) = \begin{cases} E \\ Ee^{-\alpha t} \\ Esin\beta t \\ Ecos\beta t \end{cases}$$

Using Laplace transforms, show that the charge q is given by

$$q = \frac{CE}{n}\ e^{-kt}\ [k\sin nt + n\cos nt]\ where\ k = \frac{R}{2L}\ and\ \eta^2 = \frac{1}{LC} - \frac{R^2}{L^2}$$

23. Determine the final value of $y(t)$, Given $\frac{d^2y(t)}{dt^2} - \frac{3dy}{dt} - 10y(t) = 4$

$$y(0^-) = 2, \qquad y'(0^-) = 1$$

24. For the second-order differential equation $\frac{d^2y(t)}{dt^2} - \frac{7dy(t)}{dt} + 12y(t)$

$[y(0) = 2, y(0) = 5]$, determine the response $y(t)$ solve the following simultaneous differential equations by Laplace transform

25. $\frac{dx}{dt} + 4y = 0,\ \frac{dy}{dt} - 9x = 0. \qquad given\ x = 2\ and\ y = 1\ at\ t = 0.$

Ans. $x = -\frac{2}{3}\ sin\ 6t + 2\cos 6t\ ,\ \ y = \cos 6t + 3\sin 6t$

26. $4\frac{dy}{dt} + \frac{dx}{dt} + 3y = 0,\ 3\frac{dx}{dt} + 2x + \frac{dy}{dt} = 1$ under the condition:

$$x = y = 0\ at\ t = 0$$

Ans. $x = \frac{1}{2} - \frac{1}{5}\ e^{-t} - \frac{3}{10}\ e^{-\frac{6}{11}t},\ y = \frac{1}{5}\ e^{-t} - \frac{1}{5}\ e^{-\frac{6}{11}t}$

27. $\frac{dx}{dt} + 5\,x - 2\,y = t,\ \frac{dy}{dt} + 2x + y = 0$, being given when $x = y = 0\ \ when\ t = 0.$

Ans. $x = -\frac{1}{27}(1 + 6\,t)e^{-3t} + \frac{1}{27}\ (1 + 3\,t), y - \frac{2}{27}\ (2 + 3\,t)\ e^{-3t} - \frac{2\,t}{9} + \frac{4}{27}$

28. $\frac{dx}{dt} + y = \sin t,\ \ \frac{dy}{dt} + x = \cos t$ given that $x = 2,\ and\ \ y = 0\ \ when\ t = 0.$

Ans. $x = e^t + e^{-t}, y = e^{-t} - e^t + \sin t$

29. $3\frac{dx}{dt} + 3\frac{dy}{dt} + 5\,x = 25\cos t\,,\ \ 2\frac{dx}{dt} - 3\frac{dy}{dt} = 5\sin t\ \ with\ x\ (0) = 2, y(0) = 3$

Ans. $x = 2\cos t + 3\sin t\,,\ \ y = 3\cos t + \ 2\sin 3\ \ t$

30. By Laplace transformation method, determine the response $y(t)$ for $t \geq 0$

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 5u(t)$$

31. By Laplace transform, Solve the differential equation:

$$\frac{dx}{dt} - 4x = 8, x(0^-) = x(0^+) = 2$$

32. Determine the final value of $y(t)$, given $\frac{d^2y(t)}{dt^2} - \frac{3dy}{dt} - 10y(t) = 4$

$$y(0^-) = 2, y'(0^-) = 1$$

33. By Laplace transform, determine $y(t)$, given: $\frac{d^2y(t)}{dt^2} - 3\frac{dy(t)}{dt} - 10y(t) = e^{-t}$

$$y(0^-) = 2, \qquad y'(0^-) = 1$$

34. For the second-order differential equation: $\frac{d^2y(t)}{dt^2} - 7\frac{dy(t)}{dt} + 12y(t) = 6e^{-4t}$

$y(0) = 2,\ y'(0) = 5$, determine the response $y(t)$

35. For the second-order differential equation: $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 2sin4t$

$y(0) = 2,\ y'(0) = 5$, determine the response $y(t)$

36. Solve the following differential equation using the Laplace transform method.

If $v'(0) = v(0) = 2, \quad \frac{d^2v(t)}{dt^2} + 4\frac{dv(t)}{dt} + 4v(t) = 2e^{-t}$

Ans: $(2e^{-t} + 4e^{-2t})u(t)$

37. Use the Laplace transform to solve the integro differential equation

$$\frac{dy}{dt} + 3y(t) + 2\int_0^t y(\tau)\,d\tau = 2e^{-3t} \quad y(0) = 0$$

Ans: $(-e^{-t} + 4e^{-2t} - 3e^{-3t})u(t)$

CHAPTER 5

CIRCUIT ANALYSIS BY LAPLACE TRANSFORM

5.0 Introduction

This chapter is dedicated to solving electrical circuit or networks analysis in frequency domain or s-domain which is known as Laplace Transformation method. Laplace method is linear in nature which makes its use very easy and a useful tool in circuit analysis. Consider for example Fig. 5.1 is a series R-L circuit containing a voltage source made up of a combination of a (trice-scaled-up) impulse “function” and a (twice-scaled-up) unit step function (a combination d.c voltage)

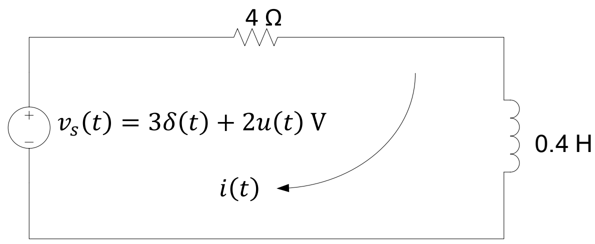


Figure 5.1

We are required to determine the current as the response (output)

Taking KVL of the loop:

$$\text{We have } 3\delta(t) + 2u(t) = 4i(t) + 0.4\frac{di(t)}{dt} \qquad 5.1$$

Taking the Laplace transform of both sides of the Eq. 5.1.

$$3 + \frac{2}{s} = 4I(s) + 0.4[sI(s) - i(0^-)]$$

A little observation on the initial condition $i(0^-)$: the source voltage, $v_s(t)$, is a sum of impulse and step signals, the former being non-zero only at $t = 0$. So, at $t = 0^-$, the sum is zero since $u(t)$ takes up non-zero value from $t = 0$. Because an inductor presents a short circuit in a steady-state, which in this case is the one prior to $t = 0^-$ (assuming the circuit has been in this condition for a long time), applying Ohm’s law (always applicable at any point in time) simply gives Eq. 5.2:

$$i(0^-) = \frac{(0-0)}{4} = 0\ \text{A} = i(0^+) \qquad 5.2$$

Because a current through an inductor cannot change in zero time (instantaneously). Even if the time was not "long" enough for the inductor to present a short circuit, the impedance would have a reactive component by $i(0^-)$ is still zero because the source voltage $v_s(0^-) = 0$ on account of the step function which takes on value of 2 only from $t = 0$ (or 0^+). Getting back to Eq. 5.2 for $i(0^+) = 0$ A, we have:

$$3 + \frac{2}{s} = 4I(s) + 0.4\,[s\,I\,(s) - 0]$$

$$\Rightarrow I(s) = \frac{\left(3 + \frac{2}{s}\right)}{(4 + 0.4s)} \qquad 5.3$$

Resolving Eq. 5.3 into partial fraction

$$= \frac{3s + 2}{s(0.4s + 4)} = \frac{\left(\frac{30s}{4}\right) + \left(\frac{20}{4}\right)}{(s + 10)}$$

$$= \frac{A}{s} + \frac{B}{s + 10}$$

$$A = \left.\frac{\left(\frac{15s}{2}\right) + 5}{s + 10}\right|_{s=0} = \frac{1}{2}$$

$$B = \left.\frac{\left(\frac{15s}{2}\right) + 5}{s}\right|_{s=-10} = \frac{-75 + 5}{-10} = 7$$

$$I(s) = \frac{\frac{1}{2}}{s} + \frac{7}{s + 10} \qquad 5.4$$

Finding the inverse Laplace transform of Eq. 5.4 we have:

$$(t) = \mathcal{L}^{-1}\, I(s) = (0.5 + 7e^{-10t})u(t)\ \text{A} \qquad 5.5$$

The first term in Eq. 5.5 i.e. 0.5, is known as the d.c component because for this term, the root (pole) and hence frequency is zero (making the period which is its reciprocal, "infinite" hence, which is its reciprocal, "infinite" hence direct current). So, the time domain expression for this component explicably written as $0.5e^{0t} = 0.5e^{0} = 0.5$ A. this portion is what remains after the transient portion $7e^{-10t}$ has died off as time tends to infinity. A simple application of Ohm's law in the steady state (inductor is short) gives $\frac{(0+2)-0}{4} = 0.5$ A, to be consistent with the foregoing observation!

Example 5.1: A resistance R in series with inductance L is connected with e.m.f $E(t)$. The current $i(t)$ is given by

$$L\frac{di}{dt} + Ri = E(t).$$

If its switch is closed at t = 0 and disconnect at t = a, find the current I in terms of it.

Solution:

Condition under which current I flows are i = 0 at t = 0,

$$E\ (t) = \{E \qquad 0 \le t \le a\}$$

$$\text{Given equation is} \qquad L\frac{di}{dt} + Ri = E\ (t) \qquad 5.6$$

Taking Laplace transform of Eq. 5.6, we have

$$\mathcal{L}\ [\overline{si} - i(0)]\ \overline{Ri} = \int_0^{\infty} e^{-st}\, E\ (t)\ dt$$

$$\mathcal{L}\ \overline{si} + \overline{Ri} = \int_0^{\infty} e^{-st}\, E\ (t)\ dt \qquad [i\ (0) = 0]$$

$$(Ls + R)i = \int_0^{\infty} e^{-st}\, E\ dt = \int_0^{\infty} e^{-st}\, E\ dt + \int_a^{\infty} e^{-st}\, E\ dt$$

$$= E\left[\frac{e^{-st}}{-s}\right]_0^a + 0 = \frac{E}{s}(1 - e^{-as}] = \frac{E}{s} - \frac{E}{s}\ e^{-as}$$

$$\bar{\imath} = \frac{E}{s(Ls + R)} - \frac{Ee^{-as}}{s(Ls + R)}$$

On inversion, we obtain

$$i = \mathcal{L}^{-1}\left[\frac{E}{s(Ls + R)}\right] - \mathcal{L}^{-1}\left[\frac{Ee^{-as}}{s(Ls + R)}\right] \qquad 5.7$$

Now we have to find the value of $\mathcal{L}^{-1}\left[\frac{E}{s\ (Ls + R)}\right]$

$$\mathcal{L}^{-1}\left[\frac{E}{s(Ls + R)}\right] = \frac{E}{L}\mathcal{L}^{-1}\left[\frac{1}{s\left(s + \frac{R}{L}\right)}\right] \quad \text{(Resolving into partial fractions)}$$

$$= \frac{E}{L} \cdot \frac{L}{R} \mathcal{L}^{-1} \left[\frac{1}{s} - \frac{1}{s + \frac{R}{L}} \right] = \frac{\mathrm{E}}{\mathrm{R}} \left[1 - e^{-\frac{R}{L}t} \right]$$

and $$\mathcal{L}^{-1} \left[\frac{Ee^{-as}}{s\,(Ls + R)} \right] = \frac{\mathrm{E}}{\mathrm{R}} \left[1 - e^{-\frac{R}{L}(t-a)} \right] u\,(t-a)$$

[By the second shifting theorem]

On substituting the values of the inverse transforms into Eq. 5.7, we have

$$i = \frac{\mathrm{E}}{\mathrm{R}} \left[1 - e^{-\frac{R}{L}t} \right] - \frac{\mathrm{E}}{\mathrm{R}} \left[1 - e^{-\frac{R}{L}(t-a)} \right] u\,(t-a)$$

Hence, $i = \frac{\mathrm{E}}{\mathrm{R}} \left[1 - e^{-\frac{R}{L}t} \right]$ for $0 \le t \le a,\quad [u\,(t-a) = 0]$

$$i = \frac{\mathrm{E}}{\mathrm{R}} \left[1 - e^{-\frac{R}{L}t} \right] - \frac{\mathrm{E}}{\mathrm{R}} \left\{ 1 - e^{-\frac{R}{L}(t-a)} \right\} \qquad for\ \ t > a$$

$[u\,(t-a) = 1]$

$$= \frac{E}{R} \left[1 - e^{-\frac{R}{L}(t-a)} - e^{-\frac{R}{L}t} \right] \Rightarrow i(t) = \frac{E}{R} - e^{-\frac{R}{L}t} \left[e^{\frac{Ra}{L}} - 1 \right] \; \boldsymbol{Ans.}$$

Example 5.2: Using the Laplace transform, find the current $i(t)$ in Fig. 5.2.

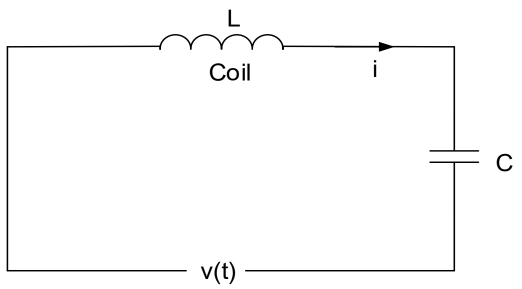


Figure 5.2

Assuming L = 1 henry, C = 1 farad, zero

initial current and charge on the capacitor, and

$$v\,(t) = t \quad when \quad 0 \le t \le 1 = 0 \quad otherwise.$$

Solution:

The differential equation for L and C circuit is given by

$$\frac{d^2q}{dt^2} + \frac{q}{C} = E \qquad 5.8$$

Putting $L = 1, C = 1, e = v\ (t)$ into Eq. 5.8, we get

$$\frac{d^2q}{dt^2} + q = v\ (t) \qquad 5.9$$

Taking Laplace Transform of Eq. 5.9 we have

$$s^2\bar{q} - sq'(0) - q^{'}(0) + \bar{q} = \int_0^\infty v\ (t)\ e^{-st}\ dt$$

Substituting $q'^{(0)} = 0, i\ (0) = q^{'}(0) = 0, we\ get$

$$s^2\bar{q} + \bar{q} = \int_0^1 te^{-st}\ dt + \int_0^\infty 0\ e^{-st}dt$$

$$(s^2 + 1)\bar{q} = \left[t\frac{e^{-st}}{-s}\right]_0^t - \int_0^1 \frac{e^{-st}}{-s}\ dt = \frac{e^{-s}}{-s} - \left[\frac{e^{-st}}{s^2}\right]_0^1 = -\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}$$

$$\bar{q} = \frac{1}{s^2 + 1}\left[-\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]$$

$$\bar{q} = \frac{-e^{-s}}{s(s^2 + 1)} - \frac{e^{-s}}{s^2(s^2 + 1)} + \frac{1}{s^2\ (s^2 + 1)}$$

Taking inverse Laplace transform, we get

$$\bar{q} = \mathcal{L}^{-1}\frac{-e^{-s}}{s\ (s^2 + 1)} - \mathcal{L}^{-1}\frac{e^{-s}}{s^2\ (s^2 + 1)} + \mathcal{L}^{-1}\frac{1}{s^2\ (s^2 + 1)} \qquad 5.10$$

We know that

$$\mathcal{L}^{-1}[e^{-s}\ f(s)] = f(t-a)u\ (t-a)$$

$$\mathcal{L}^{-1}\frac{1}{s(s^2 + 1)} = \int_0^t \sin t\ dt = [-\cos t\,]_0^t = 1 - \cos t$$

$$\mathcal{L}^{-1}\frac{1}{s^2(s^2 + 1)} = \int_0^t (1 - \cos t)\ dt = t - \sin t$$

In view of this, we have

$$\mathcal{L}^{-1}\left[\frac{-e^{-s}}{s^2(s^2+1)}\right] = -[1 - \cos(t-1)]u\ (t-1)$$

$$\mathcal{L}^{-1}\frac{e^{-s}}{s^2(s^2 + 1)} = [(t-1) - \sin(t-1)]u\ (t-1)$$

Putting into Eq. 5.10 we get

$$q = -[1 - \cos(t-1)]u\ (t-1) - [(t-1) - \sin(t-1)]u\ (t-1) + t - \sin t$$

5.1 Further Examples on Frequency Domain Circuit Analysis

Example 5.3: In the circuit of Fig. 5.3 the coil has 10 Ω resistance and a 6 H inductance. If $R = 14\ \Omega$ and the source voltage is 24 V and the switch is open at $t = 0$. Determine $i(t)$ using the Laplace transform method.

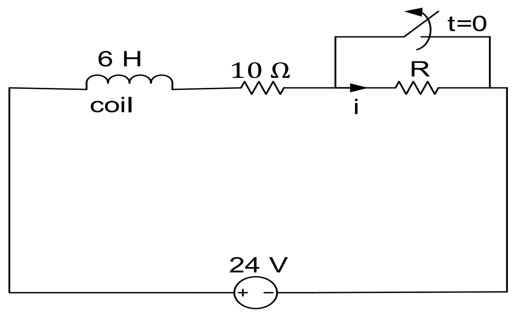


Figure 5.3

Taking the KVL @ $t = 0$,

$$i_0 = \frac{24}{10} = 2.4 \text{ A} \qquad 5.11$$

$$24 = 10i(t) + 14i(t) + 6\frac{di(t)}{dt} \qquad 5.12$$

Laplacing Eq. 5.12, we have

$$\frac{24}{s} + L\, i(0) = I_s\,(6s + 10 + 14) \qquad 5.13$$

$$\frac{24}{s} + 6(2.4) = I_s(6s + 24)$$

$$\frac{24 + 14.4s}{s} = I_s(6s + 24)$$

$$I_s = \frac{14.4s + 24}{s(6s + 24)}$$

$$I_s = \frac{2.4s + 4}{s(s + 4)} \qquad 5.14$$

Resolving Eq. 5.14 into partial fraction, we have:

$$\frac{2.4s + 4}{s(s + 4)} = \frac{A}{s} + \frac{B}{s + 4}$$

Using Cover Up-Rule

$$A = \lim_{s\to 0}\left[\frac{2.4s+4}{s+4}\right] = \frac{4}{4} = 1$$

$$B = \lim_{s\to -4}\left[\frac{2.4s+4}{s}\right] = \frac{-5.6}{-4} = 14$$

$$I_s = \frac{1}{s} + \frac{1.4}{s+4} \qquad 5.15$$

Taking the inverse Laplace transform of Eq. 5.15, i.e.

$$i(t) = \mathcal{L}^{-1} I_s = \mathcal{L}^{-1}\left[\frac{1}{s} + \frac{1.4}{s+4}\right]$$

$$i(t) = 1 + 1.4e^{-4t}\mathrm{u(t)}\ \mathrm{A}$$

Example 5.4: Find $i(t)$ i.e., the current across 12 Ω resistor at $t > 0$ in Fig. 5.4 by first Laplacing the circuit and then use the Laplace transform method to fine the current as stated.

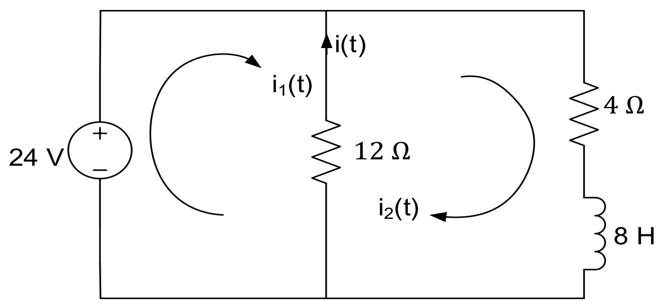


Figure 5.4

Solution:

KVL LOOP 1: $$24 = 12\big(i_1(t) - i_2(t)\big) \qquad 5.16$$

KVL L0OP 2: $$0 = 8\frac{di_2(t)}{dt} + 4i_2 + 12\big(i_2(t) - i_1(t)\big) \qquad 5.17$$

Laplacing Fig. 5.4a we have Fig. 5.4b

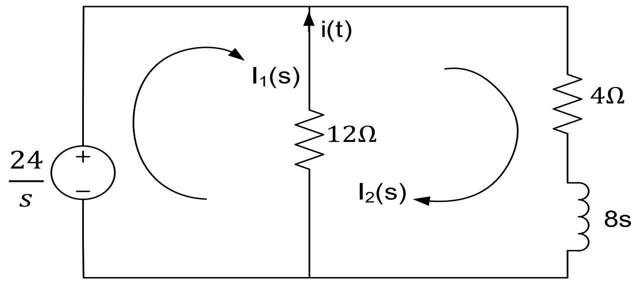


Figure 5.4b

Using MESH analysis

For mesh 1

$$\frac{24}{s} = 12I_1 - 12I_2 \qquad 5.18$$

$$12I_1 = \frac{24}{s} + 12I_2$$

$$I_1 = \frac{12I_2 + 24}{12s} = \frac{sI_2 + 2}{s}$$

$$I_1 = \frac{sI_2 + 2}{s} \qquad 5.19$$

For mesh 2

$$12I_2 - 12I_1 + 4I_2 + 8sI_2 = 0 \qquad 5.20$$

Substituting Eq. 5.18 into Eq. 5.20 we have:

$$16I_2 - 12\left(\frac{I_2\, s + 2}{s}\right) + 8\, sI_2 = 0$$

$$16I_2 - \frac{12\, sI_2 - 24}{s} + 8\;\, sI_2 =$$

$$16sI_2 - 12sI_2 - 24 + 8s^2I_2 = 0$$

$$8s^2I_2 + 16sI_2 - 12sI_2 = 24$$

$$I_2 8s^2 + 4sI_2 = 24$$

$$I_2\,(8s^2 + 4s) = 24$$

$$I_2 = \frac{24}{s(8s+4)} = \frac{3}{s(s+0.5)}$$

$$I_2 = \frac{3}{s\,(s + 0.5)} \qquad 5.21$$

Resolving Eq. 5.21 into partial fraction we have that

$$\frac{3}{s(s + 0.5)} = \frac{A}{s} + \frac{B}{s + 0.5}$$

$$A = \lim_{s \to 0} \left[\frac{3}{s + 0.5}\right] = \frac{3}{0.5} = 6$$

$$B = \lim_{s \to 0.5} \left[\frac{3}{s}\right] = \frac{3}{-0.5} = -6$$

$$I_2 = \left[\frac{6}{s} + \frac{6}{s + 0.5}\right]$$

$$i_2(t) = \mathcal{L}^{-1}\, I_2$$

$$i_2(t) = 6 - 6e^{-0.5t} A \qquad 5.22$$

$$i_2(t) = 6(1 - e^{-0.5t})u(t)\ \mathrm{A}$$

Substituting Eq. 5.22 into Eq. 5.16

$$24 = 12(i_1(t) - 6 + 6e^{-0.5t})$$

$$\frac{24}{12} = i_1(t) - 6 + 6e^{-0.5t}$$

$$i_1(t) = 2 + 6 - 6e^{-0.5t}$$

$$i_1(t) = 8 - 6e^{-0.5t}\mathrm{A} \qquad 5.23$$

But $\quad i(t) = i_1(t) - i_2(t)$

$$i(t) = 8 - 6e^{-0.5t} - 6 + 6e^{-0.5t}$$

$$i(t) = 2\ \mathrm{A}$$

Example 5.5: The circuit shown in Fig. 5.5a is under steady state with the switch at position 1. At $t = 0$ the switch, is moved to position 2. Find $i(t)$ using the Laplace transform method.

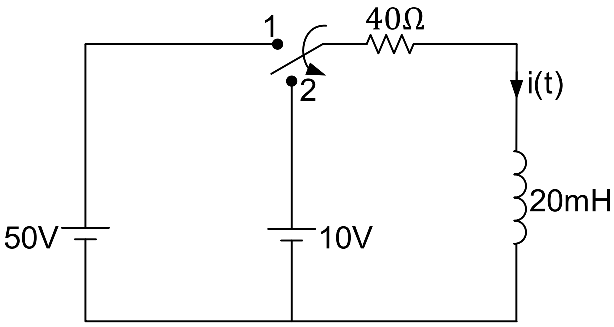


Figure 5.5a

Solution:

Laplacing Fig. 5.5a, we have Fig. 5.5b

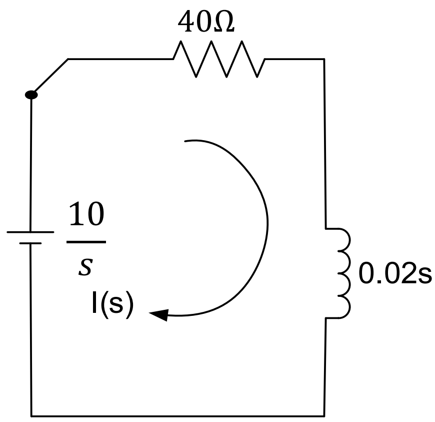


Figure 5.5b

$$i(0) = \frac{50}{40} = 1.25A \qquad 5.24$$

$$\frac{10}{s} = 40\mathrm{I} + 0.02\big(s\mathrm{I} - i(0)\big) \qquad 5.5$$

Substituting Eq. 5.24 into Eq. 5.25 we have:

$$\frac{10}{s} = 40\mathrm{I} + 0.02\ (s\mathrm{I} - 1.25)$$

Solving for I(s)

$$\frac{10}{s} = 40\text{I} + 0.02\text{sI} - 0.025$$

$$\frac{10 + 0.025s}{s} = \text{I}\ (0.025 + 40)$$

$$\text{I} = \frac{0.025s + 10}{s(0.02s + 40)} \qquad 5.26$$

Resolving Eq. 5.26 into partial fraction we have Eq. 5.27

$$I = \frac{0.025s + 10}{s(0.02s + 40)} = \frac{1.25s + 500}{s(s + 2000)}$$

$$I = \frac{1.25s + 500}{s + (s + 2000)} = \frac{A}{s} + \frac{B}{s + 2000}$$

$$A = \lim_{s \to 0} \left[\frac{1.25 + 500}{s + 2000}\right] = \frac{500}{2000} = 0.25$$

$$B = \lim_{s \to -2000} \left[\frac{1.25s + 1000}{s}\right] = \frac{-2000}{-2000} = 1$$

$$I = \frac{0.25}{s} + \frac{1}{s + 2000} \qquad 5.27$$

$$i(t) = \mathcal{L}^{-1}[\text{I}] = \mathcal{L}^{-1}\left(\frac{0.25}{s} + \frac{1}{s + 2000}\right)$$

$$i(t) = 0.25 + e^{-2000t}\text{u(t) A}$$

Example 5.6: Express the voltage of the circuit of Fig. 5.6 in the s-domain if $v(0^+) = 0$ and hence solve for $v(t)$ using the Laplace transform method. Hence obtains the steady state condition using final the value theorem.

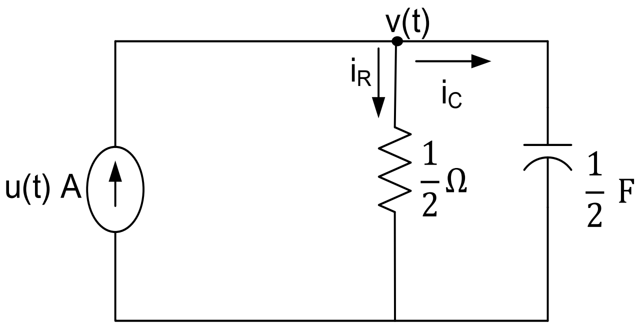


Figure 5.6

Solution:

Applying KCL into Fig. 5.6 at Node V(t), we have Eq. 5.28

$$i = i_R + i_C \quad 5.28$$

$$u(t) = \frac{V}{0.5} + 0.5\frac{dv}{dt} \quad 5.29$$

$$u(t) = 2V + 0.5\frac{dv}{dt} \quad 5.29a$$

Taking the Laplace transform of Eq. 5.9a

$$\frac{1}{s} = 2V + 0.5\,(sV - v(0^+)) \quad 5.30$$

Substituting $v\,(0^+) = 0V$ into Eq. 5.30

$$\frac{1}{s} = 2V + 0.5sV \quad 5.31$$

Solving for $V(s)$ we have Eq. 5.32

$$V(2 + 0.5s) = \frac{1}{s}$$

$$V = \frac{1}{s(0.5s + 2)}$$

$$V(s) = \frac{2}{s(s + 4)} \quad 5.32$$

Resolving Eq. 5.32 into partial fraction we have Eq. 5.33

$$V = \frac{2}{s(s+4)} = \frac{A}{s} + \frac{B}{s+4}$$

$$A = \lim_{s\to 0}\left[\frac{2}{s+4}\right] = \frac{2}{4} = 0.5$$

$$B = \lim_{s\to -4}\left[\frac{2}{s}\right] = \frac{2}{4} = -0.5$$

$$V = \frac{0.5}{s} + \frac{0.5}{s+4} \qquad 5.33$$

Taking the inverse Laplace transform of Eq. 5.33 we have Eq. 5.34

$$v(t) = \mathcal{L}^{-1}[V] = \mathcal{L}^{-1}\left(\frac{0.5}{s} - \frac{0.5}{s+4}\right)$$

$$v(t) = (0.5 - 0.5e^{-4t})u(t)\ \mathrm{V} \qquad 5.34$$

$$\therefore \qquad v(t) = 0.5\,(1 - e^{-4t})u(t)\ \mathrm{V}$$

Furthermore, we can apply final value theorem to find the steady state condition in **example 5.6.**

Fundamentally, the final value theorem states that:

$$f(\infty) = \lim_{s\to 0} s\,F(s)$$

Applying the theorem to Example 5.6 to obtain steady state value of $V(s)$

$$V_{ss} = \lim_{s\to 0} V(s)$$

From Eq. 5.32, $V_s = V = \frac{2}{s(s+4)}$

$$V_{ss} = \lim_{s\to 0}\left[\frac{2s}{s(s+4)}\right] = \frac{2}{4} = 0.5\ \mathrm{V}$$

Example 5.7: The current (in the s-domain) through a circuit is given by:

$$I_{(s)} = \frac{6}{s(s+2)(s+3)}$$

What is $i(0)$?

Solution:

Applying the final value theorem, recall

$$i(\infty) = \lim_{s \to 0} sI_{(s)} = i_{ss}$$

$$i(\infty) = \lim_{s \to 0} \left[\frac{6s}{s(s+2)(s+3)} \right] = \frac{6}{2 \times 3} = \frac{6}{6}$$

$$i(\infty) = 1\ \text{A}$$

Example 5.8: In the RL circuit of Fig. 5.7, the switch is in position 1 long enough to establish steady state-state conditions and at $t = 0$ it is switched to position 2. Find the resulting current $i(t)$ using the Laplace transform method.

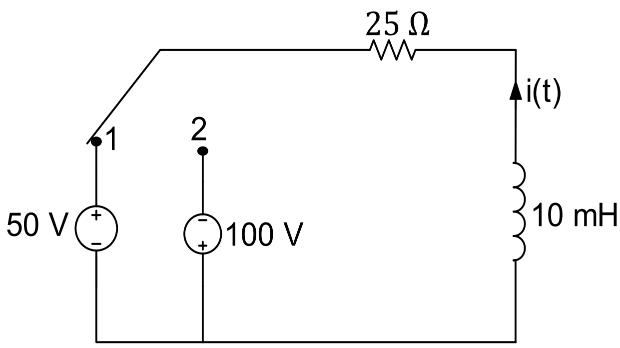


Figure 5.7

Solution

When the switch is at point 1,

$$i(0) = \frac{50}{25} = -2A \qquad 5.35$$

At point 2

Taking the KVL of Fig. 5.7

$$100 = 25i + 0.01 \frac{di}{dt} \qquad 5.36$$

Taking the Laplace transform of Eq. 5.36 we have Eq. 5.37

$$\frac{100}{s} = 25I + 0.01\ [sI - i(0)\ ] \qquad 5.37$$

Substituting Eq. 5.35 into Eq. 5.37, we have Eq. 5.38

$$\frac{100}{s} = 25I + 0.01\,[sI + 2] \qquad 5.38$$

Solving for $I(s)$ in Eq. 5.38 we have Eq. 5.39

$$\frac{100}{s} = 25I + 0.01sI + 0.02$$

$$100 = 25\,sI + 0.01s^2\,I - 0.02s$$

$$I(0.01s^2 + 25s) = 100 - 0.025$$

$$I = \frac{-0.02s + 100}{0.01s^2 + 25s} \qquad 5.39$$

Resolving Eq. 5.39 into partial fraction we have Eq. 5.40

$$I = \frac{-2s + 10000}{s(s + 2500)} = \frac{A}{s} + \frac{B}{(s + 2500)}$$

$$i(t) = \mathcal{L}^{-1}[I_s]$$

$$A = \lim_{s \to 0}\left[\frac{-25 + 10000}{s + 2500}\right] = \frac{10000}{2500} = 4$$

$$B = \lim_{s \to -2500}\left[\frac{-2s + 10000}{s}\right] = \frac{-2(-2000) + 10000}{-2500}$$

$$B = \frac{15000}{-2500} = -6$$

$$I = \frac{4}{s} - \frac{6}{s + 2000} \qquad 5.40$$

$$i(t) = \mathcal{L}^{-1}I_s$$

Taking the Laplace inverse of Eq. 5.40 we have Eq. 5.41

$$i(t) = \mathcal{L}^{-1}\left(\frac{4}{s} - \frac{6}{s + 2000}\right)$$

$$i(t) = 4 - 6e^{-2500t}u(t)\ \mathrm{A} \qquad 5.41$$

Example 5.9: In the two-mesh network in Fig. 5.8, find the currents which result when the switch is closed using the Laplace transform method.

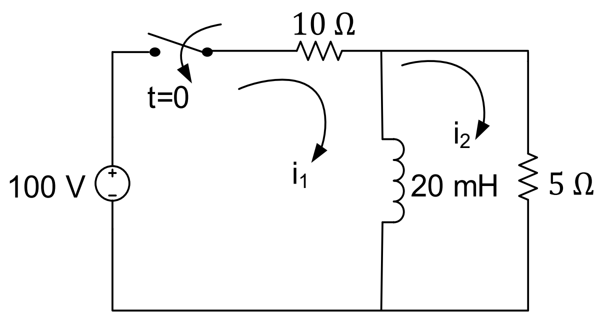


Figure 5.8

Applying KVL for mesh 1

$$\text{At LOOP 1: } 10i_1 + 0.02\left(\frac{di_1}{dt} - \frac{di_2}{dt}\right) = 100 \qquad 5.42$$

Applying KVL for mesh 2

$$\text{At LOOP 2:} \quad 10i_1 + 0.02\frac{di_1}{dt} - 0.02\frac{di_2}{dt} = 100 \qquad 5.43$$

$$0.02\left(\frac{di_2}{dt} - \frac{di_1}{dt}\right) + 5i_2 = 0$$

$$0.02\frac{di_2}{dt} - 0.02\frac{di_1}{dt} + 5i_2 = 0 \qquad 5.43a$$

Taking the Laplace transform of Eqs. 5.42 & 5.43a

$$10I_1 + 0.02sI - 0.02sI_2 = \frac{100}{s} \qquad 5.44$$

$$0.02sI_2 - 0.02sI_1 + 5I_2 = 0 \qquad 5.45$$

Solving for I_2 in equation Eq. 5.45 we have Eq. 5.46

$$0.02sI_2 - 0.02sI_1 + 5I_2 = 0$$

$$sI_2 - sI_1 + 250I_2 = 0$$

$$I_2(s + 250) = sI_1$$

$$I_2 = \frac{sI_1}{s + 250} \qquad 5.46$$

Substitute the value of Eq. 5.46 in equations recall Eq. 5.44 we have Eq. 5.47

$$I_1(10 + 0.02s) = \frac{100}{s} + 0.02sI_2 = \frac{100}{s} + 0.02s\left(\frac{sI_1}{s + 250}\right)$$

$$I_1(10 + 0.02s) = \frac{100}{s} + \frac{0.02s^2I_1}{s + 250}$$

$$I_1(10 + 0.02s) - \frac{0.02s^2I_1}{(s + 250)} = \frac{100}{s}$$

$$I_1\left[(10 + 0.02s) - \frac{0.02S^2}{(s + 250)}\right] = \frac{100}{s}$$

$$I_1\left(\frac{(s + 250)(10 + 0.2s) - 0.02s^2}{s + 250}\right) = \frac{100}{s}$$

$$I_1 = \frac{100(s + 250)}{[s(s + 250)(10 + 0.02s) - 0.02s^2]}$$

$$I_1 = \frac{100(s + 250)}{s(10s + 0.02s^2 + 2500 + 5s - 0.02s^2)}$$

$$I_1 = \frac{100(s + 250)}{s(15s + 2500)} = \frac{6.667(s + 250)}{s(s + 166.667)}$$

$$I_1 = \frac{6.667(s + 250)}{s(s + 166.667)} \qquad 5.47$$

Resolving Eq. 5.47 into partial fraction, we have Eq. 5.48

$$I_1 = \frac{6.667s + 1666.75}{s(s + 166.667)} = \frac{A}{s} + \frac{B}{(s + 166.667)}$$

$$A = \lim_{s\to 0}\left[\frac{6.667s + 1666.75}{s + 166.667}\right] = \frac{1666.77}{166.667}$$

$$A = 10$$

$$B = \lim_{s\to -166.667}\left[\frac{6.667s + 1666.75}{s}\right] = \frac{6.667(-166.667) + 1666.75}{-166.667} = -3.33$$

$$I_1 = \frac{10}{s} - \frac{3.33}{(s + 166.667\ )} \qquad 5.48$$

$$\therefore \quad i_1(t) = \mathcal{L}^{-1}\,[I_1]$$

$$i_1(t) = 10 - 3.33e^{-166.667t}\mathrm{u(t)\ A} \qquad 5.49$$

Substituting Eq. 5.47 into Eq. 5.46 we have Eq. 5.50

$$I_2 = \frac{s\,I_1}{(s+250)} = I_1 \times \frac{s}{s+250} = \frac{6.667(s+200)}{s(s+166.667)} \times \frac{s}{(s+250)}$$

$$I_2 = \frac{6.667}{s+166.667} \qquad 5.50$$

$$i_2 = \mathcal{L}^{-1}[I_2]$$

Taking the Laplace inverse of Eq. 5.50, we have Eq. 5.50a

$$i_2(t) = 6.667e^{-166.667}\mathrm{u(t)}\ \mathrm{A} \qquad 5.50\mathrm{a}$$

Example 5.10: In the two-mesh network of Fig. 5.9, there is no initial charge on the capacitor. Find the loop circuit currents i_1 and i_2 which result when the switches close at $t = 0$ using the Laplace transform method.

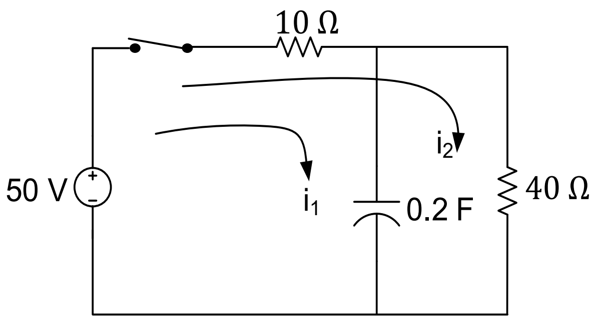


Figure 5.9

$$\text{For LOOP 1: } 10(i_1 + i_2) + \frac{1}{C}\int i_1\,dt = 5 \qquad 5.51$$

$$10i_1 + 10i_2 + 5\int i_1 dt = 50 \qquad 5.51\mathrm{a}$$

$$\text{For LOOP 2: } 10(i_1 + i_2) + 40i_2 = 50 \qquad 5.52$$

$$10i_1 + 10i_2 + 40i_2 = 50$$

$$10i_1 + 50i_2 = 50 \qquad 5.52\mathrm{a}$$

Taking the Laplace transform of Eqs. 5.51a and 5.52a respectively

$$10I_1 + 10I_2 + 5\frac{I_1}{s} = \frac{50}{s} \qquad 5.53$$

$$10I_1 + 50I_2 = \frac{50}{s} \qquad 5.54$$

From Eq. 5.54, solving for I_2, we have Eq. 5.54a

$$50I_2 = \frac{50}{s} = 10I_1$$

$$50I_2 = \frac{50 - 10sI_1}{s}$$

$$I_2 = \frac{-0.2sI_1 + 1}{s} \qquad 5.54a$$

Substituting into Eq. 5.54a into Eq. 5.53, we have Eq. 5.55

$$10I_1 - 10\left[\frac{0.2s\,I_1 + 1}{s}\right] + \frac{5I_1}{s} = \frac{50}{s}$$

$$10I_1 - \frac{2sI_1 + 10}{s} + \frac{5I_1}{s} = \frac{50}{s}$$

$$\frac{10\,sI_1 - 2s\,I_1 + 10 + 5I_1}{s} = \frac{50}{s}$$

$$8s\,I_1 + 5I_1 = 50$$

$$I_1(8s + 5) = 50 - 10$$

$$I_1 = \frac{40}{(8s + 5)} =$$

$$I_1 = \frac{5}{(s\ +\ 0.625)} \qquad 5.55$$

Taking the Laplace inverse of Eq. 5.55, we have Eq. 5.56

$$i_1 = \mathcal{L}^{-1}(I_1)$$

$$i_1 = \mathcal{L}^{-1}\left[\frac{5}{s + 0.625}\right]$$

$$i_1 = 5e^{-0.625t}u(t)\text{ A} \qquad 5.56$$

Substitute the value of I_1 in Eq. 5.55 into Eq. 5.54, we have Eq. 5.57

$$10\left[\frac{s}{s + 0.625}\right] + 50I_1 = \frac{50}{s}$$

$$\frac{50}{s + 0.625} + 50I_2 = \frac{50}{s}$$

$$50I_2 = \frac{50}{s} - \frac{50}{s + 0.625} = \frac{50(s + 0.625) - 50s}{s\,(s + 0.625)}$$

$$50I_2 = \frac{50s + 31.25 - 50s}{s(s + 0.625)}$$

$$I_2 = \frac{0.625}{s(s + 0.625)} \qquad 5.57$$

Resolving Eq. 5.57 into partial fraction, we have Eq. 5.58

$$I_2 = \frac{0.625}{s(s + 0.625)} = \frac{A}{s} + \frac{B}{s + 0.625}$$

$$A = \lim_{s \to 0} \left[\frac{0.625}{s + 0.625}\right] = \frac{0.625}{0.625} = 1$$

$$A = 1$$

$$B = \lim_{s \to 0.625} \left[\frac{0.625}{s}\right] = \frac{0.625}{-0.625} = -1$$

$$I_2 = \frac{1}{s} - \frac{1}{s + 0.625} \qquad 5.58$$

$$i_2 = \mathcal{L}^{-1} I_2$$

$$i_2 = 1 - e^{-0.625t}\mathrm{u(t)}\ \mathrm{A}$$

Example 5.11: Find $i(t)$ using Laplace transform method by first Laplacing the circuit and then taking the loop equation in the circuit of Fig. 5.10 if the initial conditions are all zero and the switch is closed at $t = 0$

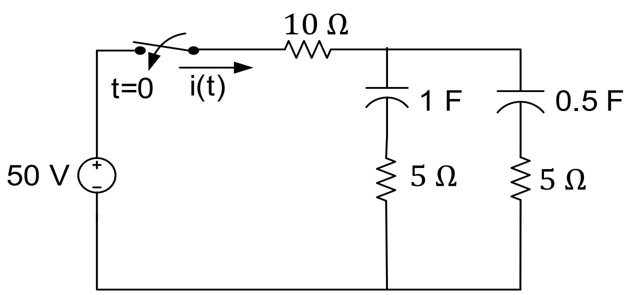


Figure 5.10

Transforming to s-domain, we have circuit of Fig. 5.10a

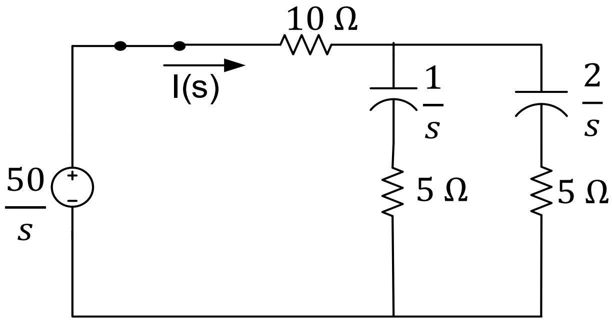


Figure 5.10a

$$V_s = I_s Z \qquad 5.59$$

$$Z = \frac{V_s}{I_s} \qquad 5.59a$$

Parallel Impedances: $$Z_q = 10 + \left(5 + \frac{1}{s}\right) // \left(\frac{2}{s} + 5\right) \qquad 5.60$$

$$= 10 + \left(\frac{\left(5 + \frac{1}{s}\right) \times \left(\frac{2}{s} + 5\right)}{5 + \frac{1}{s} + \frac{2}{s} + 5}\right)$$

$$Z_q = 10 + \left(\frac{\frac{2}{s^2} + \frac{5}{s} + \frac{10}{s} + 25}{\frac{10\, s + 3}{s}}\right)$$

$$= 10 + \left(\left[\frac{2}{s^2} + \frac{15}{s} + 25\right] \div \frac{10s + 3}{s}\right) = 10 + \left(\left[\frac{2 + 15s + 25s^2}{s^2}\right] \times \frac{s}{10s + 3}\right)$$

$$= 10 + \frac{(25s^2 + 15s + 2)s}{s^2\,(10s + 3)} = 10 + \frac{25s^2 + 15s + 2}{s(10s + 3)}$$

$$= \frac{100s^2 + 30s + 15s + 25s^2 + 2}{s(10s + 3)}$$

$$Z_q = \frac{125s^2 + 45s + 2}{s(10s + 3)} \qquad 5.60a$$

But $I_{(s)} = \frac{V_s}{Z_q}$

$$I_{(s)} = \frac{50}{s} \times \frac{s(10s+3)}{125s^2+45s+2}$$

$$I_{(s)} = \frac{50\,(10s+3)}{(125s^2+45s+2)} \qquad 5.61$$

Resolving Eq. 5.61 into partial fraction, we have Eq. 5.62

$$I_{(s)} = \frac{500s+150}{(125s^2+45s+2)}$$

$$I_{(s)} = \frac{4(s=0.3)}{(s+0.308)(s+0.052)}$$

$$= \frac{4\,(s+0.3)}{(s+0.308)(s+0.002)} = \frac{A}{(s+0.308)} + \frac{B}{(s+0.052)}$$

$$A = \lim_{s \to -0.308}\left[\frac{4(s+0.3)}{(s+0.054)}\right] = \frac{4(-0.308+0.3)}{-0.308+0.052}$$

$$A = \frac{-0.032}{-0.256} = 0.125$$

$$B = \lim_{s \to 0.052}\left[\frac{4(s+0.3)}{s+0.308}\right]$$

$$B = \frac{4\,(-0.052+0.3)}{-0.0052+0.308} = \frac{0.992}{0.256}$$

$$B = 3.875$$

$$I_{(s)} = \frac{0.125}{s+0.308} + \frac{3.875}{(s+0.052)} \qquad 5.62$$

Taking the inverse Laplace transform of Eq. 5.62 we have Eq. 5.63

$$i\,(t) = \mathcal{L}^{-1}[I_s]$$

$$i(t) = 0.125\,e^{-0.308t} + 3.875e^{-0.052t} \qquad 5.63$$

$$i(t) = 0.125e^{-0.308t} + 3.875e^{-0.052t}\ \mathrm{A}$$

Example 5.12: In the circuit of the Fig. 5.11, obtain the differential equation for i_1 and i_2. Find the current i_1 and i_2 at $t = 0$ using Laplace transform.

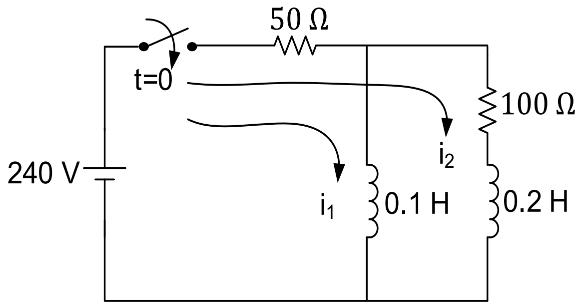


Figure 5.11

Note: To find the Laplace transform first Laplace the circuit assuming all initial conditions as shown in Eq. 5.64

$$i_1(0^+) = i_1(0^-) = 0, \;\; i_2(0^+) = i_2(0^-) = 0, \;\; \frac{di_1(0^+)}{dt} = \frac{V}{L_1} \qquad 5.64$$

Applying KVL at each of the loops:

For LOOP 1: $240 = 50i_1 + 50i_2 + 0.1\frac{di_1}{dt}$ 5.65

For LOOP 2: $240 = 50i_1 + 50i_2 + 100i_2 + 0.2\frac{di_2}{dt}$ 5.66

Laplacing the circuit of Fig. 5.11 we have Fig. 5.11a

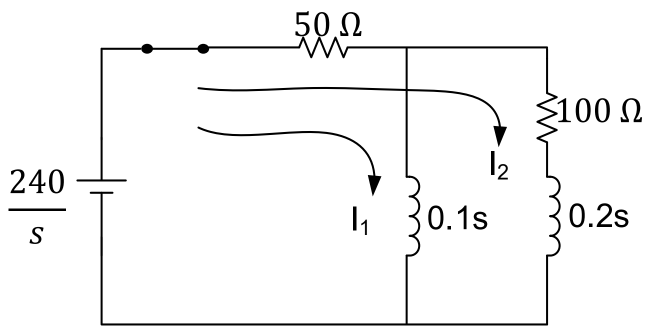


Figure 5.11a

Laplacing Eq. 5.65 we have Eq. 5.67

$$\frac{240}{s} = 50(I_1 + I_2) + 0.1sI_1 \qquad 5.67$$

Solving for I_1 in Eq. 5.67 we have Eq. 5.68

$$240 = 50s(I_1 + I_2) + 0.1s^2 I_1$$

$$240 = 50sI_1 + 50\,sI_2 + 0.1s^2 I_1$$

$$240 = I_1\,(0.1s^2 + 50s) + 50sI_2$$

$$I_1 = \frac{2400 - 500\,sI_2}{s^2 + 500s} \qquad 5.68$$

Laplacing Eq. 5.66 we have Eq. 5.69

$$\frac{240}{s} = 50\,(I_1 + I_2) + 100I_2 + 0.2sI_2 \qquad 5.69$$

$$240 = 50s(I_1 + I_2) + 100sI_2 + 0.2s^2\,I_2$$

$$1200 = 250s(I_1 + I_2) + 500\,sI_2 + s^2\,I_2 \text{ (Dividing through by 0.2)}$$

$$1200 = 250\,sI_1 + 750\,sI_2 + s^2 I_2 \qquad 5.70a$$

Substituting the value of Eq. 5.68 into Eq. 5.70a, we have Eqs. 5.70b and 5.71

$$1200 = 250s\left(\frac{2400 - 500\,sI_2}{s^2 + 500s}\right) + 750sI_2 + s^2 I_2$$

$$1200 = \frac{600000s - 125000s^2 I_2}{s^2 + 500s} + 750\,sI_2 + s^2 I_2$$

$$1200\,(s^2 + 500s) = 600000s - 125000s^2 I_2 + (750sI_2 + s^2 I_2)(s^2 + 500s)$$

$$1200\,(s^2 + 500s) = 600000s - 125000s^2 I_2 + 750sI_2(s^2 + 500s) + s^2 I_2(s^2 + 500s)$$

$$I_2\big(-12500\,s^2 + 750s(s^2 + 500s) + s^2(s^2 + 500s)\big) = 1200(s^2 + 500s) - 600000$$

$$I_2 = \frac{1200\,(s^2 + 500s) - 600000}{s^2(s^2 + 500s) + 750s(s^2 + 500s) - 125000s^2}$$

$$I_2 = \frac{s[1200\,(s + 500) - 600000]}{[s(s^2 + 500s) + 750(s^2 + 500s) - 125000s]}$$

$$I_2 = \frac{1200s + 600000 - 600000}{s\,[(s^2 + 500s) + 750\,(s + 500) - 125000]}$$

$$= \frac{1200s}{s[(s^2 + 500s) + 750\,(s + 500) - 125000]}$$

$$I_2 = \frac{1200}{s^2 + 500s + 750s + 375000 - 125000}$$

$$I_2 = \frac{1200}{s^2 + 1250s + 250000} \qquad 5.70b$$

$$I_2 = \frac{1200}{(s + 1000)(s + 250)} \qquad 5.71$$

Resolving Eq. 5.71 into partial fraction we have Eq. 5.72

$$I_2 = \frac{1200}{(s + 1000)(s + 250)}$$

$$I_2 = \frac{1200}{(s + 1000)(s + 250)} = \frac{A}{(s + 1000)} + \frac{B}{(s + 250)}$$

$$A = \lim_{s \to -1000} \left[\frac{1200}{s + 250}\right] = \frac{1200}{-1000 + 250} = \frac{1200}{-750} = -1.6$$

$$A = -1.6$$

$$B = \lim_{s \to -250} \frac{1200}{s + 1000} = \frac{1200}{-250 + 1000} = \frac{1200}{750} = 1.6$$

$$I_2 = \frac{1.6}{s + 250} - \frac{1.6}{s + 1000} \qquad 5.72$$

$$i_{2(t)} = \mathcal{L}^{-1}[I_2]$$

$$i_{2(t)} = \mathcal{L}^{-1}\left(\frac{1.6}{s + 250} - \frac{1.6}{s + 1000}\right) = 1.6e^{-250t} - 1.6e^{-1000t} \text{ A}$$

$$i_{2(t)} = 1.6(e^{-250t} - e^{-1000t})\text{u(t) A}$$

To find $i_{1(t)}$, let us substitute Eq. 5.71 into Eq. 5.69, then we will have Eq. 5.73

$$I_1 = \frac{2400 - 500\, sI_2}{s^2 + 500s}$$

$$I_1 = \frac{2400 - 500\, s\left[\frac{1200}{(s + 1000)(s + 250)}\right]}{s^2 + 500s}$$

$$I_1 = \frac{2400}{s(s + 500)} - \frac{500 \times 1200}{(s + 500)(s + 1000)(s + 250)}$$

$$I_1 = \frac{2400(s + 1000)(s + 250) - 600000s}{s(s + 500)(s + 1000)(250)} \qquad 5.73$$

Resolving Eq. 5.73 into partial fraction we have Eq. 5.74

$$\frac{2400(s+1000)(s+250)-600000s}{s(s+500)(s+1000)(250)}=\frac{A}{s}+\frac{B}{(s+500)}+\frac{C}{(s+1000)}+\frac{D}{(s+250)}$$

$$A=\lim_{s\to 0}\left[\frac{2400\times 1000\times 250}{500\times 1000\times 250}\right]=4.8$$

$$B=\lim_{s\to -500}\left[\frac{2400\times 500\times(-250)-600000\times(-500)}{(-500)\times 500\times(-250)}\right]=0$$

$$C=\lim_{s\to -1000}\left[\frac{-600000\times(-10000)}{(-1000)\times(-500)\times(-750)}\right]=-1.6$$

$$D=\lim_{s\to -250}\left[\frac{-600000\times(-250)}{(-250)\times(250)\times(750)}\right]=-3.2$$

$$I_1=\frac{4.8}{s}-\frac{1.6}{s+1000}-\frac{3.2}{s+250} \qquad 5.74$$

$$i_1(t)=\mathcal{L}^{-1}\,[I_1]$$

$$i_1(t)=(4.8-1.6e^{-1000t}-3.2e^{-250t})\mathrm{u(t)\ A}$$

Example 5.13: For the two-mesh network of Fig. 5.12, determine the values of the loop current i_1 & i_2 using Laplace transform and hence, write the s-domain equation in matrix form. Taking $Q_0 = 0$.

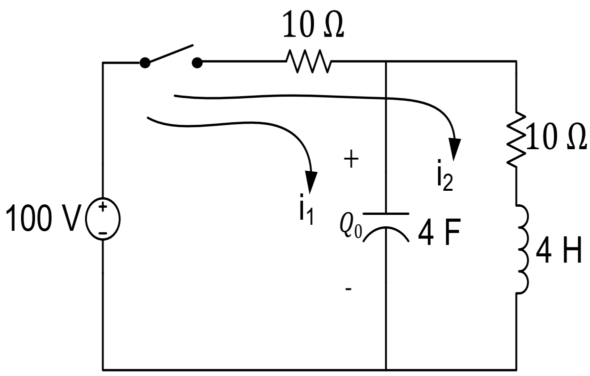


Figure 5.12

$$\text{For LOOP 1: } 10i_1+\frac{1}{4}\left(Q_o+\int_0^t (i(t)dt)\right)+10i_2=100 \qquad 5.75$$

For LOOP 2: $20i_2 + 4\frac{di2}{dt} + 10i_1 = 100$ 5.76

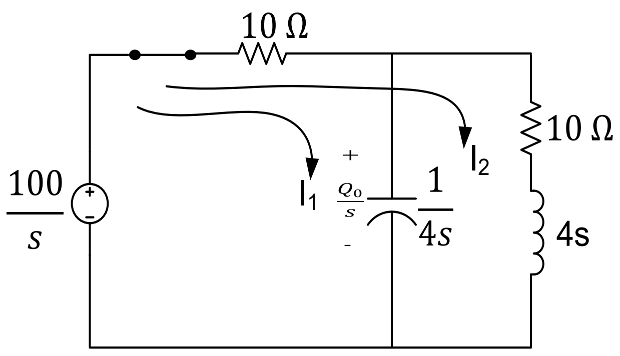


Figure 5.12a

Laplacing Eqs. 5.75 and 5.76 respectively we have Eqs. 5.77 and 5.78

$$10I_1 + \frac{I_1}{4s} + 10I_2 = \frac{100}{s} \quad 5.77$$

$$I_1 + \frac{I_1}{40s} + I_2 = \frac{100}{s}$$

$$\frac{I_1(40s+1)}{40s} + I_2 = \frac{10}{s} \quad 5.77a$$

$$20I_2 + 4sI_2 + 10I_1 = \frac{100}{s} \quad 5.78$$

$$2I_2 + 0.4SI_2 + I_1 = \frac{10}{s}$$

$$I_1 + I_2\,(2 + 0.4s) = \frac{10}{s} \quad 5.78a$$

Putting Eqs. 5.77a and 5.78a into matrix form, we have Eq. 5.79

$$\begin{bmatrix} \frac{(40s+1)}{40s} & 1 \\ 1 & (0.4s+2) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \frac{10}{s} \\ \frac{10}{s} \end{bmatrix} \quad 5.79$$

Solving Eq. 5.79 matrix using determinant method, we have Eqs. 5.79a, 5.79b and 5.79c

$$\Delta = \begin{vmatrix} \frac{40s+1}{40s} & 1 \\ 1 & (0.4s+2) \end{vmatrix} = \frac{(40s+1)(0.4s+2)}{40s} - 1$$

$$\Delta = \frac{16s^2 + 80s + 0.4s + 2}{40s}$$

$$\Delta = \frac{16s^2 + 0.4s + 2}{40s} \qquad 5.79a$$

$$\Delta I_1 = \begin{vmatrix} \frac{10}{s} & 1 \\ \frac{10}{s} & (0.4s+2) \end{vmatrix} = \frac{10}{s}(0.4s+2) - \frac{10}{s}$$

$$\Delta I_1 = 4 + \frac{20}{s} - \frac{10}{s} = \frac{4s+10}{s}$$

$$\Delta I_1 = \frac{4s+10}{s} \qquad 5.79b$$

$$\Delta I_2 = \begin{vmatrix} \frac{40s+1}{40s} & \frac{10}{s} \\ 1 & \frac{10}{s} \end{vmatrix} = \frac{10}{s}\left(\frac{40s+1}{40s}\right) - \frac{10}{s} = \frac{40s+10}{40s^2} - \frac{10}{s}$$

$$\Delta I_2 = \frac{400s + 10 - 400s}{40s} = \frac{10}{40s^2}$$

$$\Delta I_2 = \frac{1}{4s^2} \qquad 7.79c$$

Solving for $I_1$and I_2 we have Eqs. 5.80 and 5.81 respectively.

$$I_1 = \frac{\Delta I_1}{\Delta} = \frac{(4s+10)40s}{s(16s^2 + 40.4s + 2)} = \frac{40(4s+10)}{16s^2 + 40.4s + 2}$$

$$I_1 = \frac{160s + 400}{16s^2 + 40.4s + 2} = \frac{160s}{16s^2 + 40.4s + 2} + \frac{400}{16s^2 + 40.4s + 2}$$

$$I_1 = \frac{10s}{s^2 + 2.53s + 0.125} + \frac{25}{s^2 + 2.53s + 0.125}$$

$$I_1 = \frac{10s}{(s+1.263)^2 - \left(\sqrt{1.469}\right)^2} + \frac{25}{(s+1.263)^2 - \left(\sqrt{1.469}\right)^2} \qquad 5.80$$

Taking the inverse Laplace of Eqs. 5.80 we have 5.80a,

$$i_1 = \mathcal{L}^{-1}[I_1]$$

$$i_1 = 10e^{-1.263t}\cosh\sqrt{1.469t} + \frac{25}{\sqrt{1.469}}e^{-1.263t}\sinh\sqrt{1.496}\ t$$

$$= 5e^{-1.263t}(e^{1.213t} + e^{-1.212t}) + 10.3 + e^{-1.263}(e^{1.212t} - e^{-1.212t})$$

$$= 5e^{-0.05t} + 5e^{-2.475t} + 10.31e^{-0.05t} - 10.31e^{-2.475t}$$

$$i_1 = 15.31e^{-0.05t} - 5.31e^{-2.475t}u(t)\ \text{A} \qquad 5.80a$$

But
$$I_2 = \frac{\Delta I_2}{\Delta} = \frac{1}{4s^2} \div \frac{16s^2 + 40.4s + 2}{40s}$$

$$I_2 = \frac{40s}{4s^2(16s^2 + 40.4s + 2)}$$

$$I_2 = \frac{10}{s\ (16s^2 + 40.4s + 2)} \qquad 5.81$$

Resolving Eq. 5.81 into partial fraction we have Eq. 5.82

$$I_2 = \frac{10}{S\ (16s^2 + 40.4s + 2)} = \frac{A}{s} + \frac{Bs + C}{16s^2 + 40.4s + 2}$$

$$10 = A\ (16s^2 + 40.4s + 2) + (Bs + C)s$$

$$10 = 2A,$$

$$A = \frac{10}{2} = 5$$

Taking s^2:

$$0 = 16A + B \Longrightarrow\ B = -16A, \qquad A = -16 \times 5 = -80$$

Taking s:

$$40.4A + C = 0;\ \ \Longrightarrow C = -40.4A \quad C = -40.4 \times 5 = -202$$

Therefore

$$I_2 = \frac{5}{s} - \left[\frac{80s + 202}{16s^2 + 40.4s + 2}\right] \qquad 5.82$$

$$I_2 = \frac{5}{s} - \left[\frac{5s + 12.63}{s^2 + 2.525s + 0.125}\right]$$

$$I_2 = \frac{5}{s} - \left[\frac{5s + 12.63}{(s + 1.263)^2 - \left(\sqrt{1.469}\right)^2}\right]$$

$$I_2 = \frac{5}{s} - \left[\frac{5s}{(s+1.263)^2 - \left(\sqrt{1.469}\right)^2} + \frac{12.63}{(s+1.263)^2 - \left(\sqrt{1.469}\right)^2}\right] \qquad 5.82$$

a

Taking the Laplace inverse of Eq. 5.82 we have Eq. 5.83

$$i_2 = \mathcal{L}^{-1}[I_2]$$

$$i_2 = 5 - 5e^{-1.263t}\left[\frac{e^{1.212t} + e^{-1.212t}}{2}\right] - 10.42e^{-1.263t}\left[\frac{e^{1.212t} - e^{-1.212t}}{2}\right]$$

$$i_2 = 5 - 2.5e^{-0.05t} - 2.5e^{-2.475t} - 5.21e^{-0.05t} + 5.21e^{-2.475t}$$

$$i_2 = 5 - 7.71e^{-0.05t} + 2.71e^{-2.475t}\mathrm{u(t)}\ \mathrm{A} \qquad 5.83$$

Example 5.14: By the use of Laplace transform, determine the response $v(t)$ in the circuit of Fig 5.13 given $v(0^-) = 4$ V

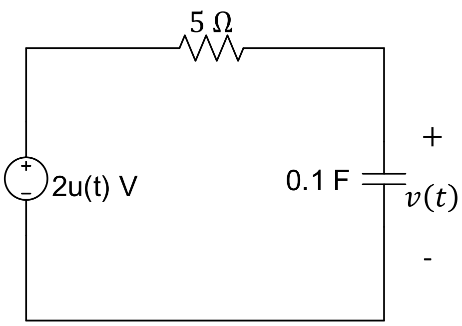


Figure 5.13

Solution:

Taking the Laplace of the Loop KVL we lave Eq. 5.84

$$\mathcal{L}\,[2u(t)] = \mathcal{L}[5i_c(t) + v(t)] \qquad 5.84$$

$$i_c = C\frac{dv(t)}{dt} \qquad 5.85$$

Applying Eq. 5.85 into Eq. 5.84 gives Eq. 5.86

$$\mathcal{L}\,[2u(t)] = \mathcal{L}\left[5(0.1)\frac{dv(t)}{dt} + v(t)\right] \qquad 5.86$$

$$\frac{2}{s} = 0.5\,[sV(s) - v(0^-)] + V(s)$$

$$V(s) = \frac{2}{s(0.5s+1)}$$

$$V(s) = \frac{4}{s(s+2)} \qquad 5.87$$

Resolving Eq. 5.87 into partial fraction gives Eq. 5.88

$$\frac{4}{s(s+2)} = \frac{A}{s} + \frac{B}{(s+2)}$$

$$A = \lim_{s\to 0}\frac{4}{(0+2)} = \frac{4}{2} = 2$$

$$B = \lim_{s\to -2}\frac{4}{(-2)} = = \frac{4}{(-2)} = -2$$

$$V(s) = \frac{2}{s} - \frac{2}{(s+2)} \qquad 5.88$$

$$v(t) = \mathcal{L}^{-1}V(s)$$

$$\Rightarrow v(t) = (2 - 2e^{-2t})u(t)\text{ V}$$

5.2 Exercise

1. By the use of Laplace transform, determine the response $v(t)$ in the circuit of Fig. A given $v(0^-) = 8$ V

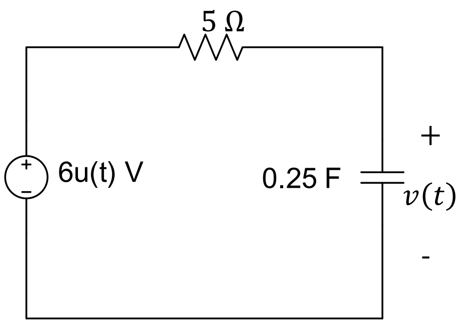


Figure A

2. By Laplace transform, determine the current $i_1 and\ i_2$ in the circuit of Fig. B

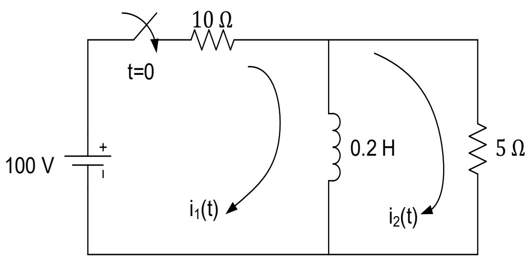


Figure B

3. For the circuit Fig. C, determine $i(t)$ as the response, by means of Laplace transform.

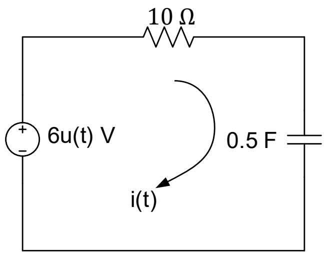


Figure C

4. For the circuit of Fig. D, determine $i(t)$ as the response by means of Laplace transform. What is its steady-state value?

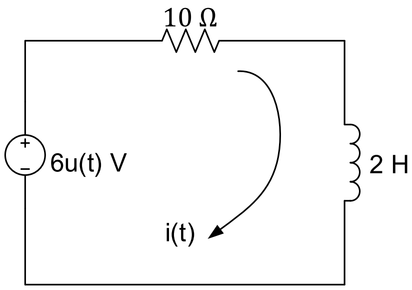


Figure D

5. For the circuit to Fig. E, determine the force response.

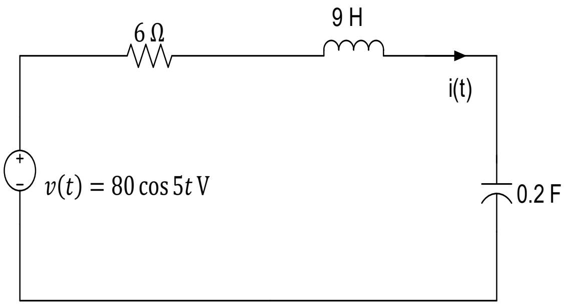


Figure E

6. Determine the current response $i(t)$ for the circuit of Fig. F, by Laplace transformation.

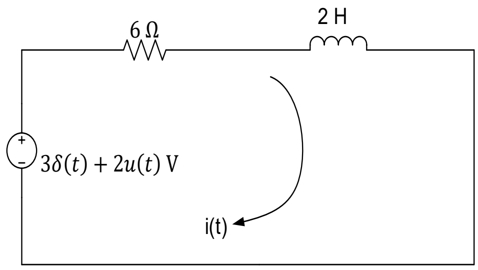


Figure F

7. What is the current response $i(t)$ for the network shown in Fig. G. Indicate time validity.

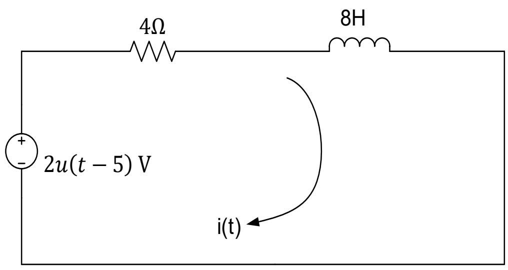


Figure G

8. For the two-mesh network of Fig. H, determine the values of the loop current $i_1 \& i_2$ using Laplace transform and hence, write the s-domain equation in matrix form.

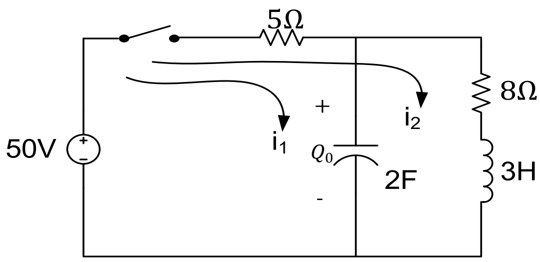


Figure H

9. For the two-mesh network of Fig. I, determine the values of the loop current $i_1 \& i_2$ using Laplace transform and hence, write the s-domain equation in matrix form.

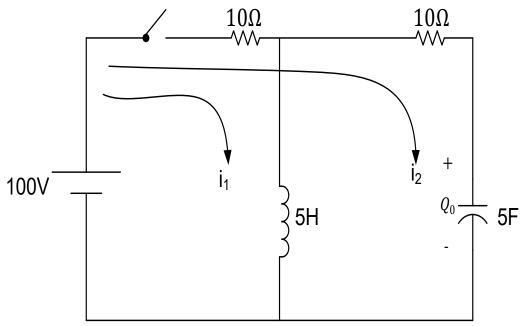


Figure I

Taking the initial conditions as follows:

$$i_1(0^+) = i_1(0^-) = 0 \quad i_2(0^+) = i_2(0^-) = 0, \qquad \frac{di_1(0^+)}{dt} = \frac{V}{L}$$

10. Without first finding $f(t)$ determine $f(0^+), f(\infty)$ for each of $F(s)$ equal to

i. $\frac{4e^{3s}(2s+30)}{s}$

ii. $\frac{(s^2-8)}{s(s^2+9)}$

11. Determine the initial value of $y(t)$, given $Y(s) = \frac{(3s^2+2)}{s^3+7s^2+12s}$

12. Without first finding $f(t)$, determine $f(0^+), \ f(\infty)$ for each of $F(s)$ equal to

(i) $\frac{5e^{-2s}(s+60)}{s}$

(ii) $\frac{(s^2+5)}{s(s^2+10)}$

CHAPTER 6

APPLICATIONS OF THE LAPLACE TRANSFORM

6.0 Introduction

Now that we have introduced the Laplace transform, let us see what we can do with it. Please keep in mind that with the Laplace transform we actually have one of the most powerful mathematical tools for analysis, synthesis, and design. Being able to look at circuits and systems in the s-domain can help us to understand how our circuits and systems really function. In this chapter we will take an in-depth look at how easy it is to work with circuits in the s-domain. In addition, we will briefly look at physical systems. We are sure you have studied some mechanical systems and may have used the same differential equations to describe them as we use to describe our electric circuits. Actually that is a wonderful thing about the physical universe in which we live; the same differential equations can be used to describe any linear circuit, system, or process. The key is the term linear.

A system is a mathematical model of a physical process relating the input to the output.

It is entirely appropriate to consider circuits as systems. Historically, circuits have been discussed as a separate topic from systems, so we will actually talk about circuits and systems in this chapter realizing that circuits are nothing more than a class of electrical systems.

The most important thing to remember is that everything we discussed in the last chapter and in this chapter applies to any linear system. In the last chapter, we saw how we can use Laplace transforms to solve linear differential equations and integral equations. In this chapter, we introduce the concept of modeling circuits in the s-domain. We can use that principle to help us solve just about any kind of linear circuit. We will take a quick look at how state variables can be used to analyze systems with multiple inputs and multiple outputs. Finally, we examine how the Laplace transform is used in network stability analysis and in network synthesis.

6.1 Circuit element models

Having mastered how to obtain the Laplace transform and its inverse, we are now prepared to employ the Laplace transform to analyze circuits. This usually involves three steps.

Steps in Applying the Laplace transform:

1. Transform the circuit from the time domain to the s-domain.

2. Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique which we are familiar with.

3. Take the inverse transform of the solution and thus obtain the solution in the time domain.

Only the first step is new and will be discussed here. As we did in phasor analysis, we transform a circuit in the time domain to the frequency or s-domain by Laplace transforming each term in the circuit.

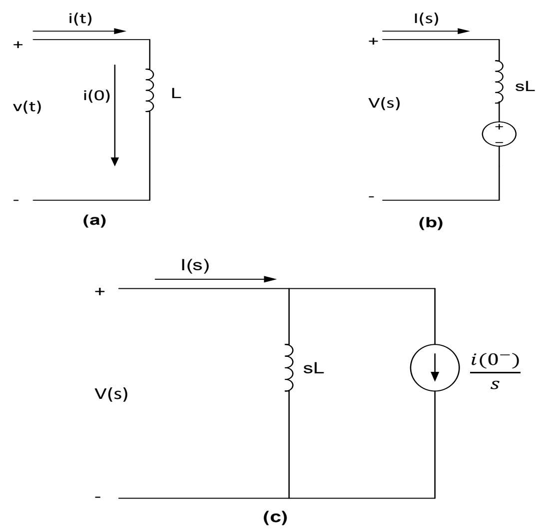


Figure 6.1 Representation of an inductor: (a) time-domain, (b, c) s-domain equivalents

For a resistor, the voltage-current relationship in the time domain is

$$v(t) = Ri(t) \tag{6.1}$$

Taking the Laplace transform, we get

$$\boxed{V(s) = RI(s)} \tag{6.2}$$

For an inductor,

$$v(t) = L\frac{di(t)}{dt} \tag{6.3}$$

Taking the Laplace transform of both sides gives

$$V(s) = L[sI(s) - i(0^-)] = sLI(s) - Li(0^-) \tag{6.4}$$

Or

$$\boxed{I(s) = \frac{1}{sL}V(s) + \frac{i(0^-)}{s}} \qquad 6.5$$

The s-domain equivalents are shown in Fig. 6.1, where the initial condition is modeled as a voltage or current source.

For a capacitor,

$$i(t) = C\frac{dv(t)}{dt} \qquad 6.6$$

which transforms into the s-domain as

$$I(s) = C[sV(s) - v(0^-)] = sCV(s) - Cv(0^-) \qquad 6.7$$

Or

$$\boxed{V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s}} \qquad 6.8$$

The s-domain equivalents are shown in Fig. 6.2. With the s-domain equivalents, the Laplace transform can be used readily to solve first- and

Representation of a capacitor: (a) time-domain (b,c) s-domain equivalent second-order circuits such as those we considered in Chapters 4 and 5, We should observe from Eqs. (6.3) to (6.8) that the initial conditions are part of the transformation. This is one advantage of using the Laplace transform in circuit analysis. Another advantage is that a complete response-transient and steady-state of a network is obtained. We will illustrate this with **Examples 6.2 and 6.3**. Also, observe the duality of Eqs. (6.5) and (6.8), confirming what we already know from the fact that L and $C, I(s)$ and $V(s)$, and $v(0)$ and $i(0)$ are dual pairs.

If we assume zero initial conditions for the inductor and the capacitor, the above equations reduce to:

$$\left.\begin{aligned} \text{Resistor: } V(s) &= RI(s) \\ \text{Inductor: } V(s) &= sLI(s) \\ \text{Capacitor: } V(s) &= \frac{1}{sC}\,I(s) \end{aligned}\right\} \qquad 6.9$$

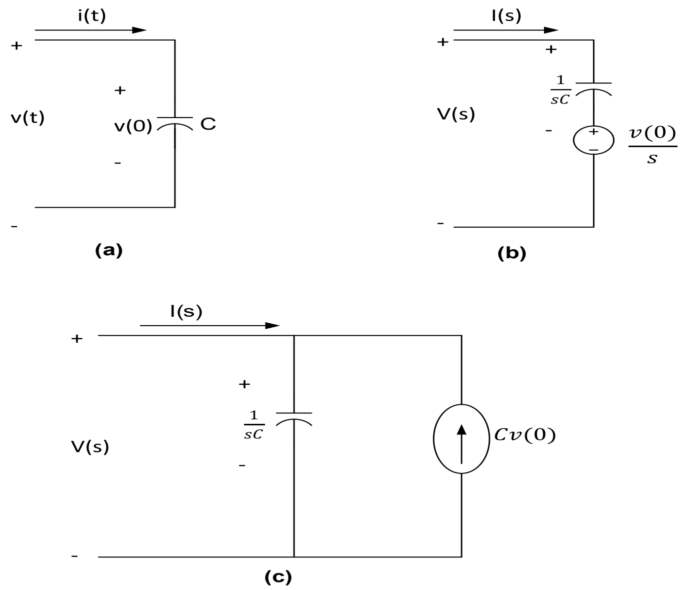


Figure 6.2 Time domain and s-domain representations of passive elements under zero initial conditions

The s-domain equivalents are shown in Fig. 6.3.

We define the impedance in the s-domain as the ratio of the voltage transform to the current transform under zero initial conditions; that is,

$$Z(s) = \frac{V(s)}{I(s)} \qquad 6.10$$

Thus, the impedances of the three circuit elements are

$$\text{Resistor: } Z(s) = R$$

$$\text{Inductor: } Z(s) = sL \qquad 6.11$$

$$\text{Capacitor: } Z(s) = \frac{1}{sC}$$

Table 6.1 summarizes these. The admittance in the s-domain is the reciprocal of the impedance, or

$$Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)} \qquad 6.12$$

The use of the Laplace transform in circuit analysis facilitates the use of various signal sources such as impulse, step, ramp, exponential, and sinusoidal.

The models for dependent sources and op amps are easy to develop drawing from the simple fact that if the Laplace transform of $f(t)$ is $F(s)$, then the Laplace transform of $af(t)$ is $aF(s)$—the linearity property. The dependent source model is a little easier in that we deal with a single value. The dependent source can have only two controlling values, a constant times either a voltage or a current. Thus,

$$\mathcal{L}[av(t)] = aV(s) \tag{6.13}$$

$$\mathcal{L}[ai(t)] = aI(s) \tag{6.14}$$

The ideal op amp can be treated just like a resistor. Nothing within an op amp, either real or ideal, does anything more than multiply a voltage by a constant. Thus, we only need to write the equations as we always do using the constraint that the input voltage to the op amp has to be zero and the input current has to be zero.

Example 6.1: Find $v_o(t)$ in the circuit of Fig. 6.3, assuming zero initial conditions

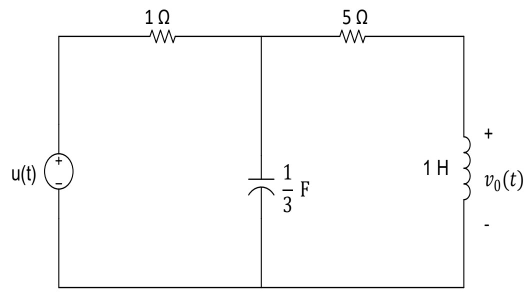


Figure 6.3

Solution:

We first transform the circuit from the time domain to the s-domain.

$$u(t) \quad \Longrightarrow \quad \frac{1}{s}$$

$$1\ \mathrm{H} \quad \Longrightarrow \quad sL = s$$

$$\frac{1}{3}\mathrm{F} \quad \Longrightarrow \quad \frac{1}{sC} = \frac{3}{s}$$

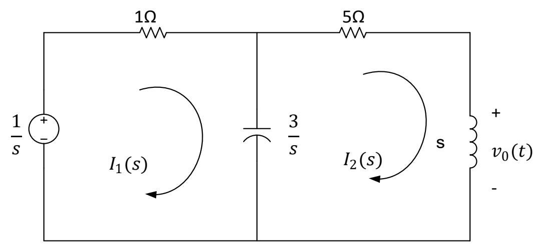


Figure 6.4 Mesh analysis of the frequency-domain equivalent of the same circuit in Fig. 6.3

The resulting s-domain circuit is in Fig 6.4. We now apply mesh analysis. For mesh 1,

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right) I_1 - \frac{3}{s} I_2 \qquad 6.15$$

For mesh 2,

$$0 = -\frac{3}{s} I_1 + \left(s + 5 + \frac{3}{s}\right) I_2$$

Or

$$I_1 = \frac{1}{3}(s^2 + 5s + 3) I_2 \qquad 6.16$$

Substituting this into Eq. (6.15),

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right) \times \frac{1}{3}(s^2 + 5s + 3) I_2 - \frac{3}{s} I_2 \qquad 6.16b$$

Multiplying Eq 6.16a by 3s gives

$$3 = \left(1 + \frac{3}{s}\right) \times s(s^2 + 5s + 3) I_2 - 9I_2 \ = (s^3 + 8s^2 + 18s) I_2$$

$$\Longrightarrow \quad I_2 = \frac{3}{s^3 + 8s^2 + 18s}$$

$$V_o(s) = sI_2 = \frac{2}{s^2 + 8s + 18} = \frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{(s+4)^2 + \left(\sqrt{2}\right)^2}$$

Taking the inverse transform yields

$$v_o(t) = \frac{3}{\sqrt{2}} e^{-4t} \sin \sqrt{2} t \ \text{ V}, \qquad t \geq 0$$

Example 6.2: Find $v_o(t)$ in the circuit of Fig. 6.6. Assume $v_o(0) = 5$ V

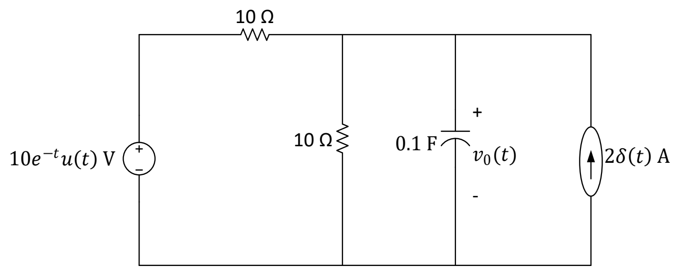


Figure 6.5

Solution

We transform the circuit to the s-domain as shown in Fig. 6.6. The initial condition is included in the form of the current source $Cv_o(0) = 0.1(5) = 0.5$ A. [See Fig. 6.2(c).] We apply nodal analysis. At the top node,

$$\frac{\frac{10}{(s+1)} - V_o}{10} + 2 + 0.5 = \frac{V_o}{10} + \frac{V_o}{\frac{10}{s}}$$

Or

$$\frac{1}{s+1} + 2.5 = \frac{2V_o}{10} + \frac{sV_o}{10} = \frac{1}{10}V_o(s+2)$$

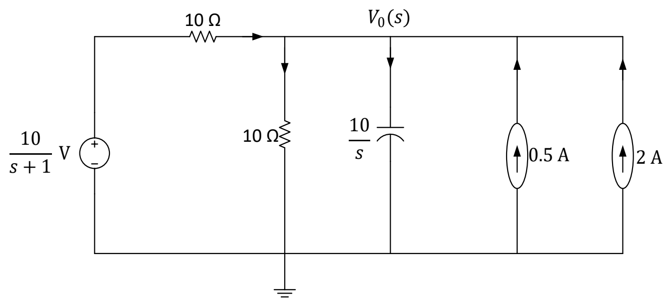


Figure 6.6

Nodal analysis of the equivalent of the circuit in Fig 6.5

Multiplying through by 10,

$$\frac{10}{s+1} + 25 = V_o(s+2)$$

Or

$$V_o = \frac{25s+35}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

Where

$$A = (s+1)V_o(s)|_{s=-1} = \left.\frac{25s+35}{(s+2)}\right|_{s=-1} = \frac{10}{1} = 10$$

$$B = (s+2)V_o(s)|_{s=-2} = \left.\frac{25s+35}{(s+1)}\right|_{s=-2} = \frac{-15}{-1} = 15$$

Thus,

$$V_o(s) = \frac{10}{s+1} + \frac{15}{s+2}$$

Taking the inverse Laplace transform, we obtain

$$v_o(t) = (10e^{-t} + 15e^{-2t})u(t)\text{ V}$$

Example 6.3: In the circuit of Fig. 6. 7(a), the switch moves from position a to position b at $t = 0$. Find $i(t)$ for $t > 0$.

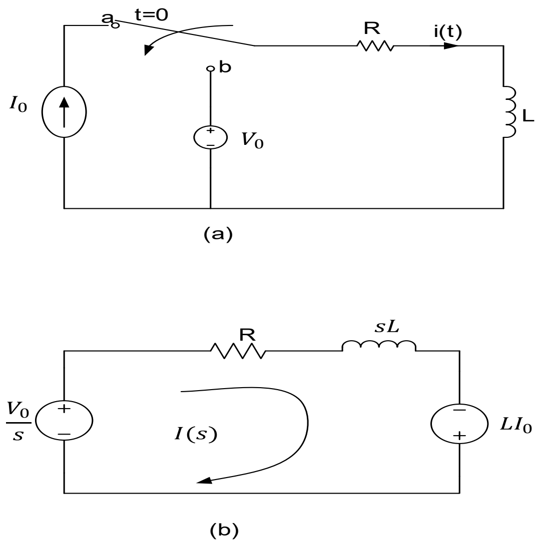


Figure 6.7

Solution:

The initial current through the inductor is $i(0) = I_o$. For $t > 0$, Fig. 6.7(b) shows the circuit transformed to the s-domain. The initial condition is incorporated in the form of a voltage source as $Li(0) = LI_o$. Using mesh analysis,

$$I(s)(R + sL) - LI_o - \frac{V_s}{s} = 0 \qquad 6.17$$

Or

$$I(s) = \frac{LI_o}{R + sL} + \frac{V_o}{s(R + sL)} = \frac{I_o}{s + \frac{R}{L}} + \frac{\frac{V_o}{R}}{s\left(s + \frac{R}{L}\right)} \qquad 6.18$$

Applying partial fraction expansion on the second term on the right hand side of Eq. (6.18) yields

$$I(s) = \frac{I_o}{s + \frac{R}{L}} + \frac{\frac{V_o}{R}}{s} - \frac{\frac{V_o}{R}}{\left(s + \frac{R}{L}\right)} \qquad 6.19$$

The inverse Laplace transform of this gives

$$i(t) = \left(I_o - \frac{V_o}{R}\right)e^{-\frac{t}{\tau}} + \frac{V_o}{R} \quad t \geq 0 \qquad 6.20$$

Where $\tau = \frac{R}{L}$. The term in parentheses is the transient response, while the second term is the steady-state response. In other words, the final value is $i(\infty) = \frac{V_o}{R}$, which we could have predicted by applying the final-value theorem on Eq. (6.18) or (6.19); that is,

$$\lim_{s \to 0} sI(s) = \lim_{s \to 0}\left[\frac{sI_o}{s + \frac{R}{L}} + \frac{\frac{V_o}{L}}{s + \frac{R}{L}}\right] = \frac{V_o}{R} \qquad 6.21$$

Equation (6.20) may also be written as

$$i(t) = I_o e^{-\frac{t}{\tau}} + \frac{V_o}{R}\left(1 - e^{-\frac{t}{\tau}}\right) \quad t \geq 0 \qquad 6.22$$

The first term is the natural response, while the second term is the forced response. If the initial condition $I_o = 0$, Eq. (6.22) becomes

$$i(t) = \frac{V_o}{R}\left(1 - e^{-\frac{t}{\tau}}\right), \quad t \geq 0 \qquad 6.23$$

which is the step response, since it is due to the step input Vo with no initial energy.

6.2 Circuit Analysis

Circuit analysis is again relatively easy to do when we are in the s-domain. We merely need to transform a complicated set of mathematical relationships in the time domain into the s-domain where we convert operators (derivatives and integrals) into simple multipliers of s and 1/s. This now allows us to use algebra to set up and solve our circuit equations. The exciting thing about this is that all of the circuit theorems and relationships we developed for dc circuits are perfectly valid in the s-domain.

Remember, equivalent circuits, with capacitors and inductors, only exist in the s-domain; they cannot be transformed back into the time domain.

Example 6.4: Consider the circuit in Fig. 6.8(a). Find the value of the voltage across the capacitor assuming that the value of $v_s(t) = 10u(t)$ V and assume that at $t = 0, -1$ A flows through the inductor and $+5$ V is across the capacitor.

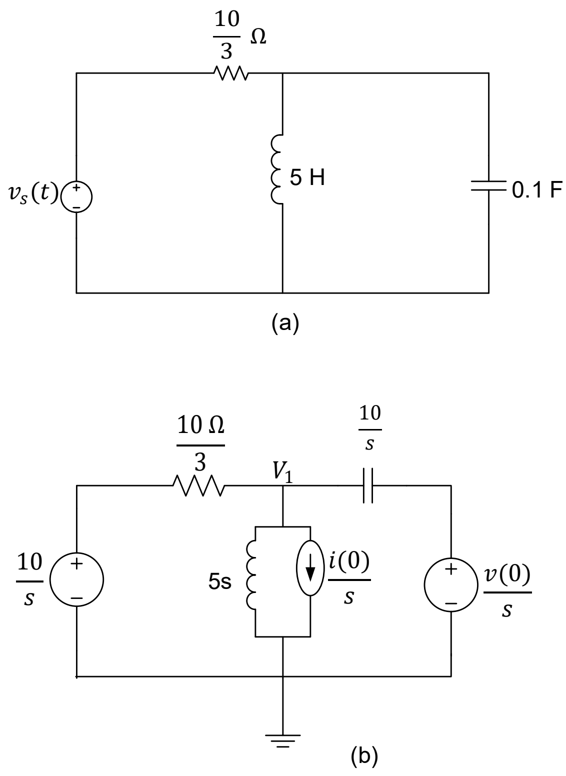


Figure 6.8

Solution:

Fig. 6.8(b) represents the entire circuit in the s-domain with the initial conditions incorporated. We now have a straightforward nodal analysis problem. Since the value of V_1 is also the value of the capacitor voltage in the time domain and is the only unknown node voltage, we only need to write one equation.

$$\frac{V_1 - \frac{10}{s}}{\frac{10}{3}} + \frac{V_1 - 0}{5s} + \frac{i(0)}{s} + \frac{V_1 - \left[\frac{v(0)}{s}\right]}{\frac{1}{(0.1s)}} = 0 \qquad 6.24$$

Or

$$0.1\left(s + 3 + \frac{2}{s}\right)V_1 = \frac{3}{s} + \frac{1}{s} + 0.5 \qquad 6.25$$

Where $v(0) = 5$ V and $i(0) = -1$ A. Simplify we get

$$(s^2 + 3s + 2)V_1 = 40 + 5s$$

Or

$$V_1 = \frac{40 + 5s}{(s+1)(s+2)} = \frac{35}{s+1} - \frac{30}{s+2} \qquad 6.26$$

Taking the inverse Laplace transform yields

$$v_1(t) = (35e^{-t} - 30e^{-2t})u(t)\ \text{V} \qquad 6.27$$

Example 6.5: For the circuit shown in Fig. 6.8, and the initial conditions used in **Example 6.4**, use superposition to find the value of the capacitor voltage.

Solution:

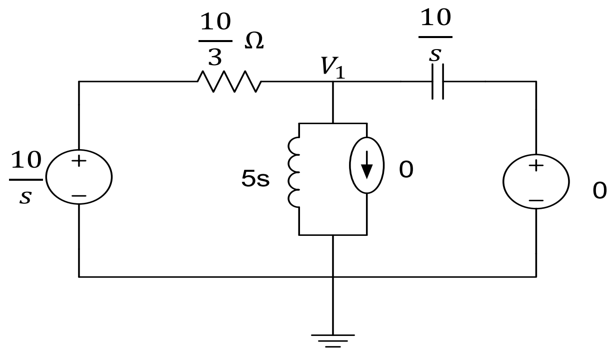


Figure 6.9a

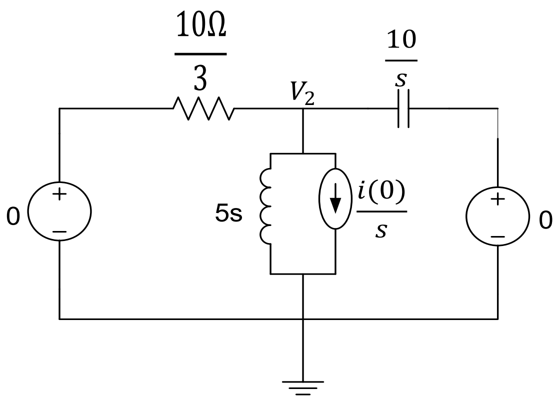


Figure 6.9b

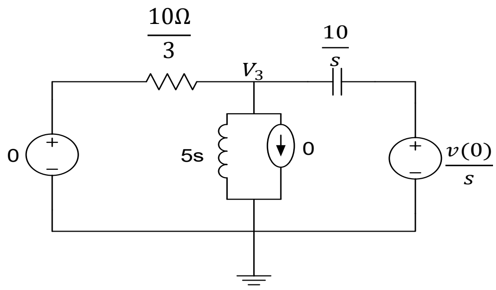


Figure 6.9c

Since the circuit in the s-domain actually has three independent sources, we can look at the solution one source at a time. Fig. 6.9 presents the circuits in the s-domain considering one source at a time. We now have three nodal analysis problems. First, let us solve for the capacitor voltage in the circuit shown in Fig. 6.9(a). (a)

$$\frac{V_1 - \frac{10}{s}}{\frac{10}{3}} + \frac{V_1 - 0}{5s} + 0 + \frac{V_1 - 0}{\frac{1}{(0.1s)}} = 0 \qquad 6.28$$

Or

$$0.1\left(s + 3 + \frac{2}{s}\right) V_1 = \frac{3}{s}$$

Simplifying we get

$$(s^2 + 3s + 2)V_1 = 30$$

$$V_1 = \frac{30}{(s+1)(s+2)} = \frac{30}{s+1} - \frac{30}{s+2} \qquad 6.28.1$$

Or

$$v_1(t) = (30e^{-t} - 30e^{-2t})u(t)\ \mathrm{V}$$

For Fig. 6.9(b) we get,

$$\frac{V_2 - 0}{\frac{10}{3}} + \frac{V_2 - 0}{5s} - \frac{1}{s} + \frac{V_2 - 0}{\frac{1}{(0.1s)}} = 0$$

Or

$$0.1\left(s + 3 + \frac{2}{s}\right)V_2 = \frac{1}{s}$$

This leads to

$$V_2 = \frac{10}{(s+1)(s+2)} = \frac{10}{s+1} - \frac{10}{s+2}$$

Taking the inverse Laplace transform, we get

$$v_2(t) = (10e^{-t} - 10e^{-2t})u(t)\ \mathrm{V} \qquad 6.28.2$$

For Fig. 6.9(c),

$$\frac{V_3 - 0}{\frac{10}{3}} + \frac{V_3 - 0}{5s} - 0 + \left(V_3 - \frac{5}{s}\right) \div \frac{1}{(0.1s)} = 0$$

Or

$$0.1\left(s + 3 + \frac{2}{s}\right)V_3 = 0.5$$

$$V_3 = \frac{5s}{(s+1)(s+2)} = \frac{-5}{s+1} + \frac{10}{s+2}$$

This leads to

$$v_3(t) = (-5e^{-t} + 10e^{-2t})u(t)\ \mathrm{V} \qquad 6.28.3$$

Now all we need to do is to add Eqs. (6.28.1), (6.28.2), and (6.28.3):

$$v(t) = v_1(t) + v_2(t) + v_3(t)$$

$$= \{(30 + 10 - 5)e^{-t} + (-30 + 10 - 10)e^{-2t}\}u(t)\ \mathrm{V}$$

Or

$$v(t) = (35e^{-t} - 30e^{-2t})u(t)\ \mathrm{V}$$

Example 6.6: Assume that there is no initial energy stored in the circuit of Fig 6.10 at $t = 0$ and that $i_s = 10u(t)$ A. (a) Find $V_o(s)$ using Thevenin's theorem. (b) Apply the initial and final value theorems to find $v_o(0^+)$ and $v_o(\infty)$. (c) Determine $v_o(t)$

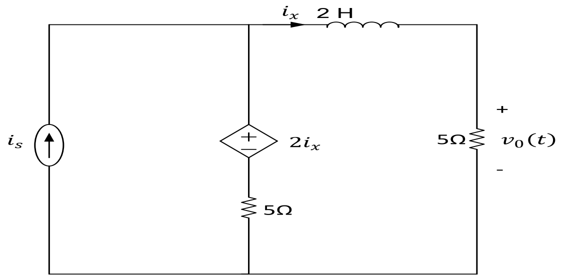


Figure 6.10

Solution:

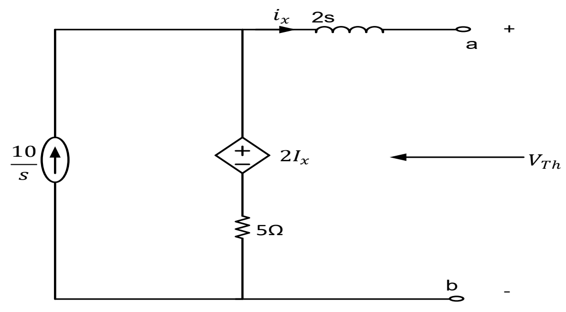
Since there is no initial energy stored in the circuit, we assume that the initial inductor current and initial capacitor voltage are zero at $t = 0$

(a) To find the Thevenin equivalent circuit, we remove the 5 Ω resistor and then find V_{oc} (V_{Th}) and I_{sc}. To find V_{Th}, we use the Laplace transformed circuit in Fig 6.11(a). Since $I_x = 0$, the dependent voltage source conditions no effect, so

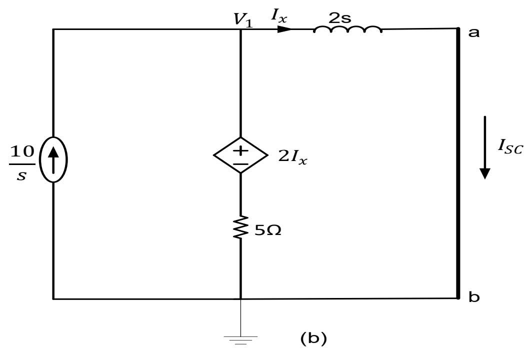
$$V_{oc} = V_{Th} = 5\left(\frac{10}{s}\right) = \frac{50}{s}$$

To find V_{Th}, we consider the circuit in Fig. 6.11(b), where we first find I_{sc}. We can use nodal analysis to solve for V_1 which then leads to $\left(I_{sc} = I_x = \frac{V_1}{2s}\right)$

$$-\frac{10}{s} + \frac{(V_1 - 2I_x) - 0}{5} + \frac{V_1 - 0}{2s} = 0$$



(a)



(b)

Figure 6.11 (a) Finding V_{TH}, (b) Determining Z_{TH}

Along with

$$I_x = \frac{V_1}{2s}$$

leads to

$$V_1 = \frac{100}{2s + 3}$$

Hence,

$$I_{sc} = \frac{V_1}{2s} = \frac{\frac{100}{(2s+3)}}{2s} = \frac{50}{s(2s+3)}$$

and

$$Z_{Th} = \frac{V_{oc}}{I_{sc}} = \frac{\frac{50}{s}}{\frac{50}{[s(2s+3)]}} = 2s + 3$$

The given circuit is replaced by its Thevenin equivalent at terminals a-b as shown in Fig. 6.12. From Fig. 6.12,

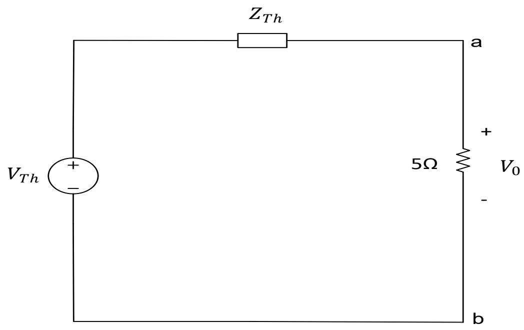


Figure 6.12 The Thevenin Equivalent of the circuit in Fig. 6.10 in s-domain

$$V_o = \frac{5}{5 + Z_{Th}} \times V_{Th} = \frac{5}{5 + 2s + 3} \times \left(\frac{50}{s}\right) = \frac{250}{s(2s + 8)} = \frac{125}{s(s + 4)}$$

(b) Using the initial-value theorem we find

$$v_o(0) = \lim_{s \to \infty} sV_o(s) = \lim_{s \to \infty} \left[\frac{125}{s + 4}\right] = \lim_{s \to \infty} \left[\frac{\frac{125}{s}}{1 + \frac{4}{s}}\right] = \frac{0}{1} = 0$$

Using the final-value theorem we find

$$v_o(\infty) = \lim_{s \to 0} sV_o(s) = \lim_{s \to 0} \left[\frac{125}{s + 4}\right] = \frac{125}{4} = 31.25 \text{ V}$$

(c) By partial fraction,

$$V_o = \frac{125}{s(s + 4)} = \frac{A}{s} + \frac{B}{s + 4}$$

$$A = sV_o(s)|_{s=0} = \left.\frac{125}{s + 4}\right|_{s=0} = 31.25$$

$$B = (s + 4)sV_o(s)|_{s=-4} = \left.\frac{125}{s}\right|_{s=-4} = -31.25$$

$$V_o = \frac{31.25}{s} - \frac{31.25}{s + 4}$$

Taking the inverse Laplace transform gives

$$v_o(t) = 31.25(1 - e^{-4t})u(t)\text{ V}$$

Notice that the values of $v_o(0)$ and $v_o(\infty)$ obtained in part (b) are confirmed.

6.3 Transfer Functions

The transfer function is a key concept in signal processing because it indicates how a signal is processed as it passes through a network. It is a fitting tool for finding the network response, determining (or designing for) network stability, and network synthesis. The transfer function of a network describes how the output behaves with respect to the input. It specifies the transfer from the input to the output in the s-domain, assuming no initial energy.

The transfer function H(s) is the ratio of the output response Y(s) to the input excitation X(s), assuming all initial conditions are zero.

Thus,

$$\boxed{H(s) = \frac{Y(s)}{X(s)}} \qquad 6.29$$

The transfer function depends on what we define as input and output. Since the input and output can be either current or voltage at any place in the circuit, there are four possible transfer functions:

$$H(s) = \text{Voltage gain} = \frac{V_o(s)}{V_i(s)} \qquad 6.30a$$

$$H(s) = \text{Current gain} = \frac{I_o(s)}{I_i(s)} \qquad 6.30b$$

$$H(s) = \text{Impedance} = \frac{\text{V(s)}}{I(s)} \qquad 6.30c$$

$$H(s) = \text{Admittance} = \frac{\text{I(s)}}{V(s)} \qquad 6.30d$$

Thus, a circuit can have many transfer functions. Note that H(s) is dimensionless in Eqs. (6.30a) and (6.30b).

Each of the transfer functions in Eq. (6.30) can be found in two ways. One way is to assume any convenient input X(s), use any circuit analysis technique (such as current or voltage division, nodal or mesh analysis) to find the output Y(s), and then obtain the ratio of the two. The other approach is to apply the ladder method, which involves walking our way through the circuit. By this approach, we assume that the output is IV as appropriate and use the basic laws of Ohm and Kirchhoff (KCL only) to obtain the input. The transfer function becomes unity divided by the input. This approach may be more convenient to use when the circuit has many loops or nodes so that applying nodal or mesh analysis becomes cumbersome. In the first method, we assume an input and find the output; in the second method, we assume the output and find the input. In both methods, we calculate H(s) as the ratio of output to input transforms. The two methods rely on the linearity property, since we only deal with linear circuits in this book. **Example 6.8** illustrates these methods.

Eq. (6.29) assumes that both X(s) and Y(s) are known. Sometimes, we know the input X(s) and the transfer function H(s). We find the output Y(s) as

$$Y(s) = H(s)X(s) \qquad 6.31$$

and take the inverse transform to get $y(t)$. A special case is when the input is the unit impulse function, $x(t) = \delta(t)$ so that $X(s) = 1$. For this case

$$Y(s) = H(s) \qquad \text{or} \qquad y(t) = h(t) \qquad 6.32$$

where

$$h(t) = \mathcal{L}^{-1}[H(s)] \qquad 6.33$$

The term $h(t)$ represents the unit impulse response—it is the time-domain response of the network to a unit impulse. Thus, Eq. (6.33) provides a new interpretation for the transfer function: $H(s)$ is the Laplace transform of the unit impulse response of the network. Once we know the impulse response $h(t)$ of a network, we can obtain the response of the network to any input signal using Eq. (6.31) in the s-domain or using the convolution integral in the time domain.

Example 6.7: The output of a linear system is $y(t) = 10e^{-t}\cos 4t\, u(t)$ when the input is $x(t) = e^{-t}u(t)$. Find the transfer function of the system and its impulse response.

Solution: If $x(t) = e^{-t}u(t)$ and $y(t) = 10e^{-t}\cos 4t\, u(t)$, then

$$X(s) = \frac{1}{s+1} \quad and \quad Y(s) = \frac{10(s+1)}{(s+1)^2 + 4^2}$$

Hence,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{10(s+1)^2}{(s+1)^2 + 16} = \frac{10(s^2 + 2s + 1)}{s^2 + 2s + 17}$$

To find $h(t)$, we write $H(s)$ as

$$H(s) = A + B \times \frac{4}{(s+1)^2 + 4^2} = \frac{10(s^2 + 2s + 1)}{s^2 + 2s + 17}$$

$$\Rightarrow A[(s+1)^2+4^2]+4B = A(s^2+2s+17)+4B \equiv 10(s^2+2s+1)$$

Constants: $17A + 4B = 10$; and Coefficients of s^2: $A = 10$

$$B = \frac{10-170}{4} = -40 \quad \Rightarrow H(s) = 10 - 40 \times \frac{4}{(s+1)^2+4^2}$$

From Table 1.2, we obtain $h(t) = 10\delta(t) - 40e^{-t}\sin 4tu(t)$

Example 6.8: Determine the transfer function $H(s) = \frac{V_o(s)}{I_o(s)}$ of the circuit in Fig. 6.13.

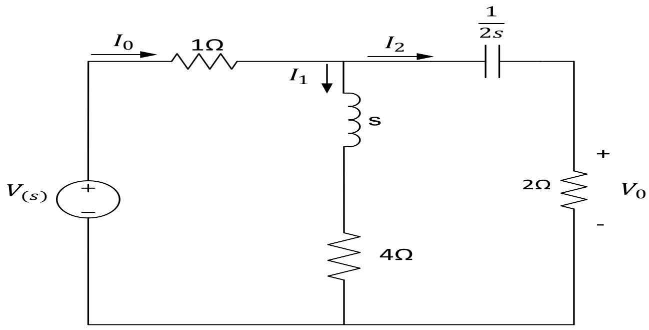


Figure 6.13

Solution

METHOD 1 By current division

$$I_2 = \frac{(s+4)I_o}{s+4+2+\frac{1}{2s}}$$

But

$$V_o = 2I_2 = \frac{2(s+4)I_o}{s+6+\frac{1}{2s}}$$

Hence,

$$H(s) = \frac{V_o(s)}{I_o(s)} = \frac{4s(s+4)}{2s^2+12s+1}$$

METHOD 2 We can apply the ladder method. We let $V_o = 1$ V. By Ohm's law

$I_2 = \frac{V_o}{2} = \frac{1}{2}$A. The voltage across the $\left(2+\frac{1}{2s}\right)$ impedance is

$$V_1 = I_2\left(2 + \frac{1}{2s}\right) = 1 + \frac{1}{4s} = \frac{4s+1}{4s}$$

This is the same as the voltage across the $(s+4)$ impedance. Hence,

$$I_1 = \frac{V_1}{s+4} = \frac{4s+1}{4s(s+4)}$$

Applying KCL at the top node yields

$$I_o = I_1 + I_2 = \frac{4s+1}{4s(s+4)} + \frac{1}{2} = \frac{2s^2+12s+1}{4s(s+4)}$$

Hence,

$$H(s) = \frac{V_o}{I_o} = \frac{1}{I_o} = \frac{4s(s+4)}{2s^2+12s+1}$$

As before

Example 6.9: For the s-domain circuit in Fig. 6.14, find: (a) the transfer function $H(s) = V_o/V_i$ (b) the impulse response, (c) the response when $v_i(t) = u(t)$ V, (d) the response when $v_i(t) = 8\cos 2t$ V.

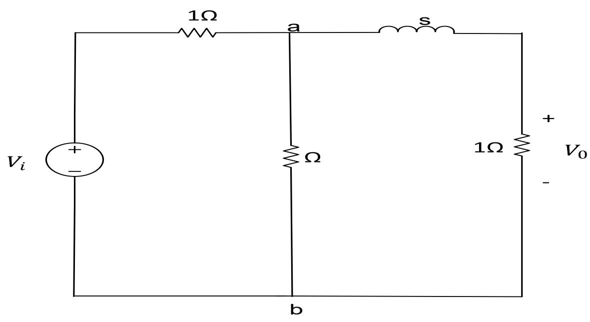


Figure 6.14

Solution

(a) Using voltage division,

$$V_o = \frac{1}{s+1}V_{ab} \qquad 6.32$$

But

$$V_{ab} = \frac{1 \parallel (s+1)}{1 + 1 \parallel (s+1)} \times V_i = \frac{\dfrac{(s+1)}{(s+2)}}{1 + \dfrac{(s+1)}{(s+2)}} \times V_i$$

Or

$$V_{ab} = \frac{s+1}{2s+3} V_i \qquad 6.32.1$$

Substituting Eq. (6.32.1) into Eq. (6.32) results in

$$V_o = \frac{V_i}{2s+3}$$

Thus, the transfer function is

$$H(s) = \frac{V_o}{V_i} = \frac{1}{2s+3}$$

(b) We may write $H(s)$ as

$$H(s) = \frac{\frac{1}{2}}{s + \frac{3}{2}}$$

Its inverse Laplace transform is the required impulse response:

$$h(t) = \frac{1}{2} e^{-\frac{3}{2}t} u(t)$$

(c) When $v_i(t) = u(t), V_i(s) = \frac{1}{s}$, and

$$V_o(s) = H(s)V_i(s) = \frac{1}{2s\left(s + \frac{3}{2}\right)} = \frac{A}{s} + \frac{B}{s + \frac{3}{2}}$$

Where

$$A = sV_o(s)|_{s=0} = \left.\frac{1}{2\left(s + \frac{3}{2}\right)}\right|_{s=0} = \frac{1}{3}$$

$$B = \left(s + \frac{3}{2}\right) V_o(s)\Big|_{s=-\frac{3}{2}} = \left.\frac{1}{2s}\right|_{s=-\frac{3}{2}} = -\frac{1}{3}$$

Hence, for $v_i(t) = u(t)$,

$$V_o(s) = \frac{1}{3}\left(\frac{1}{s} - \frac{1}{s+\frac{3}{2}}\right)$$

and its inverse Laplace transform is

$$v_o(t) = \frac{1}{3}\left(1 - e^{-\frac{3}{2}t}\right)u(t)\ \mathrm{V}$$

(d) When $v_i(t) = 8\cos 2t$, then $V_i(s) = \frac{8s}{s^2+4}$ and

$$V_o(s) = H(s)V_i(s) = \frac{4s}{\left(s+\frac{3}{2}\right)(s^2+4)}$$

$$= \frac{A}{s+\frac{3}{2}} + \frac{Bs+C}{s^2+4} \qquad 6.32.2$$

Where

$$A = \left(s+\frac{3}{2}\right)V_o(s)\Big|_{s=-\frac{3}{2}} = \frac{4s}{s^2+4}\Big|_{s=-\frac{3}{2}} = -\frac{24}{25}$$

To get B and C, we multiply Eq. (6.32.2) by $\left(s+\frac{3}{2}\right)(s^2+4)$. We get

$$4s = A(s^2+4) + B\left(s^2+\frac{3}{2}s\right) + C\left(s+\frac{3}{2}\right)$$

Equating coefficients,

$Constants$: $\quad 0 = 4A + \frac{3}{2}C \quad \Rightarrow \quad C = -\frac{8}{3}A$

s: $\quad 4 = \frac{3}{2}B + C$

s^2: $\quad 0 = A + B \quad \Rightarrow B = -A$

Solving these gives $A = -\frac{24}{25}, B = \frac{24}{25}, C = \frac{64}{25}$. Hence, for $v_i(t) = 8\cos 2t\ \ \mathrm{V}$

$$V_o(s) = -\frac{24}{25}\times\frac{1}{s+\frac{3}{2}} + \frac{24}{25}\times\frac{s}{s^2+4} + \frac{32}{25}\times\frac{2}{s^2+4}$$

and its inverse is

$$v_o(t) = \frac{24}{25}\left(-e^{-\frac{3t}{2}} + \cos 2t + \frac{4}{3}\sin 2t\right)u(t)\ \mathrm{V}$$

6.4 Exercise

1. Determine $v_o(t)$ in the circuit of Fig 6.15, assuming zero initial conditions

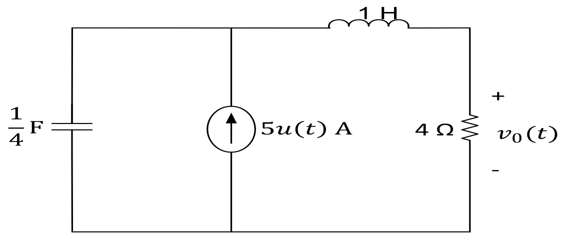


Figure 6.15

Answer: $20(1 - e^{-2t} - 2te^{-2t})u(t)$ V

2. Find $v_o(t)$ in the circuit shown in Fig. 6.16. Note that, since the voltage input is multiplied by $u(t)$, the voltage source is a short for all $t < 0$ and $i_L(0) = 0$.

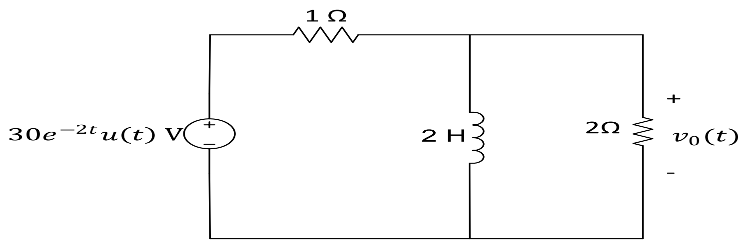


Figure 6.16

Answer: $\left(24e^{-2t} - 4e^{-\frac{t}{3}}\right)u(t)$ V

3. The switch in Fig. 6.17 has been in position b for a long time. It is moved to position a at $t = 0$. Determine $v(t)$ for $t > 0$.

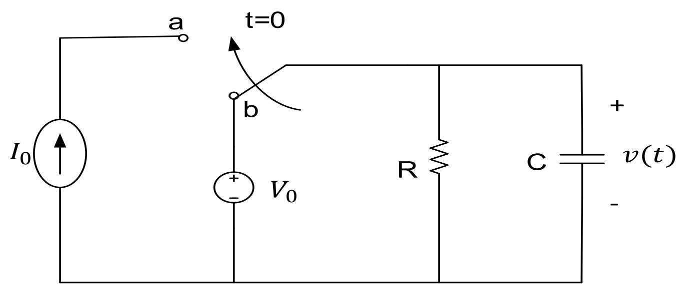


Figure 6.17

Answer: $v(t) = (V_o - I_oR)e^{-\frac{t}{\tau}} + I_oR, t > 0$, where $\tau = RC$

4. For the circuit shown in Fig. 6.17 with the same initial conditions, find the current through the inductor for all time $t > 0$.

Ans: $i(t) = (3 - 7e^{-t} + 3e^{-2t})u(t)$ A

5. For circuit shown in Fig. 6.18: The initial energy in the circuit of Fig. 6.13 is zero at $t = 0$. Assume that $v_s = 15u(t)$ V.

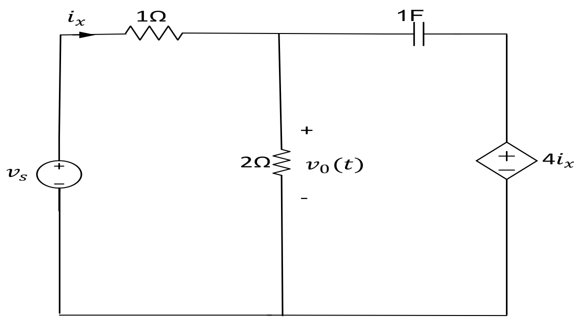


Figure 6.18

(a) Find $V_o(s)$ using the Thevenin theorem.
(b) Apply the initial- and final-value theorems to find $v_o(0)$ and $v_o(\infty)$
(c) Obtain $v_o(t)$

Ans: (a) $V_o(s) = \frac{12(s+0.25)}{s(s+0.2)}$, (b) 12 V, 10 V

6. Rework **Example 6.9** for the circuit shown in Fig. 6.19.

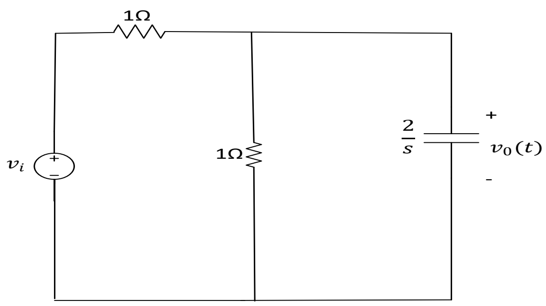


Figure 6.19

Answer:

(a) $\frac{2}{(s+4)}$ (b) $2e^{-4t}u(t)$, (c) $\frac{1}{2}(1-e^{-4t})u(t)$ V

(d) $3.2\left(-e^{-4t}+\cos 2t+\frac{1}{2}\sin 2t\right)u(t)$ V

7. The open circuit voltage ratio $\frac{V_2(s)}{V_1(s)}$ of the network shown in Fig. 6.20 is?

(a) $1+2s^2$

(b) $\frac{1}{(1+s^2)}$

(c) $1+2s$

(d) $\frac{1}{(1+2s)}$

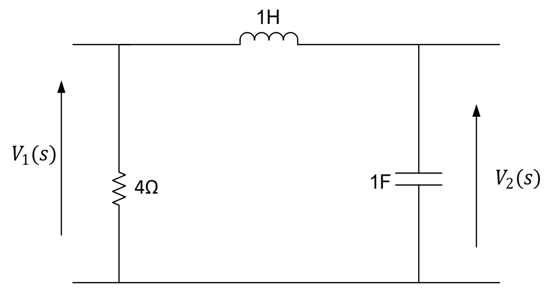


Figure 6.20

8. The transfer function of a linear system is

$$H(s)=\frac{2s}{s+6}$$

Find the output $y(t)$ due to the input $5e^{-3t}u(t)$ and its impulse response.

Answer: $-10e^{-3t}+20e^{-6t},\ t\geq 2, 2\delta(t)-12e^{-6t}u(t)$

CHAPTER 7

TRANSMISSION LINES

7.0 Introduction

There are means of relaying signals (also power) from one point to another, usually a pair of electrical conductors, with coaxial cables and twisted pair cable being some of the examples. Having said this, I must point out that the lines are not merely "wire" or cables in their simplest form, but rather are intricate cascades of electrical circuits! Bearing in mind costs, convenience and ease of calculations that involve the properties of the transmission line, they are then arranged in definite geometric patterns.

The goal of the transmission is to transport a typical signal with minimal loss. Loss there must be when we're dealing with physical realities, but the idea behind any design is to minimize such.

Up to this point in your circuit theory series, we've dealt with the more familiar low-frequency circuit where the wires that connect devices are justifiably assumed to have zero resistance, and phase delays are absent across wires. Furthermore, short circuited lines always yield zero resistance. Not so in high frequency transmission lines where the above does not obtain and we have to expect the unexpected! For example, short circuits can actually possess infinite impedance, and open circuits (the idealized model of an infinite impedance) can actually behave like short circuited wires!

For low frequency signals and d.c signals, transportation normally involves very low losses, but high frequency ones in the range of radio waves, losses are quite pronounced and the objective of the design engineer is to eliminate or minimize such. So, here, attention is focused on high frequency applications whereby the length of the line is of at least the same order of magnitude as the least the same order of magnitude as the wavelength of the signal under consideration. This is strictly with regard to systems of conductors having a forward and return path.

Areas of application include communication engineering where study is made to determine the most efficient use of power and equipment available to transfer for example, as much power as possible from the feeder line into the antenna. To avoid power wastage, a receiving antenna must be correctly matched to the line that connects it to the receiver.

To eliminate losses, we resort to "matching" the line to the load, by making the factor known as the characteristic impedance of the line, designated Z_0, equal or very close to the load impedance (Z_L). In d.c and low frequency a.c circuits earlier referred to, the characteristic impedance of parallel wires is usually insignificant and can therefore be ignored in analyzing circuit behavior. Here the phase difference between the sending and the receiving and is negligible, the period of propagation is very small compared to the period of the waveform under consideration. It can be practically assumed that the voltage along all the respective points (of a low frequency, two conductor line) are equal and in-phase with each other at any given point in time.

An idealized transmission line has an "infinite" length, this way all the energy is absorbed and more is reflected back to the source, because the characteristic (natural) impedance of the line is now matched to the frictions load impedance (Z_L)

To investigate low voltage or current changes along transmission lines, the following assumptions are made and the following parameters must be borne in mind, so that circuit analysis can be employed.

The line is made up of continuous conductors with constant cross-sectional configuration, and therefore indicating even distribution of the parameters, the problem is tracked by considering a very short length of the line that would imply a very discreet distribution of the parameters. The problem is tackled by considering a very short length of the line that would imply discrete distribution of the **parameters** which are:

1. Resistance (R): The resistance of the conductors to the flow of current.
2. Inductance (L): Associated with the time varying signal, and depends on the geometry of the cross-section of the conductors.
3. Conductance (G): Leakage current passes through the dielectric material that holds the line in position.
4. Capacitance (C): A capacitive reactance to a time-varying signal due to capacitor form from conductors and the dielectric in-between.

So, for a two-wire line, we deal with series inductance and resistance, and parallel (shunt) capacitance and conductance, because any conductor (coil) possess "natural" resistance and there is always capacitance formed wherever two conductors come close to each other!

The totality of these parameters is obtained by multiplying by the length of the line, since they are given on a per-length basis. Continuous distribution is approximated by its representation as a cascade of network of elements, with each element of length δz, (delta z).

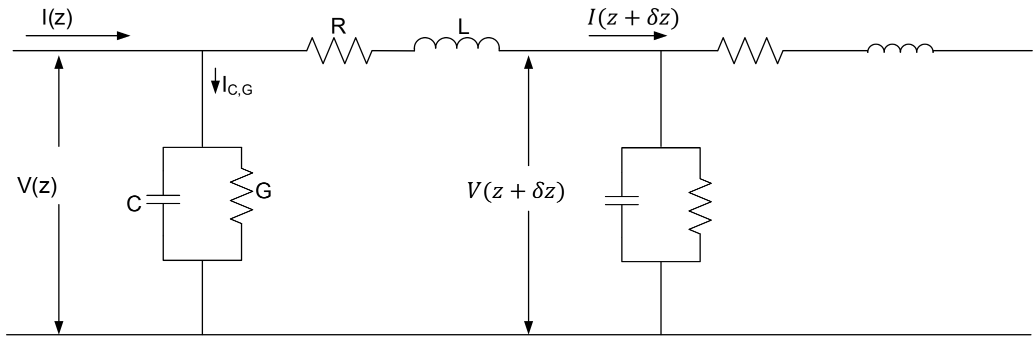


Figure 7.1 A 2-cascade representation of transmission line

Using telegrapher's equation

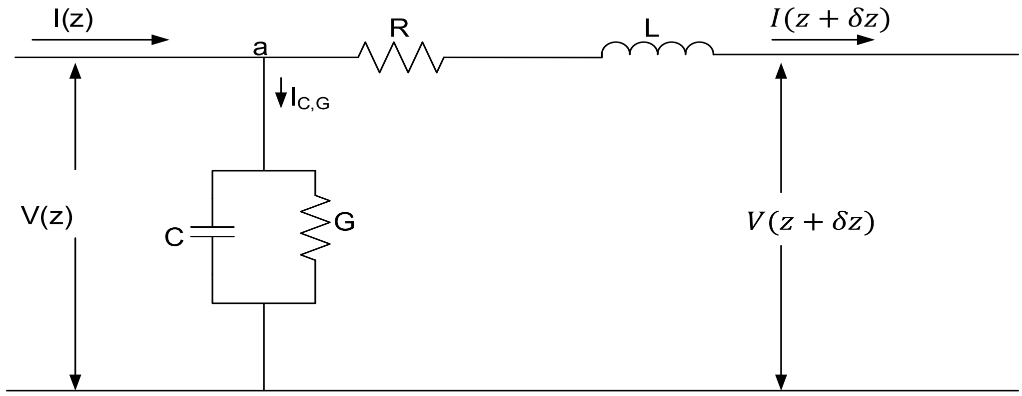


Figure 7.1a One section of the transmission line

To use telegrapher's equation, we have to consider one section of the transmission line as in Fig. 7.1a for the derivation of the characteristic impedance. The voltage on the left would be V and on the right side would be $V(z+\delta z)$. Fig. 7.1a is to be used for both the derivation methods.

The differential equations describing the dependence of the voltage and current on time and space are linear, so that a linear combination of solutions is again a solution. This means that we can consider solutions with a time dependence and the time dependence will factor out, leaving an ordinary differential equation for the coefficients, which will be phasors depending on space only. Moreover, the parameters can be generalized to be frequency-dependent.

Taking KCL at point (a) of Fig. 7.1a, the current through the parallel combination of the capacitance and admittance elements is:

$$I_{CG} = I(z) - I(z+\delta z) = C\delta z \frac{\partial V(z)}{\partial t} + G\delta z\, V(z),$$

with δz indicating per unit length basis, and with the partial derivatives noted. Voltage drops across the series combination of the resistor and inductor by KVL:

$$V_{RL} = V_R + V_L = [V(z) - V(z+\delta z)]$$

$$= R\delta z I\,(z+\delta z) + L\delta z \frac{\partial I(z+\partial z)}{\partial t}$$

Recall from first principles

$$\lim_{\delta x \to 0}\left[\frac{f(x+\delta x) - f(x)}{\delta x}\right] = \frac{df(x)}{dx}$$

So,
$$\lim_{\delta z \to 0}\left[\frac{I(z+\delta z) - I(z)}{\delta z}\right] = \frac{\partial I(z)}{\partial z}$$

So that,
$$I(z) - I(z+\delta z) \approx -\frac{\partial I(z)}{\partial z}\delta z$$

Similarly,
$$V(z) - V(z+\delta z) \approx -\frac{\partial V(z)}{\partial z}\delta z$$

$$\Longrightarrow \quad \frac{-\partial I(z)}{\partial z}\delta z = C\delta z \frac{\partial V(z)}{\partial t} + G\delta z V(z)$$

$$\frac{\partial I(z)}{\partial z} = -\left(G + C\frac{\partial}{\partial t}\right)V(z)$$

Similarly,
$$\frac{\partial V(z)}{\partial z} = -\left(R + L\frac{\partial}{\partial t}\right)I(z+\delta z)$$

$$\approx -\left(R + L\frac{\partial}{\partial t}\right)I(z) \qquad \text{for } \delta z \text{ small}$$

For sinusoidal signals, dependence on line is expressed by $e^{j\omega t}$ and derivative ∂t expressed by $j\omega$, $\left(\frac{d}{dt}e^{j\omega t} = j\omega e^{j\omega t}, \text{recall}\right)$, and partial derivatives then become total derivatives.

$$\frac{dI}{dz} = -(G + j\omega C)V \qquad 7.1$$

$$\frac{dV}{dz} = -(R + j\omega L)\, I \qquad 7.2$$

Taking the second derivatives of V, from (7.2),

$$\frac{d^2V}{dz^2} = -(R + j\omega L)\frac{dI}{dz} = (R + j\omega L)\,(G + j\omega C)V \ = \gamma^2 V \qquad 7.3$$

7.1 Propagation Constant $'\boldsymbol{\gamma}'$

Where γ^2 (gamma squared) $= (R + j\omega L)(G + j\omega C)$

Eq. (7.3) above has as its solution,

$$V = V_1 e^{-\gamma z} + V_2 e^{\gamma z} \qquad 7.4$$

Where $\qquad \gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \qquad 7.5$

In general, γ is a complex quality, and can therefore be represented by

$$\gamma = \alpha + j\beta$$

Substituting this is the expression for V,

$$V = V_1 e^{-(\alpha + j\beta)z} + V_2 e^{(\alpha + j\beta)z} \qquad 7.6$$

By a similar analysis, current is expressed with $I's$ replacing the $V's$ so, voltage at some point z down the transmission line is made up of two components, namely:

a. $V_1 e^{-(\alpha+j\beta)z} = V_1 e^{-\alpha z} e^{-j\beta z}$ whose amplitude decreases (is attenuated) as it travels down the line with z as $e^{-\alpha z}$, while $e^{-j\beta z}$ is just a phase term with no effect on the amplitude. Therefore, this component is known as the forward, or incident wave.

b. $V_2 e^{(\alpha+j\beta)z} = V_2 e^{\alpha z}\, e^{j\beta z}$ increases with increasing z, but since voltage must be attenuated as it travels along the line, z must then decrease to accommodate this fact, therefore making this component to be known as the backward, or reflected, wave, caused by a mismatch between the transmission line and the load.

So, the voltage at any point on the line a distance z from the sending end is the shin of the voltages of the incident and reflected waves at the said point.

Line parameters, α and β are determined by the **line characteristics**:

1. α is known as attenuation coefficient, and the negative/positive exponential of this is the rate at which the forward/backward wave is attenuated, and is a function of R, L, G and C, with the unit being $\mathrm{dB/m}$ (decibels per metre) or repers/m.

2. β is the phase constant and shows the phase dependence of both the incident and the reflected waves with distance z

$\beta\lambda = 2\pi \Longrightarrow \beta = \frac{2\pi}{\lambda}$, where λ (Greek alphabet lambda) is the signal wavelength.

3. γ (Gamma, Greek third alphabet) is the propagation constant, and is the complex sum of the attenuation coefficient and phase constant, where the former is the real part, and the letter the imaginary part. γ determines how the voltage (or by implication the current) along the line changes with z

7.2 Characteristic Impedance

From the Eq. (7.2), $\frac{dV}{dz} = -(R + j\omega L)I,$

$$I = -\frac{I}{R + j\omega L} \times \frac{dV}{dz}$$

Differentiating Eq. (7.4)

$$\frac{dV}{dz} = -\gamma V_1 e^{-\gamma z} + \gamma V_2 e^{\gamma z} = \gamma[V_2 e^{\gamma z} - V_1 e^{-\gamma z}]$$

And substituting in the above for

$$I = -\frac{1}{R + j\omega L} \times \gamma[V_2 e^{\gamma z} - V_1 e^{-\gamma z}] = \frac{\gamma}{R + j\omega L} \times [V_1 e^{-\gamma z} - V_2 e^{\gamma z}]$$

Substituting from Eq. (7.5) for γ,

$$\mathrm{I} = \frac{\sqrt{(R + j\omega L)(G + j\omega C}}{(R + j\omega L)} \times [V_1 e^{-\gamma z} - V_2 e^{\gamma z}]$$

$$\Longrightarrow \quad I = \sqrt{\left(\frac{G + j\omega C}{R + j\omega L}\right)} \times [V_1 e^{-\gamma z} - V_2 e^{\gamma z}] \qquad 7.7$$

By analogy with Ohm's law, $\frac{G+j\omega C}{R+j\omega L}$, is an admittance. Therefore, $\frac{R+j\omega L}{G+j\omega C}$ its reciprocal, is an impedance called the characteristic impedance of the transmission line, determined by the line parameters R, L, G & C.

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad 7.8$$

Characteristic impedance Z_o can be variously described as:

1. The value the load impedance must have to match the load to the line (to either eliminate power loss, or at least minimize same), or
2. The impedance seen from the sending end of an infinitely long line, or
3. The impedance seen looking towards the load at any point on a matched line, i.e., moving along the line produces no change in the impedance towards the load.

The transmission line is idealized as follows:

1. The line is uniform, straight and homogenous,
2. Line parameters R, L, G and C do not vary with atmospheric conditions like temperature and humidity.
3. Line parameters do not depend on frequency,
4. The analysis is applicable only between the junctions on the line because the circuit model on Fig. 7.2 (one of the cascades) is invalid across a junction

The above assumptions may be occasionally taken into consideration as we analyze transmission line.

7.3 Reflection from the Load

Shown in Fig. 7.2 where, $V_1e^{-\gamma l}$ is the incident wave, while $V_2e^{\gamma l}$ is the reflected or backward, wave on a line with total length of l. If the load has an impedance equal to the characteristic impedance Z_0, therefore say that the line is matched, and there is no reflected wave (theoretically speaking) as the incident wave is totally absorbed by the load. It, however, the load is of a value different from Z_0, then some of the incident wave would be reflected, and the amount of reflection by the load. Is expressed in terms of voltage reflection coefficient, designated by the Greek letter ρ (rho), and defined as the ratio of the reflected voltage to the incident voltage at the load terminals.

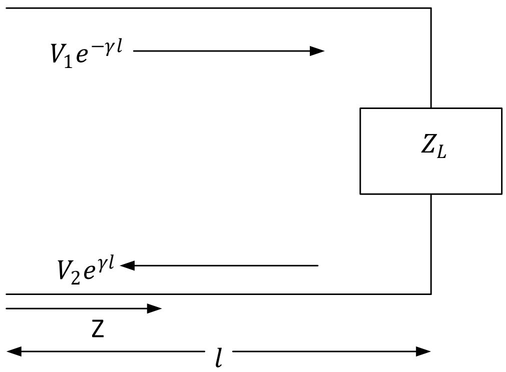


Figure 7.2 Incident wave $(V_1e^{-\gamma l})$ and Reflected wave $(V_2e^{\gamma l})$

Given that the load is at the position $z = l$,

$$V_L = V_1 e^{-\gamma l} + V_2 e^{\gamma l} \qquad 7.9$$

$$\rho = \frac{V_2 e^{\gamma l}}{V_1 e^{-\gamma l}} = \left(\frac{V_2}{V_1}\right) e^{2\gamma l} = |\rho| e^{j\psi} \qquad 7.10$$

Where the last indicates that ρ in general would be a complex quantity that can be expressed in polar form with $|\rho|$ as the magnitude and ψ as the phase angle of the reflection coefficient.

From Eqs. (7.7) and (7.8)

$$I_L = \left(\frac{V_1}{Z_0}\right) e^{-\gamma l} - \left(\frac{V_2}{Z_0}\right) e^{\gamma l} \qquad 7.11$$

Z_L (Load impedance)= $\frac{V_L}{I_L}$

And from Eqs. (7.9) and (7.11)

$$Z_L = \frac{V_L}{I_L} = \frac{V_1 e^{-\gamma l} + V_2 e^{\gamma l}}{\left[\left(\frac{V_1}{Z_0}\right) e^{-\gamma l} - \left(\frac{V_2}{Z_0}\right) e^{\gamma l}\right]}$$

Dividing through by $V_1 e^{-\gamma l}$ and multiplying by Z_0

$$Z_L = Z_0 \left(\frac{\left[1 + \left(\frac{V_2}{V_1}\right) e^{2\gamma l}\right]}{\left[1 - \left(\frac{V_2}{V_1}\right) e^{2\gamma l}\right]}\right)$$

The term in the (inner) parenthesis namely $(V_2/V_1)e^{2\gamma l}$, is simply the voltage reflection coefficient e, leading to

$Z_L = Z_0 \left(\frac{1+\rho}{1-\rho}\right)$, or rearrange

$$\rho_v = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad 7.12$$

For $Z_L = 0$ (indicating short circuit load),

$$\rho = -\frac{Z_0}{Z_0} = -1 \Longrightarrow |\rho| = 1, \qquad \mu = \pi$$

Note that $\mu \Longrightarrow \psi$ as in Eq 7.10. so, in place of ψ, we can use μ

For $Z_L = \infty$ (open circuit load):

$$\rho = \frac{Z_L}{Z_L} = 1 \Longrightarrow |\rho| = 1, \mu = 0$$

Example 7.1: If $Z_L = 75 + j50\ \Omega$, $Z_0 = 25\ \Omega$, find the reflected coefficient

$$\rho = \frac{(75 + j50 - 25)}{(75 + j50 + 25)} = \frac{50 + j50}{100 + j50} = \frac{1 + j}{2 + j}$$

$$= \frac{(1 + j)(2 - j)}{2^2 + 1} = \frac{3 + j}{5} = \frac{\sqrt{10}}{5} \angle \tan^{-1}\frac{1}{3} = 0.63\angle 18.43^{\circ}$$

Example 7.2: The lossless transmission line has characteristic impedance of 75 Ω and phase constant of 3rad/m at 100 MHz. Find inductance and capacitance of line/meter.

Solution: $Z_0 = \sqrt{\frac{L}{C}}$

$$\gamma = \beta = \omega\sqrt{LC}$$

$$\frac{Z_0}{\beta} = \frac{\sqrt{\frac{L}{C}}}{\omega\sqrt{LC}} = \frac{1}{\omega C}$$

$$\frac{75}{3} \times 2\pi f = \frac{1}{C}$$

$$\Longrightarrow \qquad 25 \times 2\pi f = \frac{1}{C}$$

$$\Longrightarrow \qquad C = \frac{1}{25 \times 6.28 \times 10^8}$$

$$\Longrightarrow \qquad C = 63.69\ \mathrm{pF/m}$$

$$Z_0{}^2 C = L$$

$$\Longrightarrow \qquad L = (75)^2 \times 63.69 \times 10^{-12} = 358\ \mathrm{nH/m}$$

Example 7.3: A lossless transmission is 80 cm long and operates at a frequency of 600 MHz the line parameters are $L = 0.25\ \mu\mathrm{H/m}$ and $C = 100\ \mathrm{pF/m}$. Find the characteristic impedance, the phase constant, and the phase velocity.

Solution:

Since the line is lossless, both R and G are zero. The characteristic impedance is

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.25 \times 10^{-6}}{100 \times 10^{-12}}} = 50\ \Omega$$

Since $$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$= j\omega\sqrt{LC}$$ we see that

$$\beta = \omega\sqrt{LC} = 2\pi\ (600 \times 10^{6})\ \sqrt{(0.25 \times 106^{-6})\ (100 \times 10^{-12})}$$

$$= 18.85\ \mathrm{rad/m}$$

Also,

$$\mathrm{V_p} = \frac{\omega}{\beta} = \frac{2\pi\ (600 \times 10^{6})}{18.85} = 2 \times 10^{8}\ \mathrm{m/s}$$

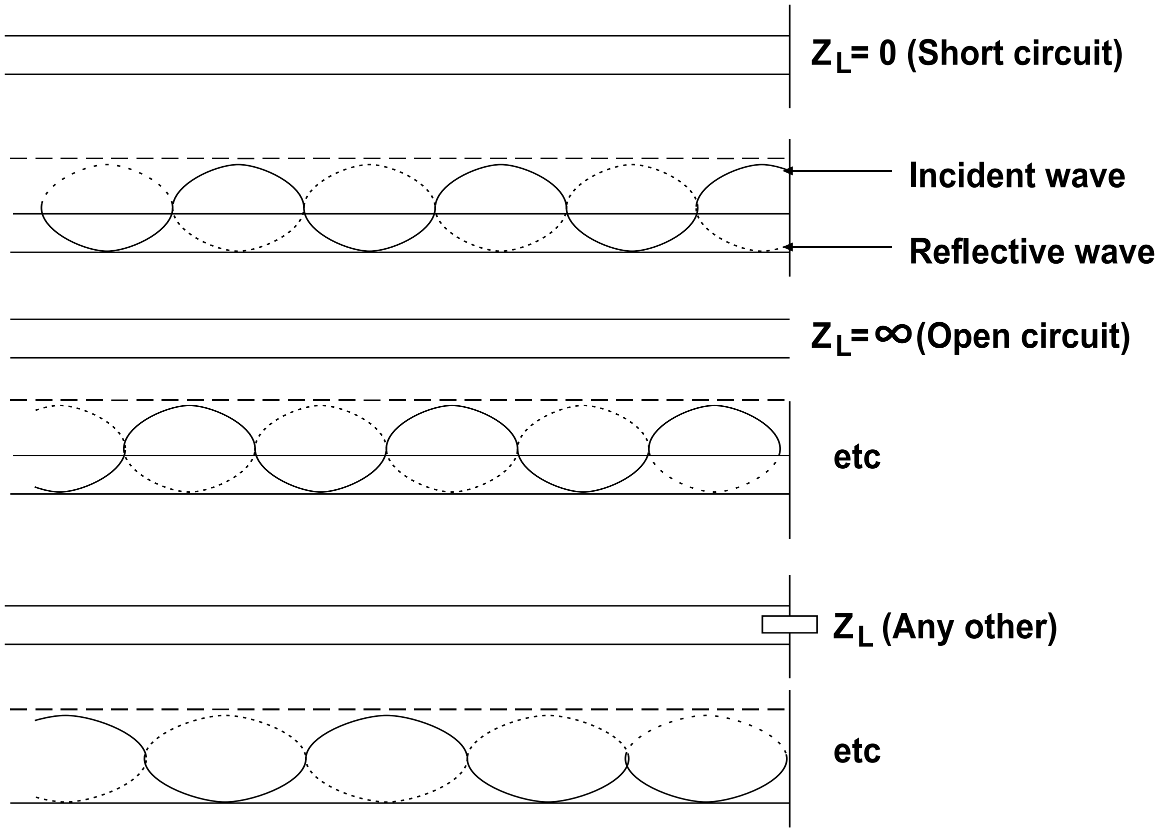


Figure 7.3 combination of 'Short circuit Impedance A ', 'Open circuit impedance B' and when the line impedance equals the load impedance C

By a similar analysis, current reflection coefficient is given by

$$\rho_I = \frac{(Z_0 - Z_L)}{Z_0 + Z_L} = -\rho_V$$

Where ρ_v stands for voltage reflection coefficient.

$$Z_L = 0 \Longrightarrow \rho_I = \frac{Z_0}{Z_0} = 1, \Longrightarrow |\rho| = 1, \mu = 0$$

Showing quality between the incident and reflected waves with no change in phase (with KCL taken at the no-load terminal).

$$Z_L = \infty \text{ (open circuit)} \Longrightarrow \rho_I = -\frac{Z_L}{Z_L} = -1 \Longrightarrow |\rho_I| = 1, \mu = \pi$$

7.4 Distortionless Line $\left(\frac{R}{L} = \frac{G}{C}\right)$

Distortionless line is the one in which attenuation constant 'α' is frequency independent while phase constant is linearly dependent on frequency.

(a) $$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)\left(\frac{RC}{L} + j\omega C\right)} \quad 7.13$$

$$= \sqrt{\frac{C}{L}}\,(R + j\omega L)$$

$$\Longrightarrow \quad \alpha = R\sqrt{\frac{C}{L}} \; and \;\; \beta = \omega\sqrt{LC} \quad 7.14$$

(b) $$V_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \quad 7.15$$

(c) $$Z_0 = R_0 + jX_0 = \sqrt{\frac{R\left(1 + \frac{j\omega L}{R}\right)}{G\left(1 + \frac{j\omega C}{G}\right)}} = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}} \quad 7.16$$

$$\Longrightarrow \quad R_0 = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}; \qquad X_0 = 0 \quad 7.17$$

A lossless line is also distortionless line, but a distortionless line is not necessarily lossless

Example 7.4: A 60 Ω distortionless transmission line has a capacitance of 0.15 $\mathrm{nF/m}$. The attenuation on the line is 0.01 dB/m. Calculate

a. the line parameters: resistance, inductance and conductance per meter of line
b. velocity of propagation
c. voltage at a distance of 1 km and 4 km with respect to sending end voltage.

Solution:
For a distortionless line,

$$\frac{R}{L} = \frac{G}{C}$$

$$Z_0 = R_0 = \sqrt{\frac{L}{C}} = 60\ \Omega$$

and

$$\alpha = R\sqrt{\frac{C}{L}} = 0.01\frac{\text{dB}}{\text{m}} = \frac{0.01}{8.69}\text{Np/m} = 1.15 \times 10^{-3}\ \text{Np/m}$$

Line parameters:

$$R = \alpha R_0 = (1.15 \times 10^{-3}) \times 60 = 0.069\ \Omega/\text{m}$$

$$L = CR_0^2 = 0.15 \times 10^{-9} \times 60^2 = 0.54\ \mu\text{H/m}$$

$$G = \frac{RC}{L} = \frac{R}{{R_0}^2} = \frac{0.059}{60^2} = 19.2\ \mu\text{S/m}$$

(b). $V = \frac{1}{\sqrt{LC}}$

$$= \frac{1}{\sqrt{0.54 \times 10^{-6} \times 0.15 \times 10^{-9}}} = 1.11 \times 10^8\ \text{m/s}$$

(c). The ratio of two voltages at a distance x apart along the line

$$\frac{V_2}{V_1} = e^{-\alpha x}$$

At 1 km

$$\frac{V_2}{V_1} = e^{-1000\alpha} = e^{-1.15} = 0.317\ or\ 31.7\%$$

At 4 km

$$\frac{V_2}{V_1} = e^{-4000\alpha} = e^{-4.6} = 0.01\ or\ 1\%$$

7.5 Low-Loss Dielectric

A low-loss dielectric is a good but imperfect insulator with a non-zero equivalent conductivity such that $t"\ \ll \in'$ or $\frac{\sigma}{\omega\in} \ll 1$. Under this condition γ can be approximated by using binomial expansion.

$$\gamma = \alpha + j\beta \equiv j\omega\sqrt{\mu\ \in'}\left[1 - \frac{j\ \in'}{2\ \in'} + \frac{1}{8}\left(\frac{\in"}{e'}\right)^2\right]$$

From which we can say

$$\alpha \cong \frac{\omega \in "}{2}\sqrt{\frac{\mu}{\in}}\left(\frac{N_p}{m}\right)\text{attenuation constant}$$

$$And \quad \beta \cong \omega\sqrt{\mu \in}\left[1+\frac{1}{8}\left(\frac{\epsilon"}{\epsilon'}\right)^2\right]\left(\frac{\text{rad}}{\text{m}}\right)\text{phase constant}$$

$$'\alpha'\text{for low} - \text{loss dielectric is a positive quantity and}$$
$$\text{is approximately directly proportional}$$
$$\text{to frequency. β deviates only very slightly from value } 2\sqrt{\mu \in}\ \text{ (lossless dielectric)}$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}}\left(1-j\,\frac{\in ''}{\in'}\right)^{-\frac{1}{2}}$$

$$\eta \simeq \sqrt{\frac{\mu}{\in'}}\left(1+j\frac{\in "}{2\in'}\right)\ (\Omega)\ \rightarrow \text{intrinsic impedance}$$

We can say that $\frac{E_x}{H_y} = \eta$ and here the electric and magnetic field intensities in lossy dielectric are not in time phase as in lossless medium.

$$V_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu \in'}}\left[1-\frac{1}{8}\left(\frac{\in "}{\in'}\right)^2\right]\text{m/s} \qquad \text{phase velocity}$$

7.6 Equivalent Circuit in Terms of Primary and Secondary Constants

Equivalent T-section of a line of length δ

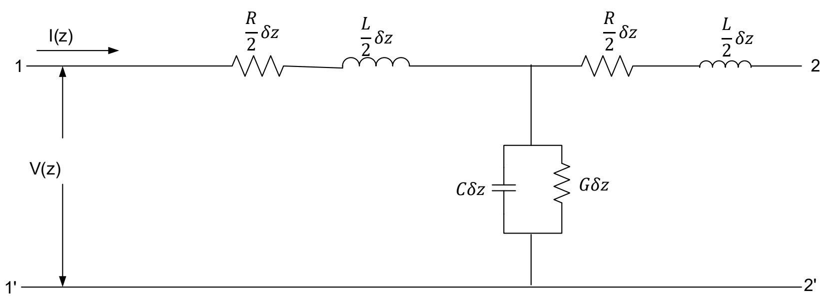


Figure 7.3 Equivalent $'T'$ Transmission Line Circuit

Equivalent π –section of a line of length δ

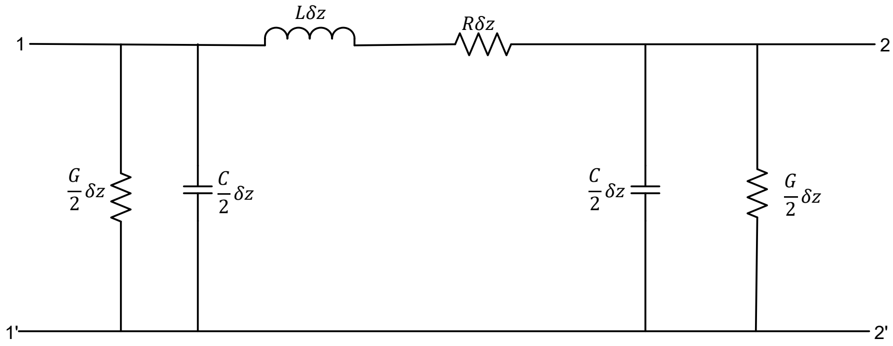


Figure 7.4 Equivalent $'\pi'$ Circuit

Here, $z = (R + j\omega L)\ \Omega; y = (G + j\omega C)\ \mho$

Secondary constants of line

a. The input impedance of line is called its characteristics impedance

$$Z_0 = \sqrt{\frac{z}{y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

b. $\gamma = \alpha + j\beta$ **(Propagation constant)**

- Real part α of γ is measured of charge in magnitude of current or voltage in each τ-section and called attenuation constant.
- Imaginary part β of γ equal difference in phase angle between the input current and the output current or the corresponding voltages and called phase shift constant.

$$\gamma = \sqrt{zy} = \sqrt{(R + j\omega L)(G + j\omega C)}$$

c. The phase shift constant or wavelength constant β indicates the amount by which the phase of an input current changes in a unit distance. In a distance equal to one wavelength λ, the phase shift is 2π radians, $\lambda = \frac{2\pi}{\beta}$, wavelength.

d. The phase velocity of propagation is

$$v_p = f\lambda = \frac{\omega}{\beta}$$

Example 7.5: An open wire transmission line has $R = 5\ \Omega/\mathrm{m}$, $L = 5.2 \times 10^{-8}\ \mathrm{H/m}$, $G = 6.2 \times 10^{-3}\ \Omega/\mathrm{m}$, $C = 2.13 \times 10^{-13}\ \mathrm{F/m}$, frequency = 4 GHz. Find $Z_0, \gamma\ and\ \mathrm{v_f}$.

Solution:

$$\text{v}_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{5.2 \times 10^{-8} \times 2.13 \times 10^{-10}}}$$
$$= 0.3 \times 10^{9} = 0.3 \times 10^{8} \text{ m/s}$$
$$\omega = 2\pi f = 2\pi \times 4 \times 10^{9} = 8\pi \times 10^{9} = 2.512 \times 10^{16} \text{ rad}$$
$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$
$$R + j\omega L = 5 + j2.512 \times 10^{10} \times 5.2 \times 10^{-8}$$
$$= 5 + j1306.24 = 1306.25 < 89.78^{\circ}$$
$$G + j\omega L = 6.2 \times 10^{-3} + j2.512 \times 10^{10} \times 2.13 \times 10^{10}$$
$$= 6.2 \times 10^{-3} + j5.35 = 8.18 < 40.79^{\circ}$$
$$\boxed{Z_0 = 12.64 < 24.49^{\circ}}$$
$$\gamma = \sqrt{(R + j\omega L)\,(G + j\omega C)}$$
$$\boxed{\gamma = 103.37 < 65.23^{\circ}}$$

Example 7.6: A typical transmission line has a resistance of 8 Ω/km, impedance of 2 mH/km, a capacitance of 0.002 μF/km and a conductance of 0.07 μs/km. Calculate the characteristic impedance, attenuation constant, phase constant of the transmission line at a frequency of 2 kHz. If a signal of 2 V is applied and the line terminated by its characteristic impedance, calculate the power delivered to load

Solution:

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$
$$= \sqrt{\frac{8 + j4\pi \times 2 \times 10^{-3} \times 10^{3}}{0.007 \times 10^{-6} + j4\pi \times 0.002 \times 10^{-6} \times 10^{3}}}$$
$$= 1.024 < -8.75^{o} \times 10^{3}\ \Omega$$
$$= (1012.1 - j155.72)\ \Omega$$

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)\,(G + j\omega C)}$$
$$\gamma = \sqrt{(8 + j4\pi \times 2 \times 10^{-3} \times 10^{3})(0.007 \times 10^{-6} + j4\pi \times 0.002 \times 10^{-6} \times 10^{3})}$$
$$= 0.02574 < 81.09 = 0.003987 + j0.02543$$

$\Rightarrow \qquad \alpha = 0.003987 \text{ Np/km}$

$$\beta = 0.02543 \text{ rad/km}$$

Input voltage $V_s = 2$ V; $l = 500$ km; $Z_0 = 1012.1\ \Omega$ (real part)

Since line is terminated in its characteristic impedance, $Z_{in} = Z_0 = Z_L$

$$\mathrm{I}_s = \frac{V_s}{Z_{in}} = \frac{2}{1024 < -8.75^o \times 10^3} = \frac{2}{1024 < -8.75^o} = 1.953 < 8.75^o \text{ mA}$$

$$I_l = I_s\, e^{-\gamma l} = (1.953 < 8.75^o) e^{(-0.003987 + j0.02543)\times 500}$$

$$|I_l| = 1.953 \times e^{-1.9935} = 0.2669 \text{ mA}$$

$$P = |I_l|^2 \text{ Real } (Z_0) = 1012.1 \times (0.2669)^2 = 72.1\ \mu\text{W}$$

$$V_p = \frac{\omega}{\beta} = \frac{4\pi \times 10^3}{0.02543} = 494.22 \text{ km/s}$$

7.7 Sending-End Impedance

To determine the degree of mismatch between the source and line, we have to know the impedance that the combination of transmission line and load presents to the source. Sending end impedance is that looking into the line from the source:

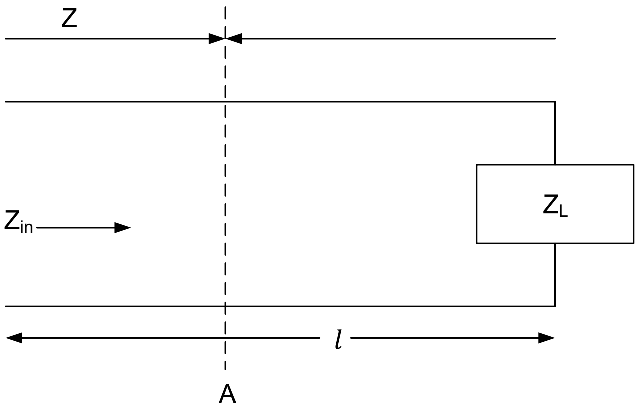


Figure 7.5 Sending end Impedance and Load Impedance

from Eqs. (7.4), (7.7) and (7.8)

$$Z_A = \frac{V_A}{I_A} = Z_0 \times \left(\frac{V_1 e^{-\gamma z} + V_2 e^{\gamma z}}{V_1 e^{-\gamma z} - V_2 e^{\gamma z}}\right)$$

From Eq. (7.10), $\frac{V_2}{V_1} = e^{-2\gamma l}$

$$\Rightarrow \qquad Z_A = Z_0 \left(\frac{e^{-\gamma z} + \rho\, e^{-2\gamma l} e^{\gamma z}}{e^{-\gamma z} - \rho\, e^{-2\gamma l}\, e^{\gamma z}}\right)$$

After dividing through by V_1

Multiplying through by $e^{\gamma l}$

$$Z_A = Z_0\left(\frac{e^{\gamma l}e^{-\gamma z} + \rho\, e^{\gamma l-2\gamma l}e^{\gamma z}}{e^{\gamma l}e^{-\gamma z} - \rho\, e^{\gamma l-2\gamma l}\, e^{\gamma z}}\right)$$

$$= Z_0\left(\frac{e^{\gamma(l-z)} + \rho e^{-\gamma(l-z)}}{e^{\gamma(l-z)} - \rho e^{\gamma(l-z)}}\right)$$

$l - z = x$ from Fig.7.5

$$\Rightarrow \qquad Z_A = Z_0\left(\frac{e^{\gamma x} + \rho e^{-\gamma x}}{e^{\gamma x} - \rho e^{-\gamma x}}\right)$$

$$\text{From} \qquad \rho_V = \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right)$$

$$Z_A = Z_0\left[\frac{e^{\gamma x} + \left[\frac{(Z_L - Z_0)}{(Z_L + Z_0)}\right]e^{-\gamma x}}{e^{\gamma x} - \left[\frac{(Z_L - Z_0)}{(Z_L + Z_0)}\right]e^{-\gamma x}}\right]$$

Multiplying through by $(Z_L + Z_0)$

$$Z_A = Z_0\left[\frac{(Z_L + Z_0)e^{\gamma x} + (Z_L - Z_0)e^{-\gamma x}}{(Z_L + Z_0)e^{\gamma x} - (Z_L - Z_0)e^{-\gamma x}}\right]$$

Factorizing,

$$Z_A = Z_0\left[\frac{Z_L(e^{\gamma x} + e^{-\gamma x}) + Z_0(e^{\gamma x} - e^{-\gamma x})}{Z_L(e^{\gamma x} - e^{-\gamma x}) + Z_0(e^{\gamma x} + e^{-\gamma x})}\right]$$

Dividing through by 2 to give hyperbolic functions

$$Z_A = Z_0\left[\frac{Z_L \cosh\gamma x + Z_0 \sinh\gamma x}{Z_L \sinh\gamma x + Z_0 \cosh\gamma x}\right]$$

Dividing through by $\cosh\gamma x$,

$$Z_A = Z_0\left[\frac{Z_L + Z_0 \tanh\gamma x}{Z_L \tanh\gamma x + Z_0}\right]$$

Putting $x = l,\ Z_A$ becomes Z_{in} (sending-end impedance)

$$\Rightarrow \qquad Z_{in} = \left[\frac{Z_0\, Z_L + {Z_0}^2 \tanh \gamma l}{Z_L \tanh \gamma l + Z_0}\right] \qquad 7.18a$$

When normalized to the characteristic impedance Z_0,

$$z_{in} = \frac{Z_{in}}{Z_0} = \left[\frac{Z_L + Z_0 \tanh \gamma l}{Z_L \tanh \gamma l + Z_0}\right]$$

Normalized, load impedance $z_L = \frac{Z_L}{Z_0}$

$$\Rightarrow \qquad z_{in} = \left[\frac{\left(\frac{Z_L}{Z_0}\right) + \tanh \gamma l}{\left(\frac{Z_L}{Z_0}\right) \tanh \gamma l + 1}\right] \qquad 7.18b$$

$$z_{in} = \frac{z_L + \tanh \gamma l}{z_L \tanh \gamma l + 1} \qquad 7.18c$$

Example 7.7: A 600 Ω lossless transmission line is fed by a 50 Ω generator. If the line is 200 m long and terminated by load of 500 Ω, determine in $dB's$.

(i) Reflection loss
(ii) Transmission loss
(iii) Return loss.

Solution:

$$\rho = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{500 - 600}{500 + 600} = \frac{-100}{1100} = \frac{-1}{100} = \frac{-1}{11}$$

i. Reflection loss $= 10 \log_{10} \frac{1}{1-|\rho|^2} = 10 \log_{10} \frac{1}{1-\frac{1}{121}} = 0.036 \text{ dB}$

ii. Transmission loss $=$ Attenuation loss $+$ Reflection loss

$$= \text{lossless} + 0.036$$
$$= 0 + 0.036 = 0.036 \text{ dB}$$

iii. Return loss $= 10 \log_{10}|\rho| = 10 \log_{10}\left(\frac{1}{11}\right) = -10.414 \text{ dB}$

7.8 Low Loss Lines

Eq. (7.5): $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$

Factoring out $j\omega L$ and $j\omega C$,

$$\gamma = \sqrt{(j\omega L)\,(j\omega C)\left(\frac{R}{j\omega L} + \frac{j\omega L}{j\omega L}\right)\left(\frac{G}{j\omega C} + \frac{j\omega C}{j\omega C}\right)}$$

$$= j\omega\sqrt{LC}\left(1+\frac{R}{j\omega L}\right)^{\frac{1}{2}}\left(1+\frac{G}{j\omega C}\right)^{\frac{1}{2}}$$

Binomial series expansion of γ gives:

$$\gamma = j\omega\sqrt{LC}\left(1+\frac{R}{2j\omega L}-\frac{1}{4}\frac{R^2}{(j\omega L)^2}\right)\times\left(1+\frac{G}{2j\omega C}-\frac{G^2}{4(j\omega C)^2}\right)$$

For low-loss lines, R and G are very small, and can therefore be ignored:

$$\Longrightarrow \qquad \gamma \approx j\omega\sqrt{LC}\left(1+\frac{R}{2j\omega L}\right)\times\left(1+\frac{G}{2j\omega C}\right)$$

$$= j\omega\sqrt{LC}\left(1+\frac{R}{2j\omega L}+\frac{G}{2j\omega C}-\frac{RG}{(2j\omega)^2 LC}\right)$$

$$= j\omega\sqrt{LC}\left(1+\frac{G}{2j\omega C}+\frac{R}{2j\omega L}-\frac{RG}{4\omega^2 LC}\right)$$

$$= j\omega\sqrt{LC}\left(1-\frac{RG}{4\omega^2 LC}-\frac{jR}{2\omega L}-\frac{jG}{2\omega C}\right)$$

$$\gamma = \alpha + j\beta = \omega\sqrt{LC}\left(j-\frac{jRG}{4\omega^2 LC}-\frac{j^2R}{2\omega L}-\frac{j^2G}{2\omega C}\right)$$

$$\alpha + j\beta = \omega\sqrt{LC}\left[\left(\frac{R}{2\omega L}+\frac{G}{2\omega C}\right)+j\left(1-\frac{RG}{4\omega^2 LC}\right)\right]$$

$$\Longrightarrow \qquad \alpha \approx \omega\sqrt{LC}\left(\frac{R}{2\omega L}+\frac{G}{2\omega C}\right), \quad \beta \approx \omega\sqrt{LC}\left(1-\frac{RG}{4\omega^2 LC}\right)$$

$$\alpha \approx \frac{R}{2}\sqrt{\frac{C}{L}}+\frac{G}{2}\sqrt{\frac{L}{C}} \qquad 7.19$$

$$\beta \approx \omega\sqrt{LC}\left(1-\frac{RG}{4\omega^2 LC}\right)$$

R and G very small, so at high frequencies:

$$\beta \approx \omega\sqrt{LC} \qquad 7.20$$

Similarly,

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{j\omega L}{j\omega C}\times\frac{j\omega C}{j\omega L}\times\left(\frac{R+j\omega L}{G+j\omega C}\right)}$$

$$= \sqrt{\frac{j\omega L}{j\omega C}\left(\frac{R}{j\omega L}+1\right)\left(\frac{G}{j\omega C}+1\right)} = \sqrt{\frac{L}{C}} \times \left(1+\frac{R}{j\omega L}\right)^{\frac{1}{2}} \times \left(1+\frac{G}{j\omega C}\right)^{\frac{1}{2}}$$

$$\approx \sqrt{\frac{L}{C}} \times \left(1+\frac{R}{2j\omega L}\right) \times \left(1-\frac{G}{2j\omega C}\right)$$

By binomial expansion, with terms in R^2, G^2 neglected

$$Z_0 = \sqrt{\frac{L}{C}} \times \left(1-\frac{G}{2j\omega C}+\frac{R}{2j\omega L}-\frac{RG}{4j^2\omega^2 LC}\right)$$

$$\approx \sqrt{\frac{L}{C}} \times \left(1-\frac{jR}{2\omega L}+\frac{jG}{2\omega C}\right)$$

$$\frac{R}{\omega L}, \frac{G}{\omega C} \text{ very small } \Rightarrow Z_0 = \sqrt{\frac{L}{C}} \qquad 7.21$$

Plugging Eq. 7.21 in 7.19

$$\alpha = \frac{R}{2Z_0} + \frac{GZ_0}{2} \qquad 7.22$$

7.9 Lines of Zero Loss

For a relatively short line and operating at very high frequencies, it is reasonable to assume zero attenuation, i.e., lossless line

$$\Rightarrow \qquad \alpha = 0 = \frac{R}{2Z_0} + \frac{GZ_0}{2} \Rightarrow {Z_0}^2 = -\frac{R}{G}$$

$$\Rightarrow \qquad Z_0 = j\sqrt{\frac{R}{G}}$$

In this case $\gamma = \alpha + j\beta = 0 + j\beta = j\beta$

Replacing γ by $j\beta$ in Eq. 7.18b

$$z_{in} = \frac{z_L + \tanh j\beta l}{1 + z_L \tanh j\beta l} \Rightarrow \frac{z_L + j\tan\beta l}{1 + jz_L \tan\beta l} \qquad 7.23$$

7.10 Quarter Wave Transformer

For a lossless line $(\alpha = 0)$ and replacing γ by $j\beta$ in Eq. (7.18a)

$$\Rightarrow \qquad Z_{in} = Z_0 \left(\frac{Z_L + j\,Z_0 \tan\beta l}{Z_0 + j\,Z_L \tan\beta l}\right)$$

$$\Rightarrow \qquad Z_{in} = Z_0 \left(\frac{\frac{Z_L}{\tan\beta l} + jZ_0}{\frac{Z_0}{\tan\beta l} + jZ_L}\right)$$

d = quarter wavelength long $\Rightarrow \tan\beta l = \tan\frac{\pi}{2} = \infty$

$$Z_{in} = \lim Z_0 \left(\frac{\frac{Z_L}{x} + j\,Z_0}{\frac{Z_0}{x} + jZ_L}\right) = Z_0\left(\frac{jZ_0}{j\,Z_L}\right) = \frac{Z_0}{Z_L}$$

$$\Rightarrow \qquad {Z_0}^2 = Z_{in}\,Z_L \qquad 7.24$$

For matching a given load to a given input impedance, a quarter wave section of lossless line is used with characteristics impedance of

$$Z_0 = \sqrt{Z_{in}\,Z_L}$$

Example 7.8: A 50 W lossless line has a length of 0.4λ. The operating frequency is 300 MHz. A load $Z_L = 40 + j30\ \Omega$ is connected at $Z = 0$, and the Thevenin equivalent source at $Z = -1$ is $12 < 0^o$ V in series with $Z_{Th} = 50 + j0\ \Omega$. Find (a) ρ; (b) S; (c) Z_{in}.

Solution:

Using 7.23,

$$Z_{in} = \frac{(Z_L + jZ_0 \tan\beta l)}{(Z_0 + jZ_L \tan\beta l)}$$

Putting $Z_L = \infty \left(\text{we know that} \frac{1}{\infty} = 0\right)$ and dividing entire by Z_L we get

$$\text{So,} \quad Z_{in} = Z_0 \frac{(1+0)}{(0 + j\tan\beta l)} = \frac{Z_0}{j\tan\beta l} = \frac{Z_0}{j\tan\beta l} = -j\,\frac{Z_0}{\tan\beta l}$$

Ans:

(a) $0.333 < 90^o$
(b) 2.00
(c) $25.5 + j5.90\ \Omega$

Example 7.9: Calculate the characteristic impedance of a quarter-wave transformer if a 120 Ω load is to be matched to a 75 Ω line.

Solution:

$$Z_0 = \sqrt{Z_L\, Z_{in}}$$

$$\Rightarrow \qquad \frac{{Z_0}^2}{Z_{in}} = Z_0$$

$$\Rightarrow \qquad Z_0 = \sqrt{120 \times 75} = 95\Omega$$

7.11 Stubs

Eq. (7.23) $\Longrightarrow z_{in} = \frac{z_L + j\tan\beta l}{1 + j\, z_L \tan\beta l}$ shows the variation of input impedance with the length of the line and this property can be used in stubs (short lengths of line) for matching applications. These are terminated in either short circuit or open circuit load.

$$\text{Open} - \text{circuit load} \Longrightarrow z_L = \infty = z_{in} = \frac{z_L}{jz_L \tan\beta l}$$

$$z_{in} = \frac{1}{j\tan\beta l} = -j\cot\beta l \qquad 7.25$$

$$\text{Short circuit load} \Longrightarrow z_{in} = \frac{0 + j\tan\beta l}{1 + 0} = j\tan\beta l \qquad 7.26$$

For lossless line:

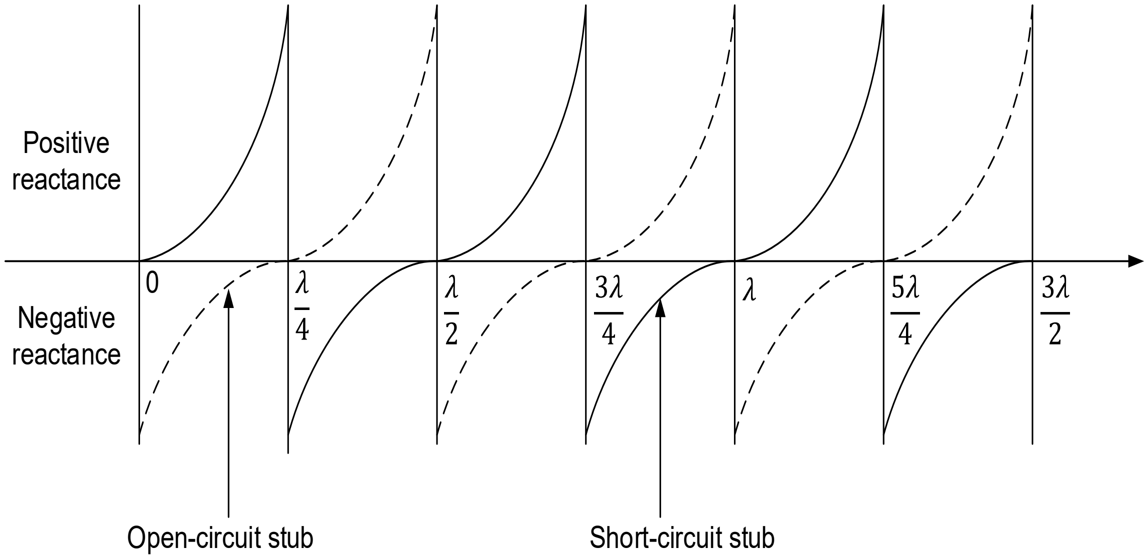


Figure 7.6 Stubs

Example 7.10: An ideal lossless $\frac{\lambda}{4}$ extension of line $Z_0 = 60\ \Omega$ is terminated with Z_L. Find Z_{in} of extension when

(i) $Z_L = 0$
(ii) $Z_L = \infty$
(iii) $Z_L = 60\ \Omega$

Solution

i. $Z_{in} = Z_0$ where $\beta l = \frac{2\pi}{\infty} \times \frac{\lambda}{4}$

$Z_{in} = j\, Z_0 \tan \beta l = \infty$ for $Z_L = 0$

ii. $Z_{in} = \frac{Z_0}{j \tan \beta l} = 0$ for $Z_{in} = \infty$

iii. $Z_L = 60\ \Omega$

$$= 60\left(\frac{60 + j60\tan(\pi/2)}{60 + j60\tan(\pi/2)}\right) = 60\ \Omega$$

7.12 Standing Waves

For a lossless line ($\alpha = 0$), the total voltage at a point z from the sending end:

$$\Longrightarrow \qquad V = \left(V_1 e^{-j\beta z} + V_2 e^{j\beta z}\right) e^{j\omega t}$$

Where $e^{j\omega t}$ indicates the time dependence.

From $\rho = \left(\frac{V_2}{V_1}\right) e^{2\gamma l}$ [Eq. (7.10)]

$$V = V_1 e^{j\omega t}\left[e^{-j\beta(l-x)} + \left(\frac{V_2}{V_1}\right) e^{j\beta l}\, e^{-j\beta x}\right]$$

For lossless line, $\rho = \left(\frac{V_2}{V_1}\right) e^{2\gamma l} = \left(\frac{V_2}{V_1}\right) e^{j2\beta l}$

Since, $\alpha = 0 \Longrightarrow V = V_1 e^{j\omega t}\, e^{-j\beta l}\left[e^{j\beta x} + e^{-j\beta x}\right]$ 7.27

This is the equation representing voltage standing wave (VSWR), made up of two component waves, one of forward direction, and the other of backward direction reflected from the load.

For a short circuit load $\rho = -1)$ and without the time dependence,

$$V = j2V_1\, e^{-j\beta l}\left[e^{j\beta x} - e^{-j\beta x}\right] = R_e V_1 e^{-j\beta l} \sin\beta x = V \qquad 7.28$$

The real part of the absolute value (modulus) of Eq. (7.28) is

$$|V| = R_e\left|\left[j2V_2 e^{-j\beta l} \sin\beta x\right]\right|$$

$$= R_e|[j2V_1(\cos\beta l - j\sin\beta l)]\sin\beta x|$$

$$= R_e\, |[2V_1(j\cos\beta l + \sin\beta l)]\sin\beta x|$$

$$|V| = 2V_1\, |\sin\beta x| \sin\beta l \qquad 7.29$$

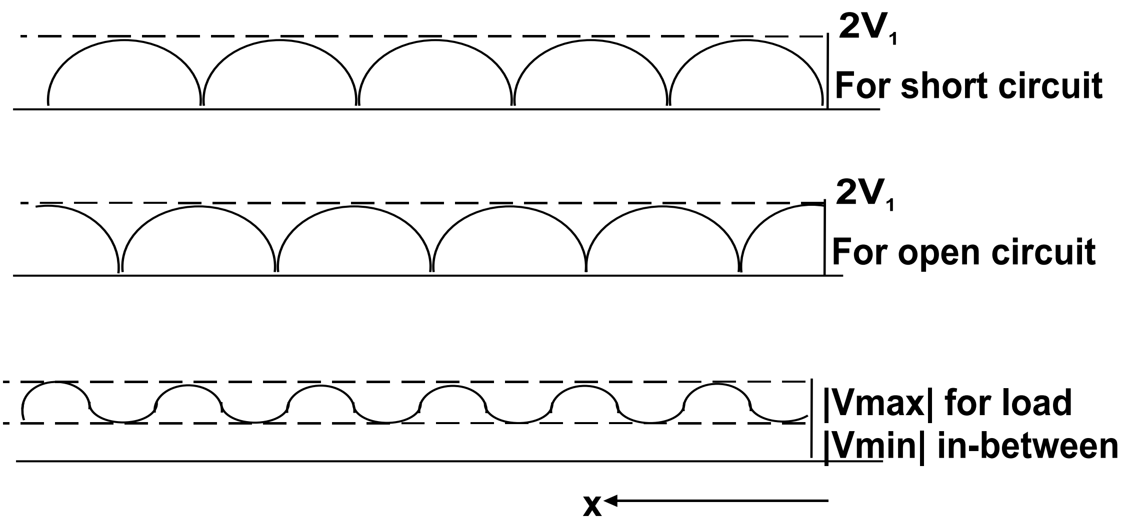


Figure 7.7 Standing Waves

For an open-circuit load $(\rho = 1)$ under the same conditions,

$$V = V_1 e^{-j\beta l}\left(e^{j\beta x} + e^{-j\beta x}\right) = 2V_1 e^{-j\beta l}\cos\beta x \Longrightarrow |V| = 2V_L|\cos\beta x|$$

For a load in between short and open circuit, say $e = 0.6 + j0.3$,

$$V = V_1 e^{-j\beta l}\left[e^{j\beta x} + (0.6 + j0.3)e^{-j\beta x}\right]$$

$$|V| = R_e\left[V_1 e^{-j\beta l}\left|e^{j\beta x} + (0.6 + j0.3)e^{-j\beta x}\right|\right]$$

From Eq. (7.29), for $(n-1)\pi = \beta x, \sin\beta x = 0$, and the next minimum occurs at

$$\frac{\pi}{\beta} = \left(\frac{\pi}{\frac{2\pi}{\lambda}}\right) = \frac{\lambda}{2} \text{ etc.}$$

It could be discerned that minima for short circuits occur at maxima for open circuit, and vice versa. Both the adjacent minima and maxima are separated by half a wavelength with the first minimum occurring at the load terminals for short circuit (maximum for open circuit). For a load in between, the minima and maxima the between zero and $2V_1$, but with adjacent minima and maxima still half a wavelength apart.

Voltage standing wave ratio (VSWR)

$$\text{By definition,} \quad \text{VSWR} \Longrightarrow S = \frac{|V_{max}|}{|V_{min}|}$$

$1 \leq S \leq \infty$ and depends on the degree of mismatch at the load (reflection coefficient).

From Eq. (7.27), plugging in $\rho = |\rho|e^{j\psi}$

$$V = V_1 e^{-j\beta l}\left[e^{j\beta x} + \rho e^{-j\beta x}\right] \qquad 7.30$$

$$V = V_1 e^{-j\beta(l-x)}\left[1 + |\rho|e^{j(\psi - 2\beta x)}\right] \qquad 7.31$$

$$|V_{max}| = V_1\,(1 + |\rho|) \qquad 7.32$$

When $(\psi - 2\beta x) = 2(m-1)\pi, m = 1,2,3, \ldots$, i.e when $2\,(m-1)$ is a positive even number, making $\cos(\psi - 2\beta x)$ positive unity,

$$|V_{min}| = V_1(1 - |\rho|) \qquad 7.33$$

When $\psi - 2\beta x) = (2m-1), m = 1, 2, 3, \ldots\ldots$ i.e when $2m - 1$ is a positive odd number, making $\cos(\psi - 2\beta x)$ negative unity,

$$S = \frac{V_1(1 + |\rho|)}{V_1(1 - |\rho|)} = \frac{1 + |\rho|}{1 - |\rho|} = S \qquad 7.34$$

$$\Rightarrow \qquad |\rho| = \frac{S-1}{S+1} \qquad 7.36$$

From Eq. (7.31) at the first voltage minimum, at $x = x_{min}$ from the load,

$$\psi - 2\beta x = \pi$$

$$\psi = 2\beta x_{min} + \pi \Longrightarrow Z = Z_{min} = \left(\frac{V}{I}\right) x_{min}$$

$$Z_{min} = \frac{V_{min}}{I_{min}} = \frac{V_1 e^{-j\beta(l-x_{min})}\left[1 + |\rho| e^{j(\psi - 2\beta x_{min})}\right]}{\left(\frac{V_1}{Z_0}\right) e^{-j\beta(l-x_{min})} - \left(\frac{V_2}{Z_0}\right) e^{j\beta(l-x_{min})}}$$

$$= \frac{V_1 e^{-j\beta(l-x_{min})}\left[1 + |\rho| e^{j(\psi - 2\beta x_{min})}\right]}{V_1 e^{-j\beta(l-x_{min})}\left[1 - |\rho| e^{j\psi - j2\beta x_{min}}\right]} Z_0$$

$$= \frac{V_1 e^{-j\beta(l-x_{min})}\left[1 + |\rho| e^{j(\psi - 2\beta x_{min})}\right]}{V_1 e^{-j\beta(l-x_{min})}\left[1 - |\rho| e^{j(\psi - j2\beta x_{min})}\right]} Z_0$$

$$= Z_0 \times \frac{1 + |\rho| e^{j\pi}}{1 - |\rho| e^{j\pi}}$$

But from trigonometry (Euler's identity), $e^{j\pi} = \cos\pi + j\sin\pi = -1$

$$Z_0 \times \frac{1 - |\rho|}{1 + |\rho|} = \frac{Z_0}{S} = Z_{min} \qquad 7.37$$

Normalized to the characteristic impedance,

$$\frac{Z_{min}}{Z_0} = z_{min} \qquad 7.37a$$

$$z_{min} = \frac{1}{S} \qquad 7.37b$$

Similarly,

$$Z_{max} = Z_0 \times \frac{1 + |\rho|}{1 - |\rho|}$$

$$Z_{max} = Z_0 \times S \qquad 7.38$$

$$S = z_{max} \qquad 7.38a$$

Example 7.11: A $50\ \Omega$ lossless transmission line is terminated by a load impedance,

$Z_L = 50 - j75\ \Omega$. If the incident power is $100\ \text{mW}$. Find the power dissipated by the load.

Solution:

The reflection coefficient $\Longrightarrow \rho = \frac{Z_L - Z_0}{Z_L + Z_0}$

$$\rho = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 - j75 - 50}{50 - j75 + 50} = 0.36 - j\ 0.48 = 0.60\ e^{-j93}$$

Then, $\langle P_t \rangle = (1 - |\rho|^2)\langle P_i \rangle = [1 - (0.60)^2](100) = 64\ \text{mW}$

Impedance at a voltage minimum/maximum

Example 7.12: A lossless transmission line of $Z_0 = 100\ \Omega$ is terminated by an unknown impedance. The termination is found to be at a maximum of the voltage standing wave and the VSWR is 5. What is the value of terminating impedance?

Solution:

We know that $Z_{max} = Z_0$. (VSWR) as the termination is at maximum of the voltage standing wave.

$$Z_{max} = 100 \times 5 = 500\ \Omega$$

7.13 Load Impedance on a Lossless Line

This can be determined if the VSWR, wavelength (λ) and distance from the load to the nearest voltage minimum are known.

Equation: $V = \left(V_1 e^{-j\beta(l-x)}\right)\left[1 + |\rho| e^{j(\psi - 2\beta x)}\right]$

$$\psi - 2\beta x = (2m - 1)\pi, m = 1, 2, 3, \ldots$$

$$m = 1 \Longrightarrow x = x_{min} \Longrightarrow \psi - 2\beta x = \pi$$

$$\Longrightarrow \quad \boxed{\psi = 2\beta x_{min} + \pi}$$

$$Z_L = Z_0 \frac{1 + \rho}{1 - \rho} = Z_0 \frac{1 + |e|\ e^{j\psi}}{1 - |e| e^{j\psi}} \qquad 7.39$$

From Eq (7.37)

$$Z_L = Z_0 \times \left[\frac{1 + \left[\frac{(S-1)}{(S+1)}\right] e^{j\psi}}{1 - \left[\frac{(S-1)}{(S+1)}\right] e^{j\psi}}\right]$$

From Eq. (7.36)

$$Z_L = Z_0 \times \left[\frac{1 + \left[\frac{(S-1)}{(S+1)}\right] e^{j(2\beta x_{min}+\pi)}}{1 - \left[\frac{(S-1)}{(S+1)}\right] e^{j(2\beta x_{min}+\pi)}}\right]$$

$$e^{j\pi} = \cos\pi + j\sin\pi = -1 \Longrightarrow Z_L = Z_0 \times \left[\frac{1 + \left[\frac{(S-1)}{(S+1)}\right] e^{j2\beta x_{min}}}{1 - \left[\frac{(S-1)}{(S+1)}\right] (e^{j2\beta x_{min}})}\right]$$

From Eq. (7.39)

$$Z_L = \frac{(S+1) + (S-1)\left(-e^{j2\beta x_{min}}\right)}{(S+1) - (S-1)(-e^{j2\beta x_{min}})} \times Z_0$$

$$= Z_0 \left[\frac{S\left(1 - e^{j2\beta x_{min}}\right) + \left(1 + e^{j2\beta x_{min}}\right)}{S(1 + e^{j2\beta x_{min}}) + (1 - e^{j2\beta x_{min}})}\right]$$

Dividing both the numerator and denominator by $e^{j\beta x_{min}}$

$$Z_L = Z_0 \left[\frac{S\left(e^{j\beta x_{min}} - e^{j\beta x_{min}}\right) + \left(e^{-j\beta x_{min}} + e^{j\beta x_{min}}\right)}{S(e^{-j\beta x_{min}} + e^{j\beta x_{min}}) + (e^{-j\beta x_{min}} - e^{j\beta x_{min}})}\right]$$

$$= Z_0 \times \frac{S(-2j\sin\beta x_{min}) + 2\cos\beta x_{min}}{S(2\cos\beta x_{min}) - j2\sin\beta x_{min}}$$

Dividing through by $2\cos\beta x_{min}$,

$$Z_L = Z_0 \left[\frac{-Sj\tan\beta x_{min} + 1}{S - j\tan\beta x_{min}}\right]$$

$$Z_L = Z_0 \left[\frac{1 - js\tan\beta x_{min}}{S - j\tan\beta x_{min}}\right] \qquad 7.40$$

Normalized load impedance, $\frac{Z_L}{Z_0}$,

$$z_L = \frac{Z_L}{Z_0} = \frac{1 - S\tanh j\beta x_{min}}{S - \tanh j\beta x_{min}} \qquad 7.41$$

Example 7.13: A $100\ \Omega$ line feeding the antenna has $\text{VSWR} = 2$ and the distance from load to the first minima is 10 cm. Design a single stub matching to make $\text{VSWR} = 1$. Given $f = 150\ \text{MHz}$

Solution:

$S = \text{VSWR} = 2$

$$|\rho| = \frac{\text{S}-1}{\text{S}+1} = \frac{1}{3} = 0.33$$

$$\text{F} = 150\ \text{MHz}$$

$$\lambda = \frac{C}{f} = 2\ \text{m}$$

We know that

$$\psi - 2\beta\ d_{min} = \pi$$

$$2\beta d_{min} = \psi - \pi = 2 \times \frac{2\pi}{2} \times 0.1 = 0.2\pi$$

The position of stub

$$l_\psi = \frac{\lambda}{4\pi}\ (\cos^{-1}(\rho) - 2\beta\ d_{min})$$

$$|l_\psi| = \frac{\lambda}{4\pi}(0.39\pi - 0.2\pi) = \frac{0.1}{4\pi} \times (0.19\pi) = 4.75\ \text{mm}$$

$$\text{Length of stub} = l_t = \frac{\lambda}{2\pi}\tan^{-1}\left(\frac{\sqrt{1-|\rho|^2}}{2|\rho|}\right)$$

$$= \frac{\lambda}{2\pi}\tan^{-1}\left(\frac{\sqrt{1-|0.33|^2}}{2(0.33)}\right) = 15\ \text{mm}$$

Example 7.14: An UHF transmission line operating at 1 GHz is connected to Z_L producing reflection coefficient $0.5\angle 30^\circ$. Design single stub matching. Find VSWR.

Solution:

$$\text{f} = 1\ \text{GHz}$$

$$\lambda = \frac{3\times 10^8}{1\times 10^9} = 0.3 \text{ m}$$

$$|\rho| = 0.5$$

$$\text{S} = \text{VSWR} = \frac{1+|\rho|}{1-|\rho|} = \frac{1.5}{0.5} = 3$$

$$\psi = 30^\circ = \frac{\pi}{6} \text{ rad}$$

$$l_s = \frac{\lambda}{4\pi}(\psi + \pi - \cos^{-1}(|\rho|)) = \frac{\lambda}{4\pi}\left(\frac{\pi}{6} + \pi - \cos^{-1}(0.5)\right)$$

$$= \frac{\lambda}{4\pi}\left(\frac{7\pi}{6} - \frac{\pi}{3}\right) = \frac{\lambda}{4\pi} \times \frac{5\pi}{6} = \frac{5\lambda}{24} = \frac{1.5}{24} = 6.25 \text{ cm}$$

$$\text{Length of stub} = \text{l}_\text{t} = \frac{\lambda}{2\pi}\tan^{-1}\left(\frac{\sqrt{1-|\rho|^2}}{2|\rho|}\right)$$

$$= \frac{\lambda}{2\pi}\tan^{-1}\left(\frac{\sqrt{1-(0.5)^2}}{2\times 0.5}\right)$$

$$= \frac{\lambda}{2\pi} \times 0.227\pi = 3.4 \text{ cm}$$

7.14 Further Examples

1. A transmission line with the characteristic impedance of 250 Ω is terminated in a load of 100 Ω. If the load is dissipating a continuous sinusoidal power of 50 watts, calculate:

(i) the reflection coefficient
(ii) Voltage standing wave ratio
(iii) reflected voltage $|V_r|$

Solution:

(i) $|\rho| = \left|\frac{100-250}{100+250}\right| = 0.43$

(ii) $S = VSWR = \frac{(|\rho|+1)}{(1-|\rho|)} = \frac{1.43}{0.57} = 2.50$

(iii) $50 = \frac{(V_{max})(V_{min})}{Z_0}$

$$50 = \frac{(V_i + V_r)(V_i - V_r)}{250}$$

$$V_i^2 - V_r^2 = 12500$$

$$V_r = \sqrt{V_i^2 - 12{,}500} = 0.43V_i$$

$$V_i^2 = (0.43V_i)^2 + 12{,}500$$

$$V_i = \pm\sqrt{\frac{12500}{(1 - 0.43^2)}}$$

$$= 123.84\ \text{V}$$

2. A lossless transmission line with $Z_0 = 60\ \Omega$ is 40 m long and operates at 3 MHz, the line is terminated with a load of $Z_L = 120 + j60\ \Omega$. Given that $u = 0.8c$ on the line, determine analytically. $c = 3 \times 10^8\ \text{m/s}$:

(i) Load admittance
(ii) Voltage reflection coefficient (magnitude & phase)
(iii) VSWR
(iv) Z_{in}
(v) Z_{max}
(vi) Z_{min}

Solution:

(i)

$$Y_L = \frac{1}{Z_L}$$

$$Y_L = \frac{1}{(120 + j60)}$$

$$= \frac{(120 - j60)}{(14400 + 3600)}$$

$$Y_L = 0.0067 - j0.00333\ \Omega$$

(ii)

$$\rho = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\rho = \frac{(120 + j60 - 80)}{(120 + j60 + 80)}$$

$$\rho = \frac{(40 + j60)}{(200 + j60)}$$

$$= \frac{(2 + j3)}{(10 + j3)}$$

$$= \frac{(2+j3)(10-j3)}{(100+9)}$$

$$= \frac{(20+9+j30-j6)}{109}$$

$$\rho = \sqrt{29^2+24^2}\angle \tan^{-1}\left(\frac{24}{29}\right)$$

$$\rho = 0.34\angle 39.64^o$$

(iii) $$VSWR = \frac{(1+0.34)}{(1-0.34)} = \frac{1.34}{0.66} = 2.03$$

(iv) $$\lambda = \frac{u}{f} = \frac{(0.8)(3\times 10^8)}{3}\times 10^{-6} = 80 \text{ m} \Rightarrow \beta l = \frac{2\pi}{\lambda}\left(\frac{40\lambda}{80}\right) = \pi$$

$$Z_{in} = Z_0\left[\frac{Z_L + jZ_0\tan\beta l}{Z_0 + jZ_L\tan\beta l}\right]$$

$$= 80\left[\frac{120+j60+j80\tan\pi}{80+j(120+j60)\tan\pi}\right]$$

$$= 80\left[\frac{120+j60}{80}\right]$$

$$Z_{in} = 120 + j60\ \Omega$$

(v) $$Z_{max} = Z_0\times S = 80(2.03) = 162.4\ \Omega$$

(vi) $$Z_{min} = \frac{Z_0}{S} = \frac{80}{2.03} = 39.41\ \Omega$$

3. A distortionless line ($RC = GL$) has $Z_0 = 80\Omega, \alpha = 25\ \text{mNP/m}, u = 0.5$, where c is the speed of the light in a vacuum. Determine

(i) R
(ii) L
(iii) G
(iv) C
(v) λ at 100MHz. ($c = 3\times 10^8$ m/s)

Solution:

$$RC = GL \Rightarrow G = \frac{RC}{L}$$

$$\Rightarrow \gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{RG}\sqrt{\left(1+\frac{j\omega L}{R}\right)\left(+\frac{j\omega C}{G}\right)}$$

$$\gamma = \sqrt{RG\left(1+\frac{j\omega L}{R}\right)\left(1+\frac{j\omega L}{R}\right)} = \sqrt{RG}\left(1+\frac{j\omega L}{R}\right) = \alpha + j\beta$$

$$\alpha = \sqrt{RG}$$

$$\beta = \sqrt{RG}\ \left(\frac{\omega L}{R}\right)$$

$$= \sqrt{RG}\left(\frac{\omega C}{G}\right)$$

$$= \omega C\sqrt{\frac{R}{G}}$$

$$= \omega\left(\sqrt{\frac{L}{C}}\right)$$

$$\beta = \omega\sqrt{LC}$$

$$Z_0 = \sqrt{\frac{R}{G}}$$

$$\alpha Z_0 = \left(\sqrt{RG}\right)\left(\sqrt{\frac{R}{G}}\right) = R$$

$$R = (25\times 10^{-3})\ (80)$$

$$R = 2\ \Omega$$

$$u = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}} \Longrightarrow \frac{Z_0}{u} = \frac{\left(\sqrt{\frac{R}{G}}\right)}{u} = \left(\sqrt{\frac{L}{C}}\right)\sqrt{LC} = L$$

$$L = \frac{80}{(0.5)(3\times 10^{8})}$$

$$G = \frac{L^2}{R} = \frac{(25\times 10^{-3})^2}{2}$$

$$= 625 \times \frac{10^{-6}}{2}$$

$$G = 312.54 \text{ V/m}$$

$$C = \frac{GL}{R} = \frac{(312.5 \times 10^{-6})(533.33 \times 10^{-9})}{2}$$

$$C = 83.33 \text{ pF}$$

$$\lambda = \frac{c}{f} = \frac{0.5 \times 3 \times 10^{8}}{100} \times 10^{-6}$$

$$\lambda = 1.5 \text{ m}$$

7.15 Exercise

1. (i) In not more than 15 words, define (explain) what is meant by transmission line
  (ii) Sketch and completely label 2 types of Transmission line
  (iii) Name and explain the parameters involved in a typical transmission line.

2. (i) Define reflection coefficient
  (ii) Under what load conditions will there be total reflection from the load
  (iii) For lines of zero loss for a quarter wave transformer, determine the expression for the characteristic impedance in forms of the input and load impedance.
  (iv) In what way does quarter the wavelength section of a transmission line act as an impedance transformer?

3. (i) What is a stub, and how is it applied in transmission lines?
  (ii) Derive the expression for reflection coefficient in terms of load and characteristic impedances.

4. What does (i) VSWR = 1 (ii) VSWR $= \infty$, signify with reference to matching of the transmission line to the load?

5. A transmission line with the characteristic impedance of $250\ \Omega$ is terminated in a load of $100\ \Omega$. If the load is dissipating a continuous sinusoidal power of 50 watts, calculate:
  (i) The reflection coefficient
  (ii) Voltage standing wave ratio
  (iii) Reflected voltage $|V_r|$

6. Two voltage waves having equal frequencies and amplitudes propagate in opposite directions in a lossless transmission line.
  (i) Determine the total voltage as a function of distance and time.
  (ii) What kind of a wave results (relating its behaviour with respect to position and time)?
  (iii) Where do the zeros in the amplitude (i.e., null position) occur?

7. A lossless transmission line of 100 cm and operates at a frequency of 300 MHz, the line parameters are $L = 0.5\ \mu\text{H/m}$ and $C = 200\ \text{pF/m}$. determine
(a) The characteristic impedance
(b) the phase constant
(c) the phase velocity.

8. (i) Define the characteristic impedance of a typical transmission line
(ii) in what other way can it be viewed

9. An airline has a characteristic impedance of 60 Ω and a phase constant of 2 rad/m at 80 MHz, calculate the inductance/meter and the capacitance/meter of the line. $(R = 0 = G, \alpha = 0)$

10. What is meant by distortionless lines?

11. A distortionless line $(RC = GL)$ has $Z_0 = 160\ \Omega, \alpha = 50m\ \text{Np/m}, u = 0.8,$ where c is the speed of the light in a vacuum. Determine R, L, G, C and λ at 100 MHz, $(c = 3 \times 10^8\ \text{m/s})$

12. (i) Show that at high frequencies:

$$(R \ll \omega L, G \ll \omega L), \gamma = \left(\frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}}\right) + j\omega\sqrt{LC}$$

(ii) Obtain a similar formula for Z_0

13. (i) Define reflection coefficient
(ii) under what load conditions will there be total reflection from the load

14. Derive the expression for reflection coefficient in terms of load and characteristic impedances.

15. (i) Define the characteristic impedance of a typical transmission line
(ii) In what other way can it be viewed

16. An airline has a characteristic impedance of 80 Ω and a phase constant of 3.5 rad/m at 100 MHz, calculate the inductance/meter and the capacitance/meter of the line. $(R = 0 = G, \alpha = 0)$

17. What is meant by distortionless lines?

18. A distortionless line $(RC = GL)$ has $Z_0 = 80\ \Omega, \alpha = 25\ m\ \text{Np/m}, u = 0.5c$, where c is the speed of the light in a vacuum. Determine R, L, G, C and λ at 100 MHz, $(c = 3 \times 10^8\ \text{m/s})$

19. An airline has a characteristic impedance of $200\ \Omega$ and a phase constant of 4rad/m at 180 MHz, Calculate the inductance/meter and the capacitance/meter of the line. $(R = 0 = G, a = 0)$

20. A distortionless line $(RC = GL)$ has $Z_0 = 120\ \Omega, \alpha = 50\ m\ \text{Np/m}, u = 0.75c,$ where c is the speed of the light in a vacuum. Determine at 160 MHz, $(c = 3 \times 10^8\ \text{m/s})$

(i) R
(ii) L
(iii) G

(iv) C
(v) λ

21. A lossless transmission line is 100 cm and operates at a frequency of 400 MHz, the line parameters are $L = 0.75\ \mu\text{H/m}$ and $C = 300\ \text{pF/m}$. determine
(a) the characteristic impedance
(b) the phase constant
(c) the phase velocity.

22. A load of $25 + j50\ \Omega$ terminates a $50\ \Omega$ line, given that the line is 60 cm long and the signal wavelength 2 m, $c = 3 \times 10^8\ \text{m/s}$. Determine analytically:
(i) The load admittance
(ii) The reflection coefficient (amplitude and phase)
(iii) Voltage Standing Wave Ratio
(iv) input impedance.

23. A lossless transmission line of characteristic impedance $150\ \Omega$ is terminated in a load of $350 + j200\ \Omega$, given that the length of the line is 80 cm and the signal wavelength is 50cm, $c = 3 \times 10^8\ \text{m/s}$ determine analytically the:
(i) load admittance
(ii) reflection coefficient VSWR
(iii) distance between the load and the nearest voltage minimum to it
(iv) normalized input impedance.

24. A transmission line with the characteristic impedance of $120\ \Omega$ is terminated in a load of $80\ \Omega$. If the load is dissipating a continuous sinusoidal power of 40 watts, calculate:
(i) The reflection coefficient
(ii) Voltage standing wave ratio
(iii) Reflected voltage $|V_r|$

25. A lossless transmission line with $Z_0 = 60\ \Omega$ is 80 m long and operates at 6 MHz, the line is terminated with a load of $Z_L = 120 + j60\ \Omega$. Given that $u = 0.5\,c$ on the line, determine analytically. $c = 3 \times 10^8\ \text{m/s}$:
(vii) Load admittance
(viii) voltage reflection coefficient (magnitude & phase)
(ix) VSWR and Z_{in}

CHAPTER 8

UHF LINES AND SMITH CHART

8.0 UHF Lines

The transmission lines are required to carry signals in ultra-high frequency (UHF) range and its characteristics is entirely different from normal telephone communication. It allows several simplifying approximations and salient features are:

(a) The line considered a considerable skin effect so that almost all the current may be assumed to pass through the outer surface of the conductor hence, the internal inductance of wires may be considered to be zero.
(b) With these high frequencies, reactance from ωL is much larger than resistance R. this is due to the fact that the a.c. resistance of the wires increases in proportion to square root of frequency $'f$ because of the skin effect while the inductive reactance increases directly as frequency f
(c) These lines are properly constructed so that the shunt conductance G may be considered to be zero at all frequencies. Transmission lines are studied under the consideration that at radio frequencies.
(d) Low dissipation lines—in which case R is small in comparison with ωL
(e) Zero dissipation lines or lossless lines—in which case R is negligible in comparison with ωL

8.1 Impedance Matching and the Smith Chart: The Key Note

At high radio frequencies, the elements like wire inductances, interlayer capacitances, and conductors and resistances have a significant yet unpredictable impact on the matching network. Higher than a few tens of megahertz, theoretical Calculations and simulations are often insufficient. In-situ RF lab measurements, along with tuning work, have to be considered for determining the proper final values. Computational values are needed to set up the type of structure and target component values.

The various ways of impedance matching, includes:

- **The Computer Simulations**: This is Complex but simple to apply, as such simulators are dedicated to differing design functions but not of that of impedance matching. Designers have to get used to the multiple data inputs that need to be entered and the correct formats. They equally need the expertise to find the useful data among the tons of results coming out. That means that a circuit-simulation software is not pre-installed on computers, unless they are dedicated to such an application.

- **The Manual Computations**: Tedious due to the length ("kilometric") of the equations and the complex nature of the numbers to be manipulated.

Instinct: This can be acquired only after one has devoted many years to the RF industry. In short, this is for the super-specialist. Smith Chart: Upon which this article concentrates.

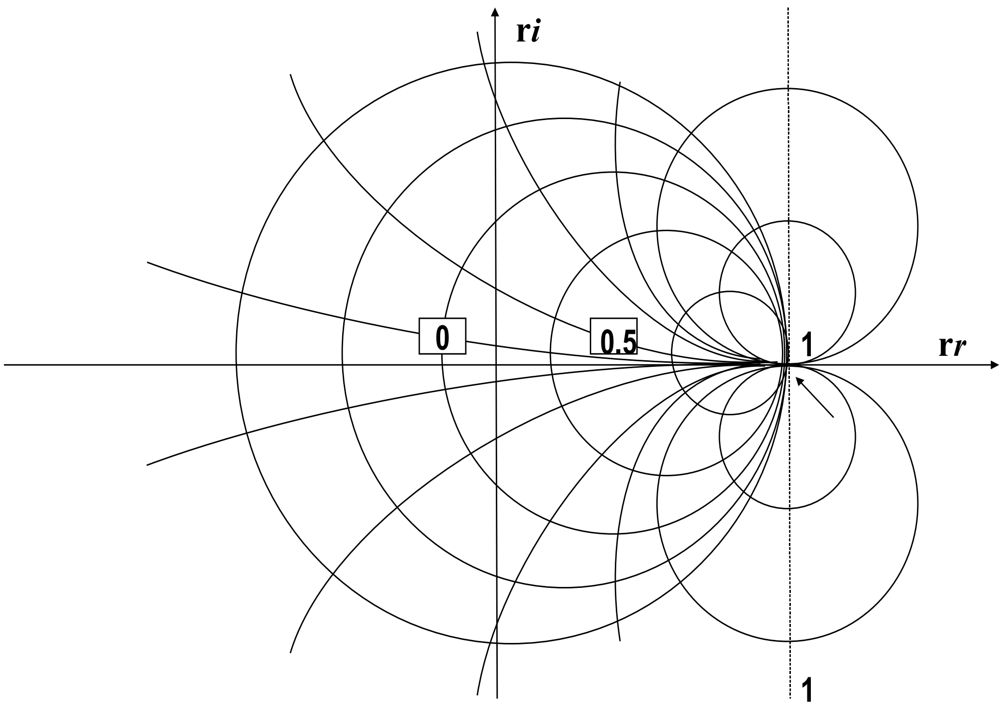


Figure 8.1. Fundamentals of Impedance and the Smith Chart.

8.2 Smith Chart

The Smith chart, invented by Phillip H. Smith (1905–1987), and T. Mizuhashi, is a graphical calculator or nomogram designed for electrical and electronics engineers specializing in radio frequency (RF) engineering to assist in solving problems with transmission lines and matching circuits. The Smith chart can be used to simultaneously display multiple parameters including impedances, admittances, reflection coefficients, scattering parameters, noise figure circles, constant gain contours and regions for unconditional stability, including mechanical vibrations analysis. The Smith chart is most frequently used at or within the unity radius region. However, the remainder is still mathematically relevant, being used, for example, in oscillator design and stability analysis. While the use of paper Smith charts for solving the complex mathematics involved in matching problems has been largely replaced by software based methods, the Smith charts display is still the preferred method of displaying how RF parameters behave at one or more frequencies, an alternative to using tabular information. Thus, most RF circuit analysis software includes a Smith chart option for the display of results and all but the simplest impedance measuring instruments can display measured results on a Smith chart display.

Smith transmission line chart is a graphical technique of solving transmission line problems. The chart consists of three sets of circular arcs and one straight line, and any value of impedance or admittance can be represented. The horizontal line that divides the chat into upper and lower halves is the locus of impedances and admittances with zero reactive and susceptive component, i.e.:

$z_L = r + j0, y_L = g + j0$ in normalized modes. ("Normalized" simply means dividing the given quantity (ohms) by the characteristic impedance Z_0). The sets of circles all have their centers on the horizontal line, and they all meet at the right side of the chart. The complete circles represent the normalized resistive or conductive components of impedance or admittance (r or g). The center of the chart has coordinates (1, 0), meaning that the complete circle represents unity normalized resistance or conductive, the zero standing for zero reactance or susceptance. The goal of matching is to change the impedance at the matching point to that of value of (1,0).

The second set of circular arcs above the horizontal line represents the positive imaginary components of impedance or admittance. The third set of arcs below the horizontal line represent the negative imaginary components of impedance or admittance.

The centers of the arcs representing the positive and negative imaginary components all lie on a vertical line at the right end.

With any circle drawn with centers at the middle of the chart (1,0), and a diameter drawn to cut across the circle, then the normalized impedance and admittance are at the opposite ends of the diameters of the circle. For example, given $z_L = 0.35 - j0.75$, on the lower half of the chart: with center at (1,0) (i.e, middle of the chart), radius at z, draw a circle. From z draw a diameter cutting the circle and ending on the other side of the circle. It is found that the diameter intersects the circle on the other side (positive imaginary components of the impedance or susceptance) at coordinates about $0.5\ \ldots. + j1.1$

Since Smith chart is a circular plot with a lot of interlaced circles on it. When used correctly, matching impedances, with apparent complicated structures, can be made without any computation. The only effort required is the reading and following of values along the circles.

The development of Smith chart was done by examining the load where the impedance must be matched. Rather than considering its impedance directly, let its reflection coefficient denote ρ, which is used to characterize a load (such admittance, gain, and transconductance). The ρ_L is more useful when dealing with RF frequencies.

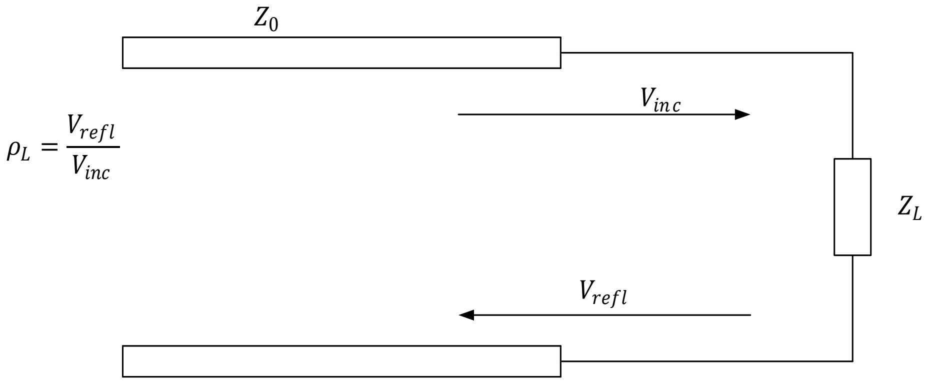


Figure 8.2 Impedance at the load

We know the reflection coefficient is defined as the ratio between the reflected voltage wave and the incident voltage wave as shown in Fig. 8.2.

The amount of reflected signal from the load is dependent on the degree of mismatch between the source impedance and the load impedance. Its expression has been defined as follows:

$$\rho_v = \frac{V_{refl}}{V_{inc}} = \frac{Z_L - Z_0}{Z_L + Z_0} = \rho_r + j\rho_i \qquad 8.1$$

Because the impedances are complex numbers, the reflection coefficient will be a complex number as well.

In order to reduce the number of unknown parameters, it is useful to freeze the ones that appear often and are common in the application. Here Z_0 (the characteristic impedance) is often a constant and a real industry normalized value, such as 50 Ω, 75 Ω, 100 Ω and 600 Ω. We can then define a normalized load impedance by:

$$z_L = \frac{Z_L}{Z_0} = \frac{(R + jX)}{Z_0} = r + jx \qquad 8.2$$

For simplification, we can rewrite the reflection coefficient formula as:

$$\rho_v = \rho_r + j\rho_i = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(Z_L - Z_0)/Z_0}{(Z_L + Z_0)/Z_0} = \frac{z - 1}{z + 1} = \frac{r + jx - 1}{r + jx + 1} \qquad 8.3$$

Clearly, we can see the direct relationship between the load impedance Z_L and its reflection coefficient. Unfortunately, the complex nature of the relation is not useful practically, so we can use the Smith chart as a type of graphical representation of the above equation.

To build the chart, the equation must be rewritten to extract standard geometrical figures.

$$z_L = r + jx = \frac{1+\rho_i}{1-\rho_i} = \frac{1+\rho_r + j\rho_i}{1-\rho_r - j\rho_i} \quad 8.4$$

And

$$r = \frac{1-\rho_r^2-\rho_i^2}{1+\ \rho_r^2 - 2.\rho_r + \rho_i^2} \quad 8.5$$

By setting the real parts and the imaginary parts of equation obtain two independent, new relationships:

$$r = \frac{1-\rho_r^2-\rho_i^2}{1+\rho_r^2 - 2.\rho_r + \rho_i^2} \quad 8.6$$

$$x = \frac{2.\rho_i}{1+\rho_r^2 - 2.\rho_r + \rho_i^2} \quad 8.7$$

Note that this equation is a relationship in the form of a parametric equation $(x-a)^2 + (y-b)^2 = R^2$ in the complex plane (ρ_r, ρ_i) of a circle centered at the coordinates $\left(\frac{r}{r}+1{,}0\right)$ and having a radius of $1/1 + r$.

$$r + r.\rho_r^2 - 2r.\rho_r + r.\rho_i^2 = 1 - \rho_r^2 - \rho_i^2 \quad 8.8$$

$$\rho_r^2 + r.\rho_i^2 - 2r.\rho_r + r.\rho_i^2 + \rho_i^2 = 1 - r \quad 8.9$$

$$(1+r).\rho_r^2 - 2r.\rho_r + (r+1)_i^2 = 1 - r \quad 8.10$$

$$\rho_r^2 - \frac{2.r^2}{r+1}\cdot\rho_r + \rho_i^2 = \frac{1-r}{1+r} \quad 8.11$$

$$\rho_r^2 - \frac{2.r^2}{r+1}\cdot\rho_r + \frac{r^2}{(r+1)^2} + \rho_i^2 - \frac{r^2}{(r+1)^2} = \frac{(1-r)}{(1+r)} \quad 8.12$$

$$\left(\rho_r - \frac{r}{r+1}\right)^2 + \rho_i^2 = \frac{1-r}{1+r} + \frac{r^2}{(1+r)^2} = \frac{1}{(1+r)^2} \quad 8.13$$

$$\left(\rho_r - \frac{r}{r+1}\right)^2 + \rho_i^2 = \left(\frac{1}{1+r^2}\right)^2 \quad 8.14$$

The points situated on a circle are all the impedances characterized by the same real impedance part value. For example, the circle, $r = 1$, is centered at the coordinates

$(0.5, 0)$ and has a radius of 0.5. It includes the point (0, 0), which is the reflection zero point (the load is matched with the characteristic Impedance). A short circuit, as a load, presents a circle centered at the coordinate (0, 0) and has a radius of 1. For an open-circuit load, the circle degenerates to a single point (centered at 1, 0 and with a radius of 0). This corresponds to a maximum reflection coefficient of 1, at which the entire incident wave is reflected totally.

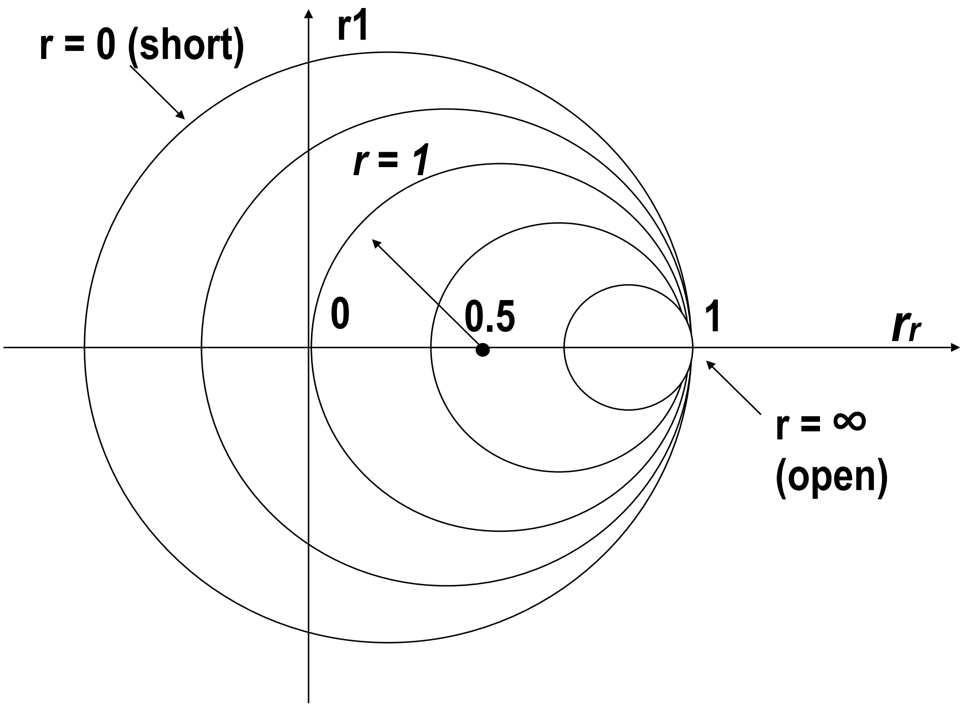


Figure 8.3 Smith Chart

When developing the Smith chart, there are certain precautions that should be noted. These are among the most important:

- All the circles have one same, unique intersecting point at the Coordinate
- The zero W circle where there is no resistance (r = 0) is the largest one.
- The infinite resistor circle is reduced to one point at (1, 0)
- There should be no negative resistance. If one (or more) should occur, we will be faced with the possibility of oscillatory conditions.
- Another resistance value can be chosen by simply selecting another circle corresponding to the new value.

8.3 Back to the Drawing Board

Moving on, we use equations above to further develop equations into another parametric equation. This results in

$$x + x.\rho_r^2 - 2.x.\rho_r + x.\rho_i^2 = 2.\rho_i \qquad 8.15$$

$$1 + \rho_r^2 - 2.\rho_r + \rho_i^2 = 2\rho_i/x \qquad 8.16$$

$$\rho_r^2 - 2.\rho_r + 1 + \rho_i^2 - \frac{2}{x}\rho_i = 0 \qquad 8.17$$

$$\rho_r^2 - 2.\rho_r + 1 + \rho_i^2 - \frac{2}{x}\rho_i + \frac{1}{x^2} - \frac{1}{x^2} = 0 \qquad 8.18$$

$$(\rho_r - 1)^2 + \left(\rho_i - \frac{1}{x}\right)^2 = \frac{1}{x^2} \qquad 8.19$$

Again, Eq. 8.19 is a parametric equation of the type $(x-a)^2 + (y-b)^2 = R^2$ in the complex plane (e_r, e_i) of a circle centered at the coordinates $(1, 1/x)$ and having a radius of $1/x$

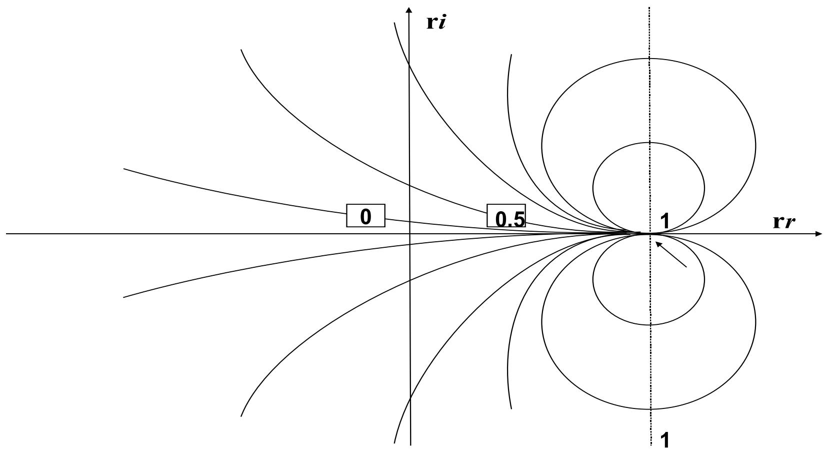


Figure 8.4 Smith chart

The points situated on a circle are all the impedances characterized by the same imaginary impedance part value x. For example, the circle $x = 1$ is centered at coordinate (1, 1) and has a radius of 1. All circles (constant x) include the

point (1,0). Differing with the real part circles, x can be positive or negative. This explains the duplicate mirrored circles at the bottom side of the complex plane. All the circle centers are placed on the vertical axis, intersecting the point 1.

8.4 Get the Picture

To complete our Smith chart, we superimpose the two circles' families. It can then be seen that all of the circles of one family will intersect all of the circles of the other family. Knowing the impedance, in the form of $r + jx$, the corresponding reflection coefficient can be determined. It is only necessary to find the intersection point of the two circles corresponding to the values r and x

8.5 The Reciprocation

The reverse operation is also possible. Knowing the reflection coefficient, find the two circles intersecting at that point and read the corresponding values r and x on the circles. The procedure for this is as follows:

- Determine the impedance as a spot on the Smith chart.
- Find the reflection coefficient (G) for the impedance.
- Having the characteristic impedance and G, find the impedance.
- Convert the impedance to admittance.
- Find the component values for the wanted reflection coefficient (in particular the elements of a matching network).

8.6 To Extrapolate

Because the Smith chart resolution technique is basically a graphical method, the precision of the solutions depends directly on the graph definitions. Here is an example that can be represented

Let's compare the two coordinates:

$$0.35 - j0.75 \text{ and } 0.5 \ldots\ldots + j\ 1.1$$

$$z_L = 0.35 - j0.75 = \sqrt{0.35^2 + 0.75^2}\angle \tan^{-1}\frac{0.75}{0.35}$$

$$= 0.828\angle - 64.98^{\circ}$$

$$\frac{1}{z_L} = \frac{1}{0.828\angle - 64.98^{\circ}} = \frac{1\angle 64.98^{\circ}}{0.828}$$

$$= \frac{\cos 64.98^{\circ} + j\sin 64.98^{\circ}}{0.828} =$$

$$0.51 + j\ 1.095\ \cong 0.5 \ldots . + j\ 1.1 = y$$

So, z_L and its reciprocal y, which is the normalized admittance, are found to lie at the opposite ends of the diameters of the circle centered at coordinates (1,0)! Herein lies one great use of which the smith chart is made.

8.7 Black Magic Design

The Smith chart is plotted on the complex reflection coefficient plane in two dimensions and is scaled in normalised impedance (the most common), normalised admittance or both, using different colours to distinguish between them. These are often known as the Z, Y and YZ Smith charts respectively. Normalised scaling allows the Smith chart to be used for problems involving any characteristic or system impedance which is represented by the center point of the chart. The most commonly used normalization impedance is 50 ohms. Once an answer is obtained through the graphical constructions described below, it is straightforward to convert between normalised impedance (or normalised admittance) and the corresponding unnormalized value by multiplying by the characteristic impedance (admittance). Reflection coefficients can be read directly from the chart as they are unitless parameters.

The Smith chart has a scale around its circumference or periphery which is graduated in wavelengths and degrees. The wavelengths scale is used in distributed component problems and represents the distance measured along the transmission line connected between the generator or source and the load to the point under consideration. The degrees' scale represents the angle of the voltage reflection coefficient at that point. The Smith chart may also be used for lumped-element matching and analysis problems.

Use of the Smith chart and the interpretation of the results obtained using it requires a good understanding of AC circuit theory and transmission-line theory, both of which are prerequisites for RF engineers.

As impedances and admittances change with frequency, problems using the Smith chart can only be solved manually using one frequency at a time, the result being represented by a point. This is often adequate for narrow band applications (typically up to about 5% to 10% bandwidth) but for wider bandwidths it is usually necessary to apply Smith chart techniques at more than one frequency across the operating frequency band. Provided the frequencies are sufficiently close, the resulting Smith chart points may be joined by straight lines to create a locus.

The Normalised Impedance Smith chart

A wave travels down a transmission line of characteristic impedance Z_0, terminated at a load with impedance Z_L and normalised impedance z = Z_L/Z_0. There is a signal reflection with coefficient ρ. Each point on the Smith chart simultaneously represents both a value of z_L (bottom left), and the corresponding value of ρ (bottom right), related by $z_L = \frac{1+\rho}{1-\rho}$

Using transmission-line theory, if a transmission line is terminated in an impedance (Z_T) which differs from its characteristic impedance (Z_0), a standing wave will be formed on the line comprising the resultant of both the incident or forward (v_i) and the reflected or reversed (v_r) waves. Using complex exponential notation:

$$\text{The normalised impedance } \Rightarrow z_L = \frac{R + j\omega L}{Z_0}$$

The SI unit of impedance is the ohm with the symbol of the uppercase Greek letter omega (Ω) and the SI unit for admittance is the Siemens with the symbol of an upper case letter S or moh (℧). Normalised impedance and normalised admittance are dimensionless. Actual impedances and admittances must be normalised before using them on a Smith chart. Once the result is obtained it may be de-normalized to obtain the actual result.

The Complete Smith Chart

Black Magic Design

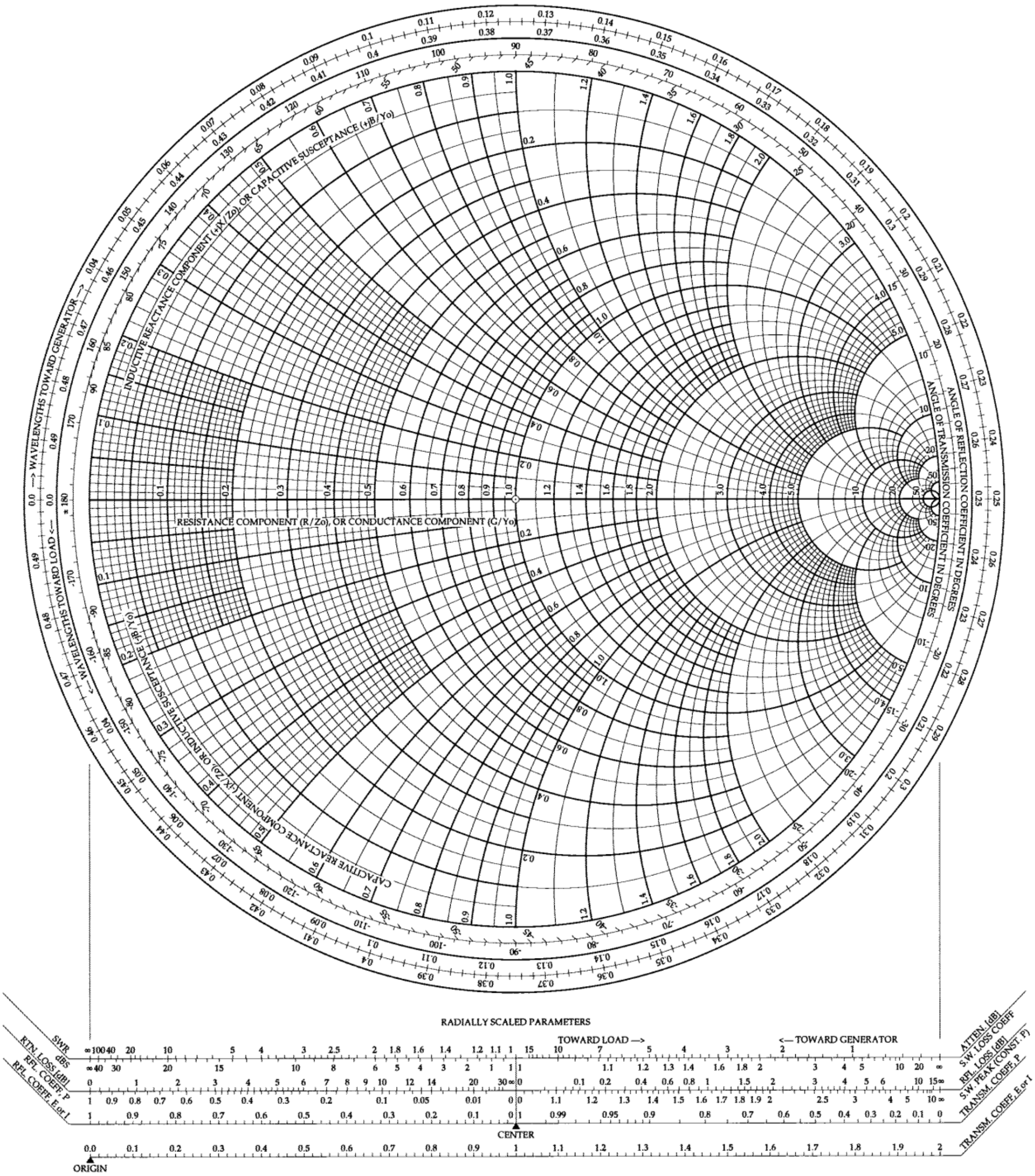


Figure 8.5

Example 8.1: A lossless transmission line of characteristics impedance $100\,\Omega$ is terminated in a load of $300 + j\ 150\ \Omega$. Determine with the use of Smith Chart:

i. The reflection coefficient
ii. The load admittance
iii. VSWR
iv. Distance between the load and the nearest voltage minimum to it.
v. Normalized input impedance z_{in}, given that the length of the line is 92cm and the signal wavelength is 40 cm.

Solution:

Normalized load impedances,

$$z_L = \frac{Z_L}{Z_0} = \frac{(300 + j150)}{100}$$

$$z_L = 3 + j1.5$$

We plot the normalized impedance in the smith chart as shown in the Fig.8.6.

a circle drawn through z_L cuts out $S = 3.8 = \text{VSWR}$ next, draw a line from z_L through the center (1,0) to **angle of reflection coefficient** (the innermost ring on the outside of the chart), giving $\psi = 16^0$.

The modulus (absolute value) of the reflection coefficient, $|\rho|$, measured on the scale in Fig. 8.6 gives approximately 0.6.

$$\Longrightarrow \rho = |\rho|e^{j\psi}$$

$$\rho = 0.6e^{j16^o}$$

$$\rho = 0.6e^{j0.28}$$

Load admittance y_L across the circle from z_L on the opposite side is approximately

$y_L = 0.26 - j0.14$.

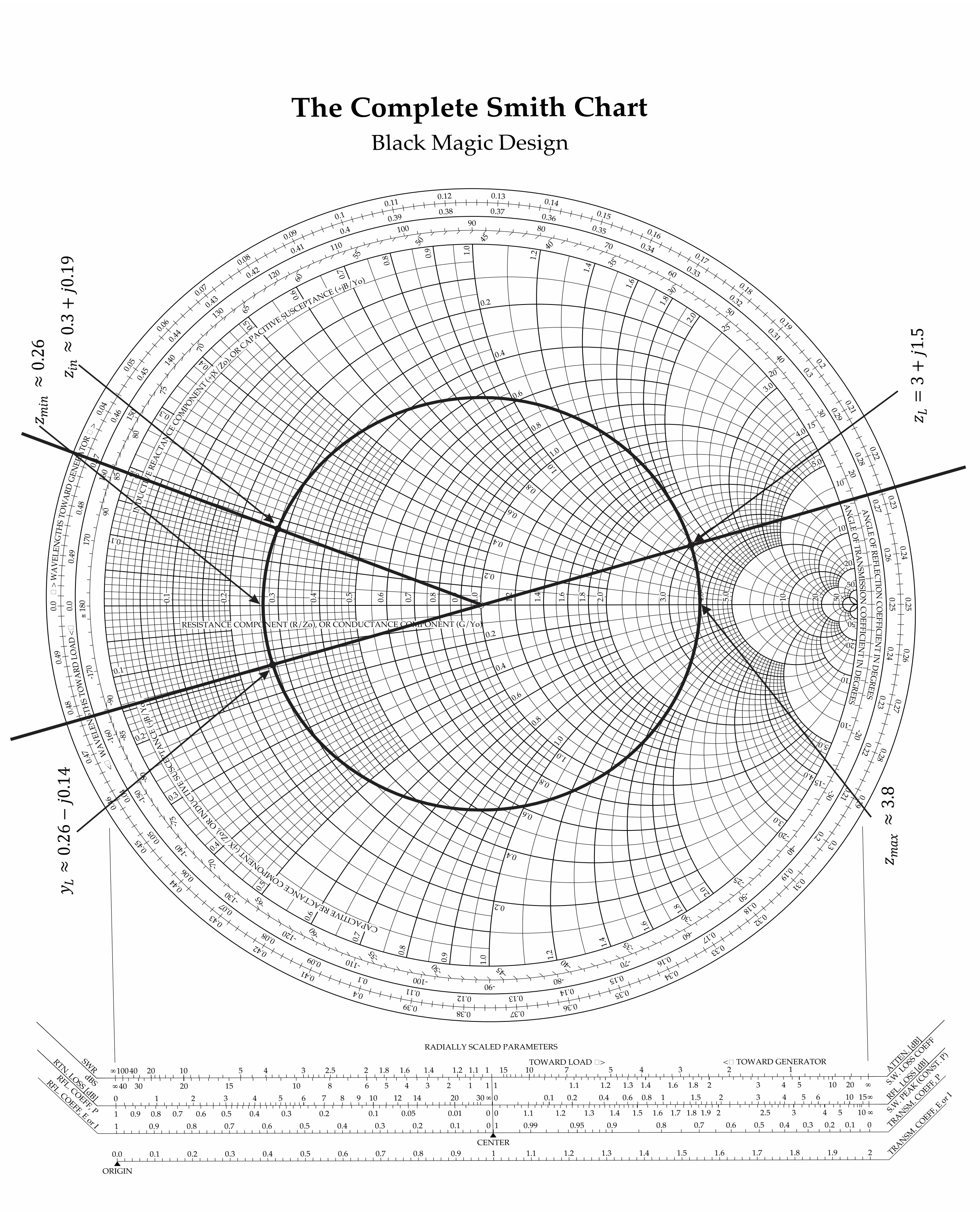

The Complete Smith Chart
Black Magic Design
$z_{min} \approx 0.26$
$z_{in} \approx 0.3 + j0.19$
$z_L = 3 + j1.5$
$y_L \approx 0.26 - j0.14$
$z_{max} \approx 3.8$
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
ANGLE OF TRANSMISSION COEFFICIENT IN DEGREES
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
RADIALLY SCALED PARAMETERS
TOWARD LOAD
TOWARD GENERATOR
SWR
dBS
RTN. LOSS [dB]
RFL. COEFF, P
RFL. COEFF, E or I
ATTEN. [dB]
S.W. LOSS COEFF
RFL. LOSS [dB]
S.W. PEAK (CONST. P)
TRANSM. COEFF, P
TRANSM. COEFF, E or I
CENTER
ORIGIN

Figure 8.6

The length of the line is defined as: Number of wavelengths $= \left(\frac{92}{40}\right)\lambda = 2.3\lambda$

Wavelength towards generator (WTG) reading at the load point extension line is 0.23λ

Therefore, z_{in} is at $2.3\lambda + 0.23\lambda = 2.53\lambda$

1 revolution $= 0.5\lambda$ (half a wavelength)

$$\Longrightarrow 2.53\lambda = 5revs + 0.03\lambda$$

z_{in} is therefore, at 0.03 on WTG scale of Fig. 8.6, reading $z_{in} \approx 0.26 + j0.19$

$z_{min} = \frac{1}{s}[\mathbf{Eq.(7.31)}]$ lies on the horizontal line to the left of (1,0) x_{min} is determined by reading the distance from z_L to v_{min} on WTG scale, and v_{min} is at point C on the chart which reads (0,0) on the scale.

$z_{min} \Longrightarrow v_{min} \approx 0.26$ from the load.

Example 8.2: A lossless transmission line of length 0.45λ and characteristic impedance $75\ \Omega$ is terminated in an impedance $195 + j135\ \Omega$. Find using Smith Chart:

(a) The voltage reflection coefficient
(b) The standing wave ratio
(c) Input impedance
(d) Location of voltage maximum on line
(e) z_{min}

Solution: given

$$Z' = 0.44\lambda$$

$$Z_o = 75\ \Omega$$

$$z_L = 195 + j135\ \Omega$$

(a) Let us first find the voltage reflection coefficient

1. Plot a smith chart $z_L = \frac{Z_L}{Z_0} = 2.6 + j1.8$ see point P

SMITH CHART

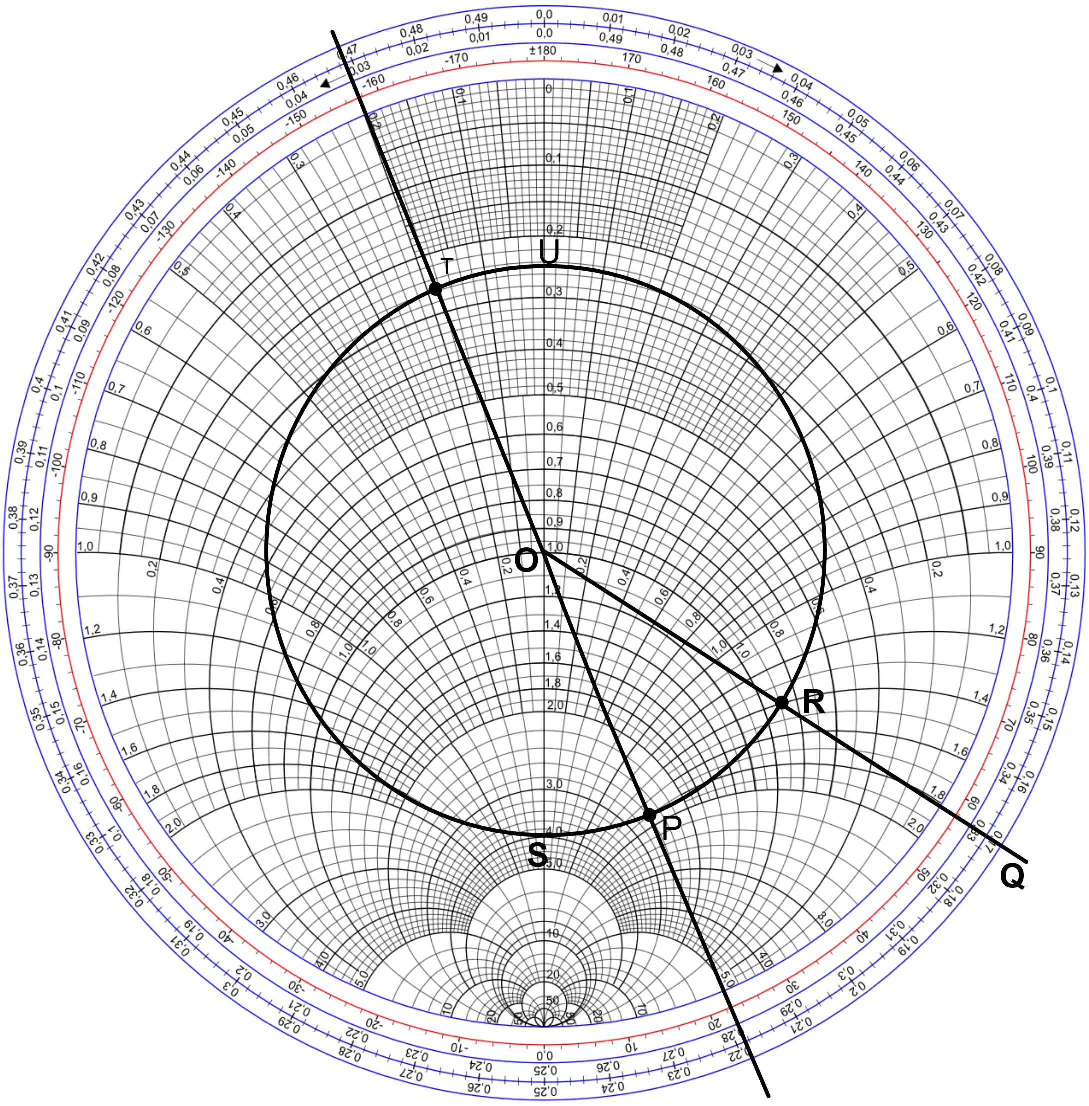


Figure 8.7

The load admittance is at point T which reads $y_L = 0.26 - j0.18$

2. With center at origin, draw a circle of radius $Op = |\rho| = 0.6$ Measured from the voltage or current reflection coefficient scale of the complete smith chart shown in Fig.8.5 which is the radius of chart $OP_{SC} =$ unity).
3. Draw line (straight) OP and extend it to OP on the periphery. Read 0.220 on WTG scale.
4. Phase angle ψ of reflection coefficient note that $4\pi \to 1rev\ of\ 2 \times 360^o$

$$\psi = (\text{change in wavelength}) \times 4\pi$$

$$= (0.25 - 0.22) \times 4\pi$$

$$0.12\pi \text{ (rad)}$$

$$\psi = 21^o$$

Therefore $$\rho = |\rho|e^{j\psi}$$

$$\rho = |0.6|e^{j0.37}$$

or

$$\rho = 0.6 \angle 21^o$$

(b) The $|\rho| = 0.6$ circle intersects with positive real-axis OP' at $r = S = 4$. Two, voltage standing wave ratio is 4

(c) To find input impedance more P' by total of 0.45λ WTG, i.e. $0.45\lambda + 0.22\lambda = 0.67\lambda = 1rev + 0.17\lambda$ or $(0.5 - 0.22) + 0.17$ or first to 0.5 (i.e $0.0, 0.0$) and then further to 0.17 to Q, $z_{in} = 0.9 + j1.4$ which is at point R

(d) Maximum voltage (z_{max}) is the same thing as the VSWR i.e the voltage standing wave. Therefore, $z_{max} = 4$

(e) The minimum voltage (z_{min}) is at point U on the smith chart. $z_{min} = 0.25$

Example 8.3: A lossless transmission line of characteristic impedance $125\,\Omega$ is terminated in a load of $350 - j200\ \Omega$. By the smith chart determine the:

(a) The load Admittance
(b) Reflection coefficient
(c) VSWR
(d) Normalize input impedance. Given that the length of the line is 1.6λ
(e) Distance between the load and the nearest voltage minimum to it

Solution: given

$$Z' = 1.6\lambda$$

$$Z_o = 125\Omega$$

$$Z_L = 350 - j200\ \Omega$$

(a) Just as in the foregoing **Example 8.2** solution, let us first of all find the voltage reflection coefficient by plotting the load impedance on the smith chart.

SMITH CHART

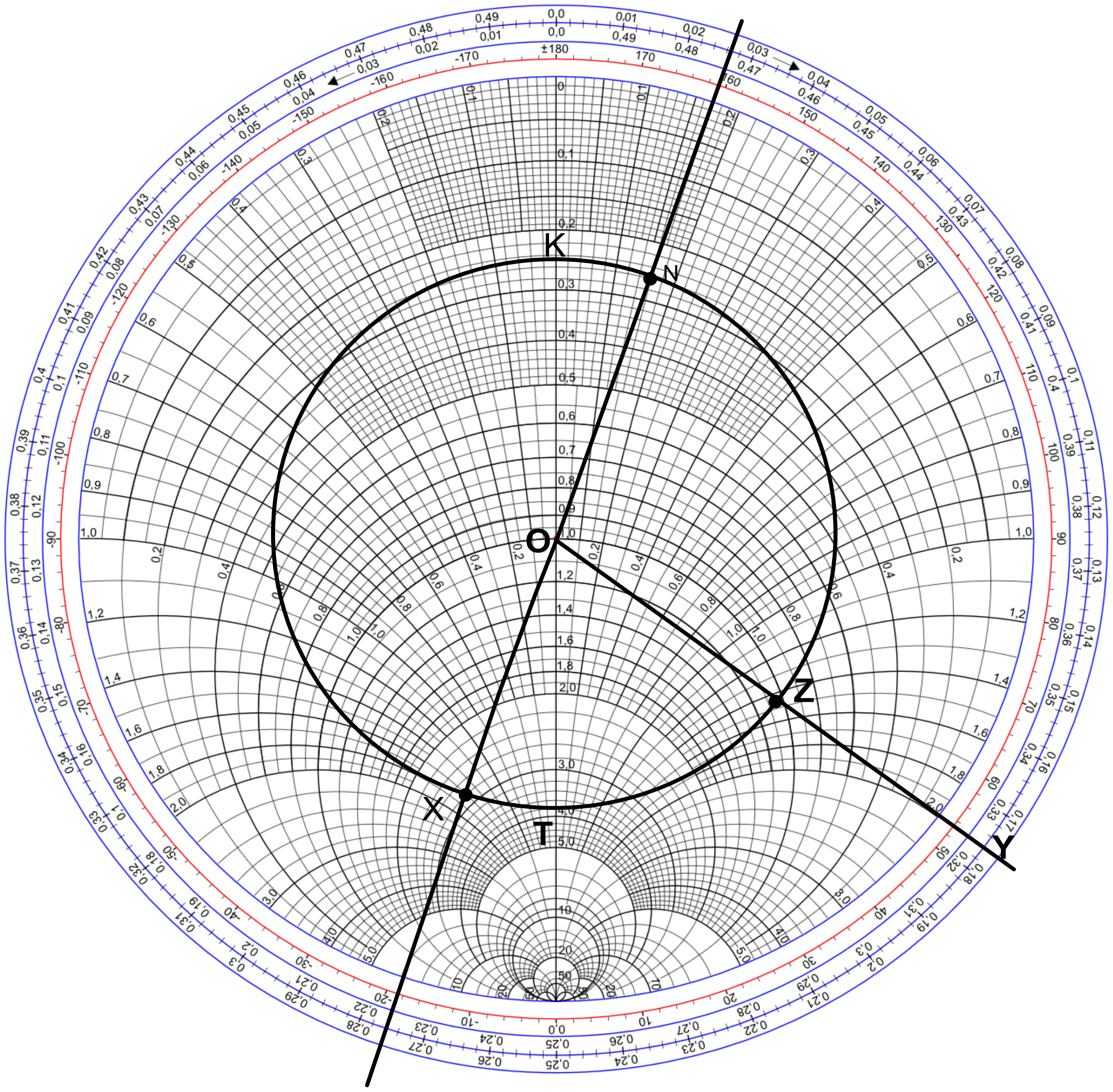


Figure 8.8

1. Plot a smith chart $z_L = \frac{Z_L}{Z_0} = 2.8 - j1.6$ see point

The load impedance is at point N which reads $y_L = 0.26 + j0.15$

(b)

2. With center at origin, draw a circle of radius $OX = |\rho| = 0.58$ Measured from the voltage or current reflection coefficient scale of the complete smith chart shown in Fig.8.5 which is the radius of chart $OX_{SC} =$ unity).
3. Draw line (straight) OX and extend it to OX on the periphery. Read 0.224 on **wavelength towards load (WTL)** scale.
4. Phase angle ψ of reflection coefficient note that $4\pi \rightarrow 1rev\ of\ 2x360^o$

$$\psi = (\text{change in wavelength}) \times 4\pi$$

$$= (0.25 - 0.224) \times 4\pi$$

$$0.026\pi \text{ (rad)}$$

$$\psi = -18^o$$

Therefore

$$\rho = |\rho| e^{j\psi}$$

$$\rho = |0.58| e^{-j0.32}$$

or

$$\rho = 0.58\ \angle{-18^o}$$

(c) The $|\rho| = 0.58$ circle intersects with positive real-axis OP' at $r = T = 3.8$. Two, voltage standing wave ratio is 3.8

(d) To find input impedance more X' by total of 1.6λ "WTL", which is the anticlockwise i.e. $1.6\lambda + 0.224\lambda = 1.824\lambda = 3\,rev + 0.324\lambda$ or first to 1.5 (i.e. $0.0, 0.0$) and then further to 0.324 to Y which reads input impedance $z_{in} = 1 + j1.4$ at point Z

(e) The minimum voltage (z_{min}) is at point K on the smith chart. $z_{min} = 0.26$

Example 8.4: A lossless transmission line with $Z_0 = 60\ \Omega$ is 80 m long and operates at 6 MHz, the line is terminated with a load of $Z_L = 120 + j60\ \Omega$.

Given that $u = 0.5c$ on the line, $c = 3 \times 10^8$ m/s determine by the use of smith chart:

(i) Load admittance
(ii) Voltage reflection coefficient
(iii) VSWR
(iv) z_{in}
(v) z_{max}
(vi) z_{min}

Solution:

Given that $u = 0.5c$ on the line, $c = 3 \times 10^8$ m/s

$$\text{The line wavelength}(\lambda) = \frac{c}{f} = \frac{0.5 \times 3 \times 10^8}{6 \times 10^6} = 25 \text{ m}$$

$$Z_o = 60\ \Omega$$

$$Z_L = 120 + j60\ \Omega$$

$$z_L = \frac{Z_L}{Z_0} = \frac{120 + j60}{60} = 2 + j1\ \Omega$$

SMITH CHART

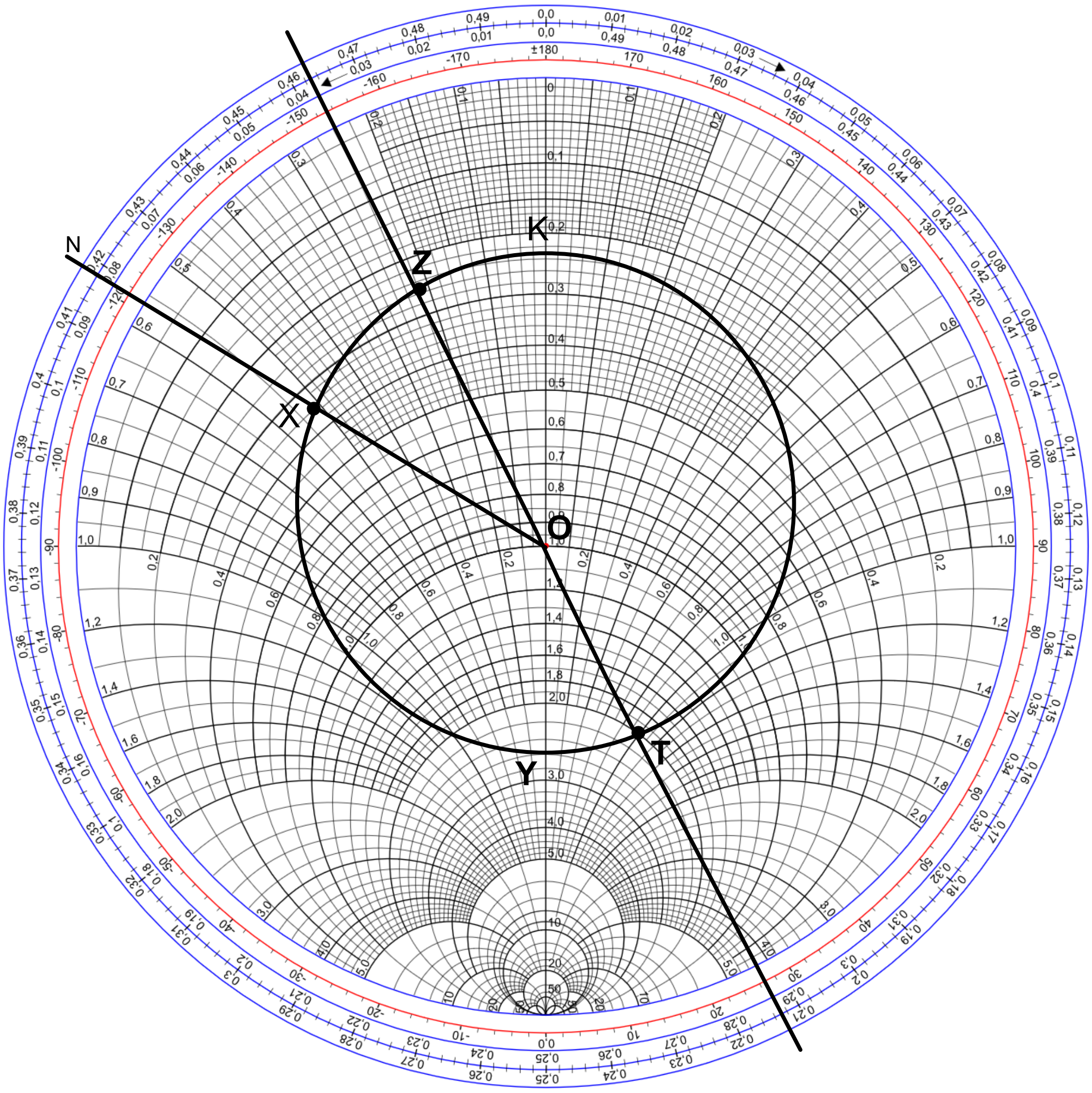


Figure 8.9

(i) Load admittance is at point Z which reads $y_L = 0.4 - j0.2\ \mho$

(ii) Phase angle ψ of reflection coefficient note that

$$4\pi \rightarrow 1rev \quad of \quad 2 \times 360^o$$

$$\psi = (\text{change in wavelength}) \times 4\pi$$

$$= (0.25 - 0.218) \times 4\pi$$

$$= 0.032\pi\ (\text{rad})$$

$$\psi = 28^o$$

Therefore

$$\rho = |\rho| e^{j\psi}$$

$$\rho = |0.44| e^{j0.49}$$

or

$$\rho = 0.44\ \angle 28^o$$

(iii) VSWR is at point Y on the chart at it reads 2.6

(iv) To find input impedance (z_{in}) more T' by total of 3.2λ WTG, i.e. $3.2\lambda + 0.214\lambda = 3.414\lambda = 6rev + 0.414\lambda$ or first to 3.0 (i.e. $0.0, 0.0$). And then further to 0.414 to N which reads input impedance $z_{in} \approx 0.35 - j0.5$ as seen in point X

(v) z_{max} is at point Y i.e., 2.6

(vi) z_{min} is at point K i.e., ≈ 0.24

8.8 Uses of Smith Chart

1. For impedance, the intersection of the circles with the line $r + j0$ gives the maximum impedance on the line at the intersection to the right of (1,0) and the minimum impedance at the intersection to the left. Voltage minima occur at impedance maxima.
2. For admittance, the intersection of the circles with the line $g + j0$ gives the maximum admittance at the intersection to the right of (1,0) and the minimum admittance at the intersection to the left. Voltage minima are at admittance maxima.

A locus of points on a Smith chart covering a range of frequencies can be used to visually represent:

3. how capacitive or how inductive a load is across the frequency range
4. how difficult matching is likely to be at various frequencies

5. how well matched a particular component is!

8.9 Exercise

1. A lossless transmission line with $Z_0 = 60\ \Omega$ is 80 m long and operates at 6 MHz, the line is terminated with a load of $Z_L = 120 + j60\ \Omega$.
Given that $u = 0.5c$ on the line, $c = 3 \times 10^8\ m/s$ determine by the use of Smith chart:

(i) Load admittance
(ii) voltage reflection coefficient (magnitude & phase)
(iii) VSWR
(iv) z_{in}

2. A lossless transmission line with $Z_0 = 80\ \Omega$ is 40 m long and operates at 3 MHz, the line is terminated with a load of $Z_L = 120 + j60\ \Omega$.
Given that $u = 0.8c$ on the line, $c = 3 \times 10^8\ \mathrm{m/s}$ determine by the use of smith chart:

(vii) Load admittance
(viii) Voltage reflection coefficient
(ix) VSWR
(x) z_{in}
(xi) z_{max}
(xii) z_{min}

3. A lossless transmission line with $Z_0 = 100\ \Omega$ is 40 m long and operates at 3 MHz, the line is terminated with a load of $Z_L = 300 + j150\ \Omega$.
Given that $u = 0.8c$ on the line, $c = 3 \times 10^8\ \mathrm{m/s}$ determine by the use of smith chart:

(i) load admittance
(ii) Voltage reflection coefficient
(iii) VSWR
(iv) z_{in}
(v) z_{max}
(vi) z_{min}

4. A lossless transmission line of characteristic impedance 150 Ω is terminated in a load of 350 - j200 Ω, 92 cm long and operates at 3 MHz Determine using Smith Chart:
(a) Load impedance

(b) reflection coefficient
(c) VSWR

5. A lossless transmission line with $Z_0 = 100\ \Omega$ is 80 m long and operates at 6 MHz, the line is terminated with a load of $Z_L = 300 - j150\ \Omega$.
Given that $u = 0.8c$ on the line $c = 3 \times 10^8\ \mathrm{m/s}$, determine by the use of smith chart:

(i) load impedance
(ii) Voltage reflection coefficient
(iii) VSWR
(iv) z_{in}
(v) z_{max}
(vi) z_{min}

6. A lossless transmission line with $Z_0 = 60\ \Omega$ is 85 m long and operates at 5 MHz, the line is terminated with a load of $Z_L = 120 + j60\ \Omega$.
Given that $u = 0.8c$ on the line, $c = 3 \times 10^8\ \mathrm{m/s}$,determine by the use of smith chart:

(i) Load admittance
(ii) Voltage reflection coefficient
(iii) VSWR
(iv) z_{in}
(v) z_{max}
(vi) z_{min}

7. A load of $25 + j50\ \Omega$ terminates at $50\ \Omega$ line, use Smith chart to determine:

(i) The load admittance
(ii) The reflection coefficient (amplitude and phase)
(iii) Voltage Standing Wave Ratio
(iv) Input impedance, given that the line is 60 cm long and the signal wavelength 2 m
(i) Load Impedance
(ii) Reflection coefficient
(iii) VSWR
(iv) Distance between the load and the nearest voltage minimum to it
(v) Normalized input impedance. Given that the length of the line is 80 cm and the signal wavelength is 50 cm

MULTICHOICE QUESTION A

1. The Laplace transform of the function $f(t)$ is defined to be (a) $\int_{0^-}^{\infty} f(t)\,dt$ (b) $\int_{0^-}^{\infty} e^{-st} f(t)\,dt$ (c) $\int_{0^-}^{\infty} f(t)\, e^{-st}$ (d) $\int_{-\infty}^{0^+} e^{-st} f(t)\,dt$
2. ***s*** in the function in the options in **question 1:** (1) may be complex number (2) may be real number (3) is cyclic frequency (4) complex frequency (a) 1&2 above (b) 2&3 (c) 1, 2 & 3 (d) 1, 2 & 4
3. Laplace transform of $f(t)$ is a function of (a) time (b) complex frequency (c) time and complex frequency (d) none of these
4. For complex frequency $s = \sigma + j\omega$, the following represents the nature of $\sigma{:}\,\sigma$ (1) has a damping effect (2) causes the convergence of the integral $\int_{t}^{-\infty} f(t)e^{-st}dt$ (3) < 0: (a) 1,2,3 (b) 1,2 (c) 2,3 (d) 1,3
5. Double integration of a unit step function leads to (a) an impulse (b) a parabola (c) a ramp (d) a doublet
6. The open circuit voltage ratio $V_2(s)/V_1(s)$ of the network shown in fig A 1 is (a) $1 + 2s^2$ (b) $1/(1+s^2)$ (c) $1 + 2s$ $(d) 1/(1+2s)$

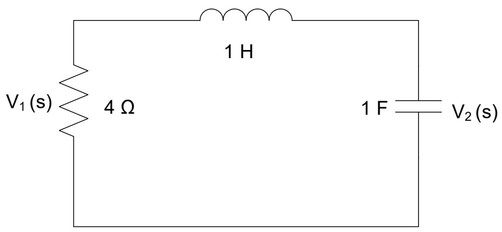


Figure A1

7. The response of an initially relaxed linear circuit to a signal V_s is $e^{-2t}u(t)$. If the signal is changed to $\left(V_s + 2\frac{dv_s}{dt}\right)$, the response will be (a) $-4e^{-2t}u(t)$ (b) $-3e^{-2t}u(t)$ (c) $4e^{-2t}u(t)$ (d) $5e^{-2t}u(t)$
8. A first-order linear system is initially relaxed system for a unit step signal $u(t)$. The response if $V_1(t) = (1 - e^{-3t})$ for $t > 0$. If a signal $3u(t) + \delta(t)$ is applied to the same initially relaxed system, the response will be: (a) $(3 - 6e^{-3t})u(t)$ (b) $(3 - 3e^{-3t})u(t)$ (c) $3u(t)$ (d) $(3 + 3e^{3t})u(t)$
9. The Laplace transform of $(t^2 - 2t)u(t-1)$ is: (a) $(2e^{-s}/s^2) - (2e^{-s}/s^1)$ (b) $\left(\frac{2e^{-2s}}{s^3} - \frac{2e^{-t}}{s^2}\right)$ (c) $(2e^{-s}/s^1) - (e^{-s}/s)$ (d) none of the above

10. The unit impulse response of a system is $c(t) = 4e^{-t} + 6e^{-2t}$. The step response of the same system for $t \geq 0$ is: (a) $-3e^{-2t} + 4e^{-t} + 1$ (b) $-3e^{-2t} - 4e^{-t} + 1$ (c) $-3e^{-2t} - 4e^{-t} - 1$ (d) $3e^{-2t} + 4e^{-t} - 1$
11. Given $I(s) = (10s + 4)/s(s + 1)(s^2 + 4s + 5)$ the final value of $i(t)$is: (a) 4/5 (b) 5/4 (c) 4 (d) 5
12. Given $F(s) = (s + 4)/s(s + 2)$ final and initial values of $f(t)$ will be: (a) 1,1 (b) 1,2 (c) 2,2 (d) 2,1
13. The d.c gain of a system represented by the transfer function $10(s + 1)(s + 2)$ is (a) 1 (b) 2 (c) 5 (d) 10
14. The current response for the circuit shown in fig A2 is: (a) $1 - e^{-t}$ (b) $1 + e^{-(t-s)}u(t - 5)$ (c) $(1 - e^{-t})u(t - 5)$ (d) $1 - e^{-(t-3)}$

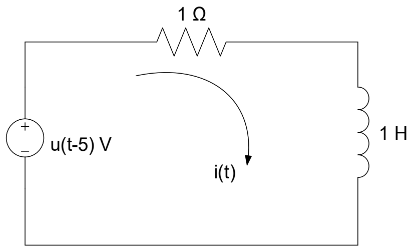


Figure A2

15. The response of an initially relaxed system to a unit ramp excitation is $(t + e^{-t})$, its step response is: (a) $\frac{1}{2}t^2 - e^{-t}$ (b) $1 - e^{-t}$ (c) $-e^{-t}$ (d) t
16. The impulse response of an RL circuit is: (a) rising exponential function (b) decaying exponential function (c) step function (d) parabolic function
17. The Laplace transform of a rectangular current pulse of duration T and magnitude t is (a) $1/s$ (b) $(1/s)e^{-sT}$ (c) $(1/s)e^{sT}$ (d) $\left(\frac{1}{s}\right)/(1 - e^{-sT})$
18. Given $\mathcal{L}[x(t)] = X(s), \mathcal{L}[x(t - t)]$ equals (a) $e^{sT}X(s)$ (b) $e^{-sT}X(s)$ (c) $X(s)/(1 + e^{sT})$ (d) $X(s)/(1 - e^{-sT})$
19. The response of a network for $t \geq 0$ is $v(t) = Kte^{-\propto t}$, with $\propto$ real and positive. The value of t that results in maximum value of $v(t)$ is (a) $(\propto)$ (b) $2 \propto$ (c) $1/\propto$ (d) $\propto^2$
20. Given $H(s) = a/(s^2 + a^2)$ then the final value of $h(t)$ is (a) zero (b) indeterminate (c) unity (d) ∞(undefined)
21. The Laplace transform of a unit ramp function at $t = a$ is (a) $1/(s + a)^2$ (b) $e^{-as}/(s + a)^2$ (c) a/s^2 (d) e^{-as}/s^2
22. The Laplace transform of the voltage across a capacitor of 0.5F is: $V_c(s) = (s + 1)/(s^2 + s + 1)$. Then current $i(0^+)$ through the capacitor is (a) 0 A (b) 0.5 A (c) 2 A (d) 1 A

23. The response of an initially relaxed linear constant parameter network to a unit impulse applied at $t = 0$ is $4e^{-2t}u(t)$. The response to a unit step function will therefore be: (a) $2[(1 - e^{-2t})u(t)]$ (b) $4|e^{-t} - e - 1^{-2t}|u(t)$ (c) $\cos 2t$ (d) $(1 - 4e^{-4t})u(t)$

24. The closed loop transfer function of a control system is given by: $\frac{Y(s)}{G(s)} = 2(s-1)/(s+2)(s+1)$. Given that $g(t) = u(t)$, then response $y(t)$ is (a) $-3e^{-2t} + 4e^{-t} - 1$ (b) $-3e^{-2t} - 4e^{-t} + 1$ (c) undefined (infinity) (d) zero

25. Given $X(s) = \frac{(5-s)}{(s^2-s-2)}, x(t)$ is: (a) $e^{2t}u(t) - 2e^{-t}u(t)$ (b) $-e^{2t}u(-t) + 2e^{-t}u(t)$ (c) $-e^{2t}u(-t) - 2e^{-t}u(t)$ (d) none of the above

26. Given $Y(s) = \frac{(s+5)}{(s+1)(s+3,)}, y(t)$ is: (a) $2e^{-t} - e^{-3t}$ (b) $2e^{-t} + e^{-3t}$ (c) $e^{-t} - e^{-3t}$ (d) $e^{-t} + e^{-3t}$

27. Given $X(s) = \frac{(2s+1)}{(s^2+8s^2+16s+5)}, x(\infty)$ equals (a) ∞ (b) unity (c) zero (d) 2

28. Reflection coefficient for the transmission line shown in fig A3, is (a) 1 (b) -1 (c) 0 (d) 0.5

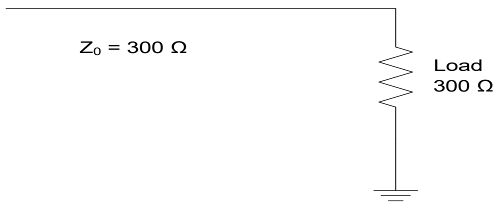


Figure A3

29. For a 200Ω line with a pure capacitive impedance of $-j200\,\Omega$ determine the reflection (a) 0 (b) 1 (c) $1\angle -\pi/2$ (d) $1\angle 90^o$

30. Transmission of power to a load over a transmission line achieves optimum value when standing wave ratio (SWR) is (a) $2:1$ (b) $1:2$ (c) $1:1$ (d) $1:\sqrt{2}$

31. For an open-circuited load, voltage reflection coefficient is (a) $|\rho| = 1, \beta = \pi$ (b) $|\rho| = 0, \beta = \pi$ (c) $|\rho| = 1, \beta - \pi$ (d) $|\rho| = 1, \beta = \pi$

32. On the smith chart, the area on the upper half stand for (a) inductive reactance and capacitive reactance (b) inductive reactance and capacitive susceptive (c) inductive susceptance and capacitive susceptance

33. For a given voltage signal: V_{max} (a) $V_1(1 - |\rho|)$ (b) $V_1(1 + |\rho|)$ (c) $V_1(1 - |\rho|)^2$ (d) $V_1(1 + |\rho|)^2$

34. The inverse Laplace transform of $F(s)$ is (a) $\mathcal{L}\{F(s)\} = f(s)$ (b) $\mathcal{L}^{-1}\{F(s)\} = f(t)$ (c) $\mathcal{L}\{F(s)\} = f(t)$ (d) $\mathcal{L}\{F(s)\}^{-1} = f(t)$

35. Laplace transform of 1 is (a) $s(s>0)$ (b) $s(s<0)$ (c) $\frac{1}{s}(s>0)$ (d) $\frac{1}{s}(s<0)$

36. $\mathcal{L}(e^{at}) =$ (a) $1/(s-a)(s<a)$ (b) $1/(s-a)(s>a)$ (c) $1/(s+a)(s>a)$ (d) $(s-a)(s>a)$

37. Laplace transform of e^{-at} is (a) $1/(s-a)(s<-a)$ (b) $1/(s+a)(s>a)$ (c) $1/(s-a)(s>a)$ (d) $1/(s+a)(s>-a)$

38. $\mathcal{L}(e^{at}) =$ (a) $2/(s-a)^2$ (b) $2/(s-a)$ (c) $1/(s-a)^2$ (d) $1/(s+a)^2$

39. Laplace transform of $\sin at$ is (a) $s/(s^2-a^2)$ (b) $s/(s^2+a^2)$ (c) $a/(s^2-a^2)$ (d) $a/(s^2+a^2)$

40. $\mathcal{L}\{e^{at}\sin bt\} =$ (a) $a/[(s-b)^2+a^2]$ (b) $b/[(s-a)^2+b^2]$ (c) $a/[(s-b)^2+b^2]$ (d) $b/[(s-a)^2+a^2]$

41. Laplace transform of $\cos bt$ is (a) $s/(s^2-b^2)$ (b) $s/(s^2+b^2)$ (c) $b/(s^2-b^2)$ (d) $b/(s^2+b^2)$

42. $\mathcal{L}\{e^{at}\cos bt\} =$ (a) $a/[(s-b)^2+a^2\,]$ (b) $b/[(s-a)^2+b^2]$ (c) $(s-a)/[(s-a)^2+b^2]$ (d) $(s-b)/[(s-a)^2+a^2]$

43. Laplace transform of $\sinh bt$ is (a) $b/(s^2-b^2)(s>|b|)$ (b) $b/(s^2+b^2)(s>|b|)$ (c) $b/(s^2-b^2)(s<|b|)$ (d) $b/(s^2+b^2)(s<|b|)$

44. $\mathcal{L}\,[\cosh at] =$(a) $a/(s^2-a^2)(s>|a|)$ (b) $s/(s^2+a^2)(s>|a|)$ (c) $s/(s^2-a^2)(s>|a|)$ (d) $a/(s^2+a^2)(s>|a|)$

45. Laplace transform of $e^{-at}f(t)$ is (a) $F(s-a)$ (b) $2F(s+a)$ (c) $e^{as}F(s-a)$ (d) $e^{as}F(s+a)$

46. $\mathcal{L}^{-1}(1/s) =$ (a) $\delta(t)$ (b) t (c) 1 (d) e^t

47. Laplace transform of $e^{2t}+4t^3$ is (a) $\frac{1}{(s+2)}+\frac{6}{s^3}$ (b) $\frac{1}{(s-2)}+\frac{24}{s^4}$ (c) $\frac{1}{(s+2)}+\frac{24}{s^4}$ (d) $\frac{1}{(s-2)}+\frac{6}{s^3}$

48. $\mathcal{L}\,\{2\sin 3t+3\cos 3t\} =?$ (a) $\frac{2}{(s^2-9)}+\frac{3s}{(s^2-9)}$ (b) $\frac{2}{(s^2+9)}+\frac{3s}{(s^2+9)}$ (c) $\frac{6}{(s^2+9)}+\frac{3s}{(s^2+9)}$ (d) $\frac{6}{(s^2-9)}+\frac{3s}{(s^2-9)}$

49. The response of an initially relaxed linear circuit to a signal V_s is $e^{-3t}u(t)$. If the signal is changed to $V_s+2\frac{dV_s}{dt}$, the response becomes (a) $-3e^{-2t}u(t)$ (b) $-4e^{-2t}u(t)$ (c) $5e^{-2t}u(t)$ (d) $4e^{-2t}u(t)$

50. A first order linear system is initially relaxed for a unit step signal $u(t)$. The response is $v_1(t) = (1-e^{-4t})$ for $t>0$. If a signal $4u(t)+\delta(t)$ is applied to the same initially relaxed system, the response will be (a) $(4-8e^{-4t})u(t)$ (b) $(4-4e^{-4t})u(t)$ (c) $(4+4e^{-4t})u(t)$ (d) $4u(t)$

51. Laplace transform of $(t^2-2t)u(t-1)$ is (a) $\frac{2e^{-s}}{s^3}-\frac{2e^{-s}}{s^2}$ (b) $\frac{2e^{-s}}{s^3}-\frac{e^{-s}}{s}$ (c) $\frac{2e^{-s}}{s^3}-\frac{2e^{-s}}{s^2}$ (d) none of the above

52. The unit impulse response of a system is $y(t) = -4e^{-t} + 3e^{-2t}$. The step response of the same system for $t > 0$ will be (a) $4e^{-2t} - 3e^{-2t} + 1$ (b) $4e^{-2t} - 3e^{-2t} - 1$ (c) $-4e^{-2t} + 3e^{-2t}$ (d) $-4e^{-2t} - 3e^{-2t} + 1$

53. The final value of $y(t)$ for $Y(s) = \frac{(10s+4)}{s(s+1)(s^2+4s+5)}$ will be (a) 0 (b) ∞ (c) 0.8 (d) $4,5$

54. The unit step response of a network is $(1 - e^{-at})$. Its unit impulse response will be (a) ae^{-at} (b) $\frac{1}{ae^{-t/a}}$ (c) $\frac{e^{-\frac{t}{a}}}{a}$ (d) $(1-a)e^{-at}$

55. The voltage through a resistor with current $i(t)$ in the s-domain is (a) $sRI(s)$ (b) $s^2RI(s)$ (c) $RI(s)$ (d) $V/I(s)$

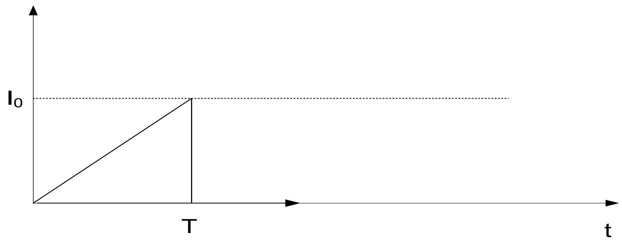


Figure A4

56. Laplace transform of the waveform shown Fig A4 is (a) $\frac{I_0}{Ts^2} + \left(\frac{I_0}{Ts}\right)e^{-Ts}$ (b) $\frac{I_0}{Ts^2} + (I_0/Ts^2)e^{-Ts}$ (c) $\frac{I_0}{Ts^2} + \left(\frac{I_0}{Ts^2}\right)e^{-Ts}(1+sT)$ (d) $\left(\frac{I_0}{Ts^2}\right) - I_0$

57. Given $Y(s) = \frac{(s+2)}{s(s+1)}$, the initial and final values of $y(t)$ will be respectively: (a) 2,1 (b) 1,2 (c) 2,2 (d) 1,1

58. The impedance of a $10 - H$ inductor is (a) $\frac{10}{s}$ (b) $\frac{s}{10}$ (c) $\frac{1}{10s}$ (d) $10s$

59. Laplace transform of $e^{-at}f(t)$ is (a) $F(s-a)$ (b) $F(s+a)$ (c) $e^{as}F(s-a)$ (d) $e^{as}F(s+a)$

60. $\mathcal{L}^{-1}\left(\frac{1}{s}\right) =$ (a) $\delta(t)$ (b) t (c) 1 (d) e^t

MULTICHOICE QUESTION B

1. The Laplace transform of the function $f(t)$ is defined to be (a) $\int_{0^-}^{\infty} f(t)\ dt$ (b) $\int_{0^-}^{\infty} e^{-st} f(t)dt$ (c) $\int_{0^-}^{\infty} f(t)e^{-st}$ (d) $\int_{-\infty}^{0^-} e^{-st} f(t)dt$
2. S above (1) may be complex number (2) may be real or complex number (3) is cyclic frequency (4) is complex frequency (a) 1&2 above (b) 2&3 (c) 1,2&3 (d) 1,2&4

3. Laplace transform of $f(t)$ is a function of (a) time (b) complex frequency (c) time &complex frequency (d) none of the above
4. The inverse Laplace transform of $F(s)$ is (a) $\mathcal{L}\{F(s)\} = f(s)$ (b) $\mathcal{L}^{-1}\{F(s)\} = f(t)$ (c) $\mathcal{L}\{F(s)\} = f(t)$ (d) $\mathcal{L}\{F(s)\}^{-1} = f(t)$
5. Laplace transform of 1 is (a) $s(s > 0)$ (b) $s(s < 0)$ (c) $\frac{1}{s}(s > 0)$ (d) $1/s(s < 0)$
6. $\mathcal{L}(e^{at}) =$ (a) $1/(s-a)\ (s < a)$ (b) $1/(s-a)(s > a)$ (c) $1/(s+a)(s > a)$ (d) $(s-a)(s > a)$
7. Laplace transform of e^{-at} is (a) $1/(s-a)(s < -a)$ (b) $1/(s+a)(s > a)$ (c) $1/(s-a)(s > a)$ (d) $1/(s+a)(s > -a)$
8. $\mathcal{L}(te^{at}) =$(a) $2/(s-a)^2$ (b) $2/(s-a)$ (c) $1/(s-a)^2$ (d) $1/(s+a)^2$
9. Laplace transform of $\sin at$ is (a) $s/(s^2-a^2)$ (b) $s/(s^2+a^2)$ (c) $a/(s^2-a^2)$ (d) $a/(s^2+a^2)$
10. $\mathcal{L}\{e^{at}\sin bt\} =$ (a) $a/[(s-b)^2+a^2]$ (b) $b/[(s-a)^2+b^2]$ (c) $a/[(s-b)^2+b^2]$ (d) $b/[(s-a)^2+a^2]$
11. Laplace transform of $\cos bt$ is (a) $s/(s^2-b^2)$ (b) $s/(s^2+b^2)$ (c) $b/(s^2-b^2)$ (d) $b/(s^2+b^2)$
12. $\mathcal{L}\{e^{at}\cos bt\} =$ (a) $a/[(s-b)^2+a^2]$(b) $b/[(s-a)^2+b^2]$(c) $(s-a)/[(s-a)^2+b^2]$ (d) $(s-b)/[(s-a)^2+a^2]$
13. Laplace transform of $\sin h\ \ bt$ is (a) $b/(s^2-b^2)(s > |b|)$ (b) $b/(s^2+b^2)(s > |b|)$ (c) $b/(s^2-b^2)(s < |b|)$ (d) $b/(s^2+b^2)(s < |b|)$
14. $\mathcal{L}[\cosh at] =$ (a) $a/(s^2-a^2)(s > |a|)$ (b) $s/(s^2+a^2)(s > |a|)$ (c) $s/(s^2-a^2)(s > |a|)$ (d) $a/(s^2+a^2)(s > |a|)$
15. Laplace transform of $e^{-at}f(t)$ is (a) $F(s-a)$ (b) $F(s+a)$ (c) $e^{as}F(s-a)$ (d) $e^{as}F(s+a)$
16. $\mathcal{L}^{-1}(1/s) =$ (a) $\delta(t)$ (b) t (c) 1 (d) e^t
17. Laplace transform of $e^{2t} + 4t^3$ is (a) $\frac{1}{(s+2)} + 6/s^3$ (b) $\frac{1}{(s-2)} + 24/s^4$ (c) $\frac{1}{(s+2)} + 24/s^4$ (d) $\frac{1}{(s-2)} + 6/s^3$
18. $\mathcal{L}\{2\sin 3t + 3\cos 3t\} =$ (a) $\frac{2}{(s^2-9)} + 3s/(s^2-9)$ (b) $\frac{2}{(s^2+9)} + 3s/(s^2+9)$ (c) $\frac{6}{(s^2+9)} + 3s/(s^2+9)$ (d) $\frac{6}{(s^2-9)} + 3s/(s^2-9)$
19. The response of an initially relaxed linear circuit to a signal V_s is $e^{-2t}u(t)$. If the signal is changed to $V_s + 2\frac{dVs}{dt}$, the response becomes (a) $-3e^{-2t}u(t)$ (b) $-4e^{-2t}u(t)$ (c) $5e^{-2t}u(t)$ (d) $4e^{-2t}u(t)$
20. The first order linear system is initially relaxed for a unit step signal $u(t)$. The response if $V_1(t) = (1 - e^{-4t}\ for\ t > 0)$. If a signal $4u(t) + \delta(t)$ is applied to the same initially relaxed system, the response will be (a) $(4 - 8e^{-4t})u(t)$ (b) $(4 - 4e^{-4t})u(t)$ (c) $(4 + 4e^{-4t})u(t)$ (d) $\big(4u(t)\big)$

21. Laplace transform of $(t^2-2t)u(t-1)$ is (a)$\frac{2e^{-s}}{s^3}-2e^{-s}/s^2$ (b) $\frac{2e^{-s}}{s^3}-\frac{e^{-s}}{s}$ (c) $\frac{2e^{-2s}}{s^3}-\frac{2e^{-s}}{s^2}$ (d) none of the above
22. The unit impulse response of a system is $y(t)=-4e^{-t}+6e^{-2t}$. The step response of the same system for $t>0$ will be (a) $4e^{-2t}-3e^{-2t}+1$ (b) $4e^{-2t}-3e^{-2t}-1$ (c) $-4e^{-2t}+3e^{-2t}+1$ (d) $-4e^{-2t}-3e^{-2t}+1$
23. The final value of $y(t)$ for $Y(s)=\frac{(10s+4)}{s(s+1)(s^2+4s+5)}$ will be (a) 0 (b) ∞ (c) 0.8 (d) 4.5
24. The unit step response of a network is $(1-e^{-at})$. Its unit impulse response will be (a) ae^{-at}(b) $1/ae^{-t/a}$ (c) $e^{-t/a}/a$ (d) $(1-a)e^{-at}$
25. The impulse response of an $R-L$ circuit is a (a) rising exponential function (b) decaying exponential function (c) step function (d) parabolic function
26. Laplace transform of the waveform in Fig 9 is ………

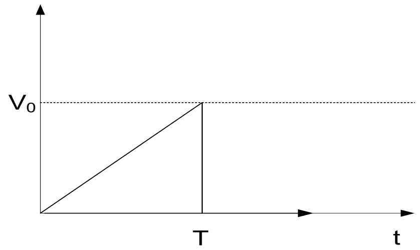


Figure 9

(a) $\frac{V_0}{Ts^2}+\left(\frac{V_0}{Ts}\right)e^{-Ts}$ (b) $\frac{V_0}{Ts^2}+\left(\frac{V_0}{Ts^2}\right)e^{-Ts}$ (c) $\frac{V_0}{Ts^2}+\left(\frac{V_0}{Ts^2}\right)e^{-Ts}(1+sT)$ (d) $\left(\frac{V_0}{Ts^2}\right)-V_0$

27. Given $Y(s)=(s+2)/s(s+1)$, the initial and final values of $y(t)$ will be respectively: (a) 2,1 (b) 1,2 (c) 2,2, (d) 1,1
28. A linear time-invariant system has impulse response $e^{2t}, t>0$. Given zero initial conditions and imput of $3e^{3t}$ the output for $t>0$ is (a) e^{5t} (b) $e^{3t}-e^{2t}$ (c) $e^{3t}+e^{2t}$ (d) none of the above
29. The d.c gain of a system represented by the transfer function $8/(s+1)(s+2)$ is (a) 2 (b) 1 (c) 4 (d) 8

(For **Question 30&31)** the plot of the signal $y(t)$ is in Fig.10

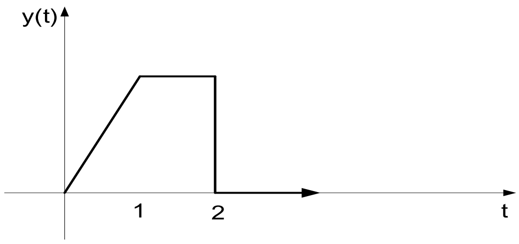
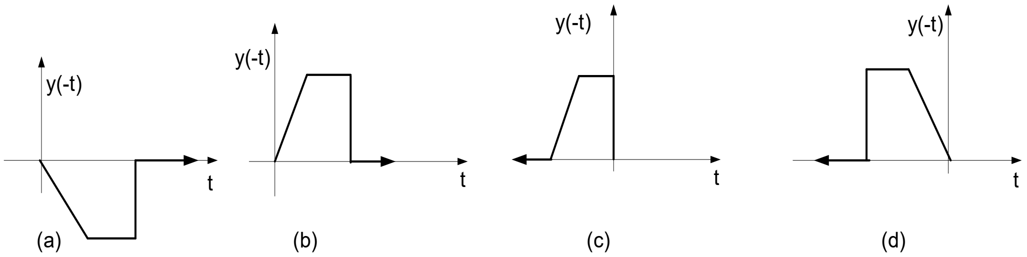
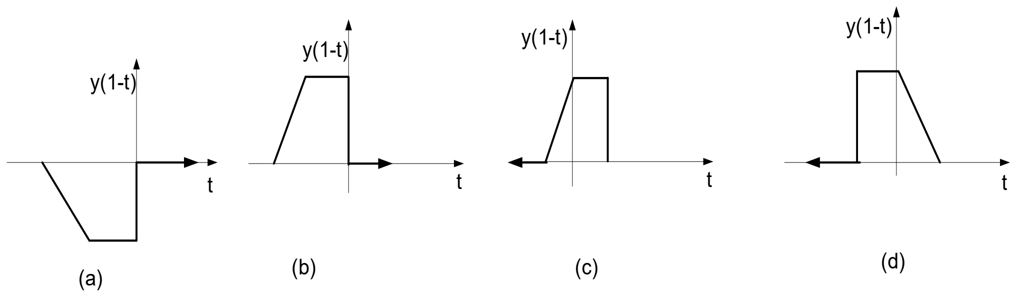


Figure 10

30. Then the pot of $y(-t)$ will be:



31. The plot of $y(1-t)$ will be:



32. Voltage transfer function of a simple $R-C$ integrator has: (a) a finite zero and a pole at the origin (b) a finite zero and a pole at infinity (c) a zero at the origin and a finite pole (d) a zero at infinity and a finite pole.
33. The current response for the circuit shown in Fig.11 is:

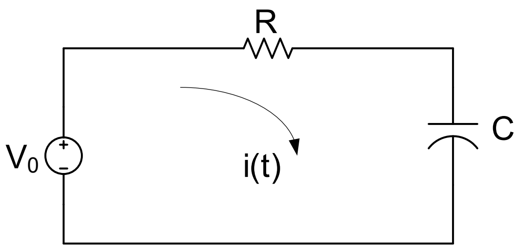


Figure 11

(a) V_0/R (b) $\left(\frac{V_0}{R}\right)\left(e^{-t/RC}\right)$ (c) $\left(\frac{V_0}{R}\right)\left(1-e^{-t/RC}\right)$ (d) $\left(\frac{V_0}{R}\right)\left(1+e^{-t/RC}\right)$

34. The response $x(t)$ of a network is expressed by $\frac{d^2x(t)}{dt^2}+x(t)$. If $v(t)=Ke^{-2t}$, then the dominant solution of x (A's are constants) for $t>0$ resembles (a) A_1e^t (b) A_2e^{-t} (c) A_1e^{2t} (d) A_2 cost $+ A_3 sint$

35. A unit step function $u(t-a)$ is applied to the circuit in Fig 12. The current response $i(t)$ is given by:

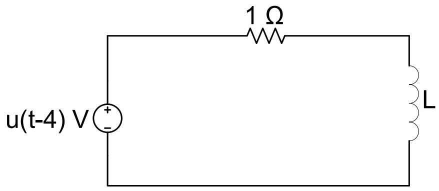


Figure 12

(a) $1-e^{-t}$ (b) $[1-e^{-(t-4)}]u(t-4)$ (c) $1-e^{-(t-4)}$ (d) $(1-e^{-t})u(t-4)$

36. The response of an initially relaxed system to a unit ramp excitation is $(t+e^{-t})$. Its step response will be (a) 1-e^{-t} (b) $\frac{t^2}{2}-e^{-t}$ (c) $t(d)-e^{-t}$
37. The response of a network is $i(t)=Ate^{-at}$ for $t\geq 0$, with a real and positive. The value of t for which $i(t)$ will be maximum is (a) a^2 (b) $2a$ (c) $1/a$ (d) a
38. A rectangular pulse of duration T and magnitude I has the Laplace transform (a) $(1/s)e^{-sT}$ (b) $(I/s)e^{sT}$ (c) (I/s) (d) $\left(\frac{I}{s}\right)(1-e^{-sT})$
39. Given $L[f(t)]=F(s), L[f(t-T)]$ equals (a) $e^{-sT}F(s)$ (b) $e^{sT}F(s)$ (c) $F(s)/(1-e^{sT})$ (d) $F(s)/(1+e^{sT})$
40. The unit impulse response of a linear time-invariant system is unit stop function $u(t)$. For $t>0$, the response of the system to an excitation $e^{-at}u(t), a>0$, will be (a) ae^{-at} (b) $a(1-ae^{-at})$ (c) $1-ae^{-at}$ (d) $(1-e^{-at})/a$
41. For $F(s)=\frac{2(s+1)}{(s^2+2s+5)}, f(0^+), f(\infty)$, are respectively, (a) 2,0 (b) 0,2 (c) 2/5,0 (d) 1,0
42. Laplace transform of a unit ramp function at $t=a$, is (a) a/s^2 (b) e^{-as}/s^2 (c) $e^{-as}/(s+a)^2$ (d) $1/(s+a)^2$
43. For a voltage across a capacitor of value $0.5F, V_c(s)=1/(s^2+1)$. Then $i_C(0^+)$ is (a) 0 (b) 2A (c) 0.5A (d) 1A
44. The poles of the transfer function $(s+1)/(s^2-5s+6)(s-4)$ are (a) 1 (b) 1,2,3,4 (c) 1,5,6,4 (d) 2,3,4
45. The voltage through a resistor with current $i(t)$ in the s-domain is (a) $sRI(s)$ (b) s^2RI (c) $RI(s)$ (d) $V/I(s)$
46. The current through an $R-L$ series circuit with input voltage $v(t)$ is given in the s domain as: (a) $V(s)\left(R+\frac{1}{sL}\right)$ (b) $V(s)(R+sL)$ (c) $V(s)/(R+1/sL)$ (d) $V(s)(R+sL)$
47. The impedance of a 10F capacitor is (a) 10/s (b) s/10 (c) 1/10s (d) 10s
48. The impedance of a 10-H inductor is (a) $10/s$ (b) $s/10$ (c) $1/10s$ (d) $10s$

49. The Laplace transform of $u(t-2)$ is (a) $1/(s+2)$ (b) $1/(s-2)$ (c) e^{2s}/s (d) e^{-2s}/s
50. The inverse Laplace transform of $(s+2)[(s+2)^2+4]$ is (a) $e^{-t}\cos 2t$ (b) $e^{-t}\sin 2t$ (c) $e^{-2t}\cos 2t$ (d) $e^{-2t}\sin 2t$
51. Reflection coefficient for the transmission line shown in Fig 13 is………

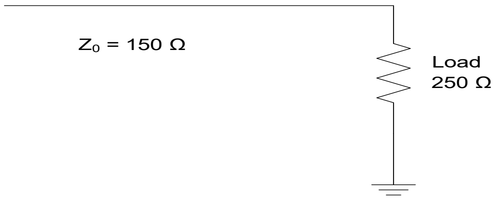


Figure 13

(a) 1 (b) -1 (c) 0 (d) 0.5
52. For a -400Ω line with a pure capacitive impedance of $-j400\Omega$, the reflection coefficient is (a)0 (b) 1 (c) ∞ (d) -1
53. Transmission of power to a load over a transmission line achieves optimum value standing wave ratio (SWR) is (a) $2:1$ (b) $1:2$ (c) $1:1$ (d) $1:\sqrt{2}$
54. For an open circuited load, voltage reflection coefficient is $|\rho|, \beta$, respectively, (a) $1, \pi$ (b) $0, \pi$ (c) $-1, \pi$ (d) 1,0
55. For a given voltage signal, V_{max} =(a) $V_1(1+|\rho|)$ (b) $V_1(1+|\rho|)$ (c) $V_1(1-|\rho|)^2$ (d) $V_1(1+|\rho|)^2$
56. For a reflection coefficient of $0.6/45^0$ VSWR equals (a) 1.67 (b) 0.25 (c) 4 (d) 6
57. For a VSWR of 5, the magnitude of the reflection coefficient is (a) 0.2 (b) 2 (c) 0.33 (d) 0.67
58. For a characteristic impedance of 200Ω and voltage standing wave ratio of 4, the normalizes minimum, maximum impedances will be respectively, (a) $50\Omega, 200\Omega$ (b) $0.25, 4$ (c) $200, 800$ (d) $2\Omega, 8\Omega$

(**For Questions 59 and 60**) A transmission line has a characteristic impedance of 100Ω and load impedance of $35-75\Omega$. On the smith chart.

59. The corresponding load admittances is (a) $10+j11\Omega$ (b) $0.5+j1.1$ (c) $1.1+j0.51$ (d) $50+j110$
60. The voltage standing wave ratio is (a) 4.6 (b) 3.6 (c) 5.6 (d) 6.6

LAPLACE TRANSFORM CONCEPT

1. Find the Laplace transform of $(1+\cos 2t)$ using first principle.
2. Find the Laplace Transform of $\cos^2 t$.
3. Find the Laplace Transform of $4t^{-\frac{1}{2}}$

4. ind the Laplace Transform of $5t^2 \cos 2t$
5. Find the Laplace Transform of: $8 + t + 9t^2 + t^3$
6. $\sin 4t$ using first principles
7. $4 \cos 4t$ using Euler's rule
8. $t^3 e^{-2t}$
9. $7sin^3 2t$
10. $6e^{-4t} cos^2 t$

SOLUTION OF DIFFEREBTIAL EQUATIONS BY LAPLACE TRANSFORMS

Solve the following differential equations using Laplace Transform:

1. $\frac{d^2y}{dx^2} + y = 0,\ where\ \ y = 1\ \ and\ \ \frac{dy}{dx} = -1\ \ at\ \ x = 0$
2. $\frac{d^2y}{dx^2} - 4y = 0,\ where\ \ y = 0\ \ and\ \ \frac{dy}{dx} = -6\ \ at\ \ x = 0.$
3. $\frac{d^2y}{dx^2} + y = 0,\ where\ \ y = 1,\ \frac{dy}{dx} = 1\ \ at\ \ x = 0$
4. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0,\ where\ \ y = 2,\ \frac{dy}{dx} = -4\ \ at\ \ x = 0$
5. $y'' + 2y' + y = te^{-t}\ \ if\ y(0) = 1,\ y^{'}(0) = -2$
6. $\frac{d^2y}{dx^2} + y = x \cos 2x\ ,where\ \ y = \frac{dy}{dx} = 0\ at\ t = 0.$
7. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - y = x^2 x^{2x},\ where\ y = 1, \frac{dy}{dx} = 0,\ \frac{d^2y}{dx^2} = -2\ at\ \ x = 0$
8. $\frac{dx}{dt} + 4y = 0,\ \frac{dy}{dx} - 9x = 0.\qquad given\ x = 2\ and\ y = 1\ at\ t = 0.$
9. $4\frac{dy}{dt} + \frac{dx}{dt} + 3y = 0,\ \frac{3\,dx}{dt} + 2x + \frac{dy}{dt} = 1$, under the condition $x = y = 0\ at\ t = 0$
10. $\frac{dx}{dt} + 5x - 2y = t,\quad \frac{dy}{dt} + 2x + y = 0$ being given when $x = y = 0\ \ when\ t = 0$
11. $\frac{dx}{dt} + y = \sin t,\ \ \frac{dy}{dt} + x = \cos t$ given that $x = 2,\ and\ \ y = 0\ \ when\ t = 0.$
12. $3\frac{dx}{dt} + 3\frac{dy}{dt} + 5x = 25 \cos t,\ \ 2\frac{dx}{dt} - 3\frac{dy}{dt} = 5 \sin t$

$$with\ x\,(0) = 2, y(0) = 3$$

13. For the second-order differential equation $\frac{d^2y(t)}{dt^2} - \frac{7dy(t)}{dt} + 12y(t) = 6e^{-4t}$
$y(0) = 2, y'(0) = 5$, determine the response $y(t)$

14. For the second-order differential equation $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 2sin4t$
$y(0) = 2, y'^{(0)} = 5$, determine the response $y(t)$

15. Solve the following differential equation using the Laplace transform method.
$\frac{d^2v(t)}{dt^2} + 4\frac{dv(t)}{dt} + 4v(t) = 2e^{-t}$, If $v(0) = v(0) = 2$

APPLICATION OF LAPLACE TRANSFORM

1. In the circuit of figure 5.3 of your textbook, the coil has $10\ \Omega$ resistance and a 8 H inductance. If $R = 14\ \Omega$ and the source voltage is 30 V and the switch is open at $t = 0$. Determine $i(t)$ using Laplace transform method.
2. The circuit shown in Fig. N is under steady state with the switch at position 1. At $t = 0$ the switch, is moved to position 2. Find $i(t)$ using Laplace transform method.

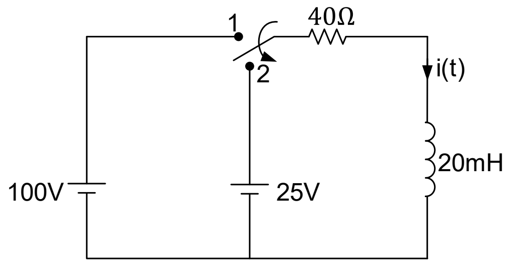


Figure N

3. In the RL circuit of Fig. O below the switch is in position 1 long enough to establish steady state-state conditions and $t = 0$ it is switched to position 2. Find the resulting current $i(t)$ using the Laplace transform method.

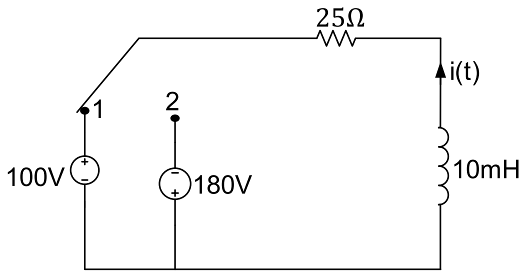


Figure O

4. Find $i(t)$ using Laplace transform method by first Laplacing the circuit of Fig.R and then taking the loop equation in the circuit of figure below if the initial conditions are all zero and the switch is closed at $t = 0$

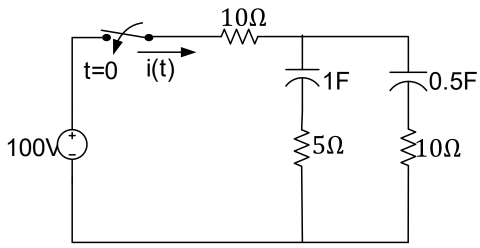


Figure R

5. In the circuit of the Fig.S, obtain the differential equation for i_1 and i_2. Find the current i_1 and i_2 at $t = 0$ using Laplace transform. Given that all initial conditions are

$$i_1(0^+) = i_1(0^-) = 0 \quad i_2(0^+) = i_2(0^-) = 0, \qquad \frac{di_1(0^+)}{dt} = \frac{V}{L_1}$$

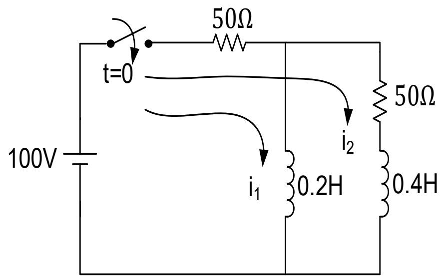


Figure S

6. For the two-mesh network of Fig.P, determine the values of the loop current i_1 & i_2 using Laplace transform and hence, write the s-domain equation in matrix form.

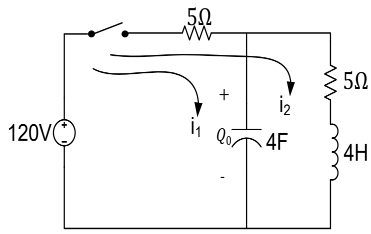


Figure P

7. Determine the Laplace transform of the ramp function in Fig. L.

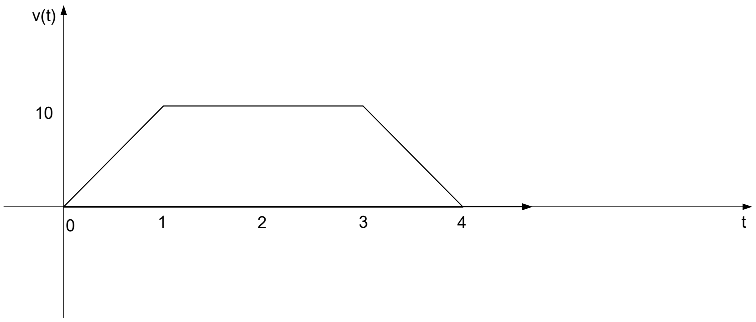


Figure L

8. The step response of a system is given by $f(t) = 2t^2 + 3t + 1$. Determine its impulse response.

9. Determine the Laplace transform of the ramp function in Fig. M.

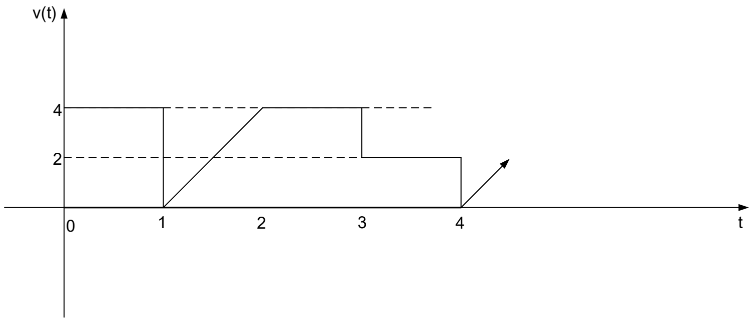


Figure M

10. Determine the Laplace transform of the function in Fig. N

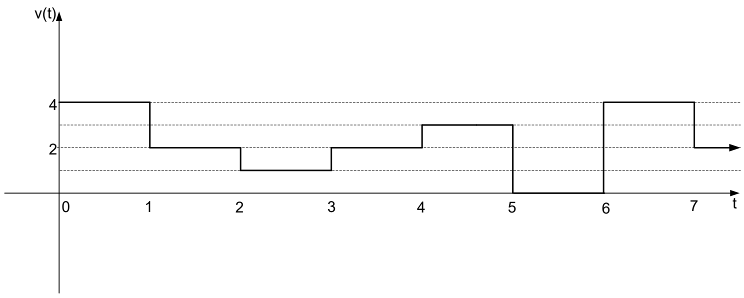


Figure N

11. Find the Laplace transform of $f(t) = (\cos 3t + e^{-4t})u(t)$

12. Find the Laplace transform of $f(t) = t^2 \cos 4t\, u(t)$

13. Find the Laplace transform of the function $h(t)$ in Fig. O.

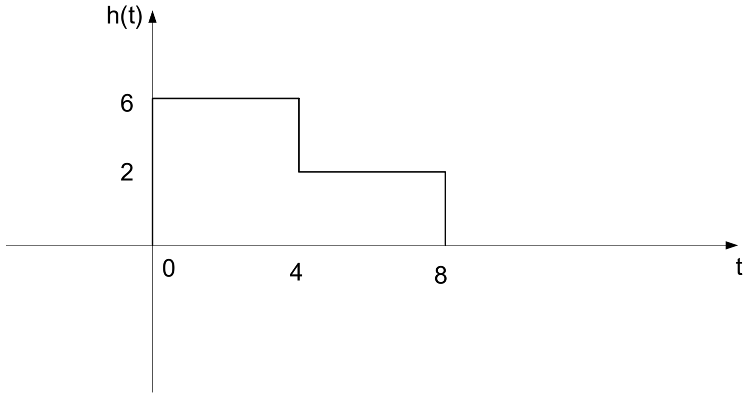


Figure O

Transmission Lines Solution with Smith Chart

1. A lossless transmission line of characteristics impedance $75\ \Omega$ is terminated in a load of $300 + j\ 150\ \Omega$. determine the:

i. Reflection coefficient

ii. The load admittance

iii. VSWR

iv. Distance between the load and the nearest voltage minimum to it.

v. Normalized input impedance Z_{in}, given that the length of the line is 92 cm and the signal wavelength is 40 cm.

Use the smith chart in Fig. Q3 to answer question 1

2. A lossless transmission line of characteristic impedance $125\ \Omega$ is terminated in a load of $350 + j200\ \Omega$. By the smith chart determine the:

(a) The load impedance

(b) Reflection coefficient

(c) VSWR

(d) Normalize input impedance. Given that the length of the line is 1.6λ

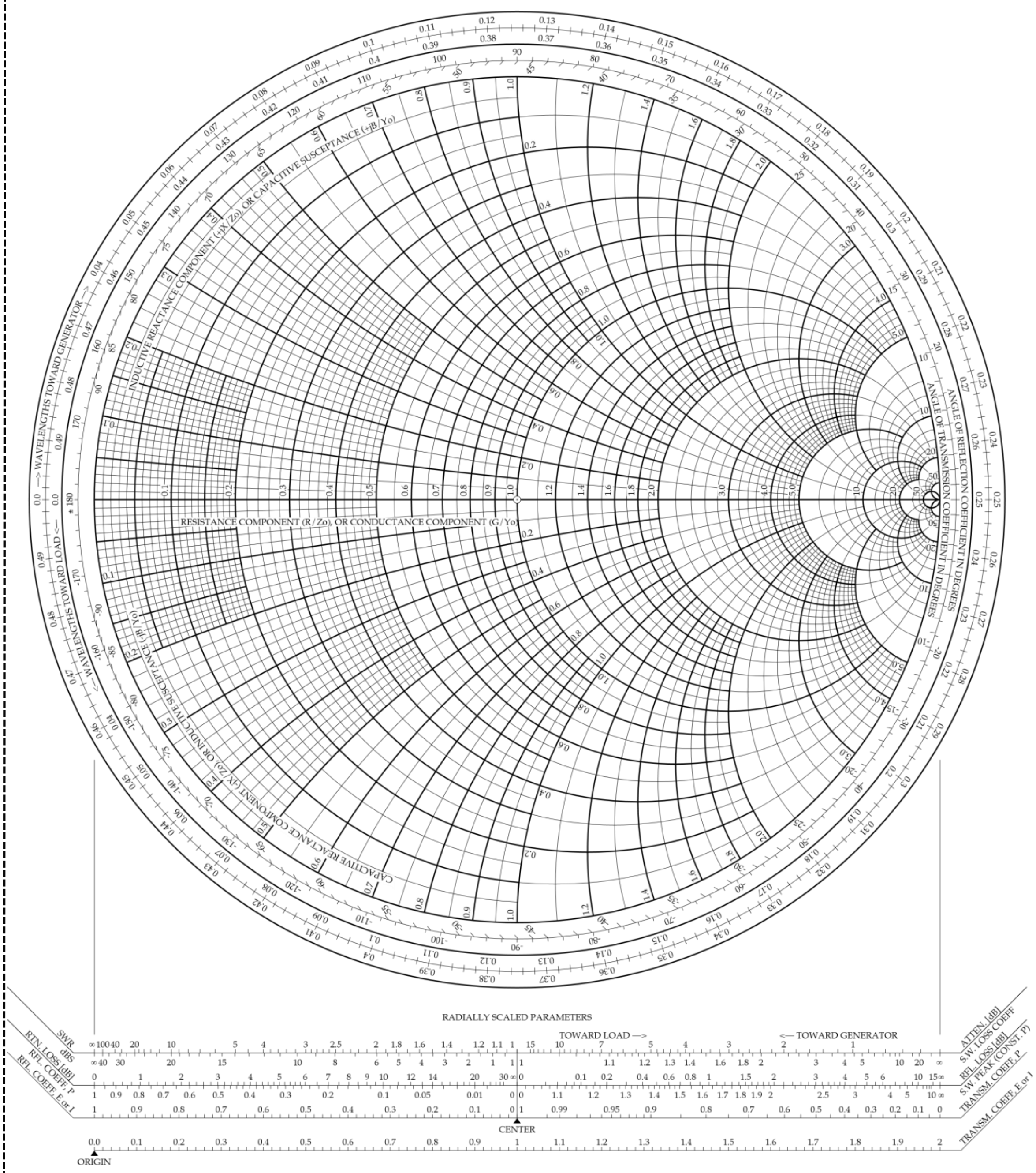
(e) Distance between the load and the nearest voltage minimum to it.

Use the smith chart in Fig. Q2 to answer question 2

3. A lossless transmission line with $Z_0 = 100\ \Omega$ is 80 m long and operates at 6 MHz the line is terminated with a load of $Z_L = 120 + j60\ \Omega$.

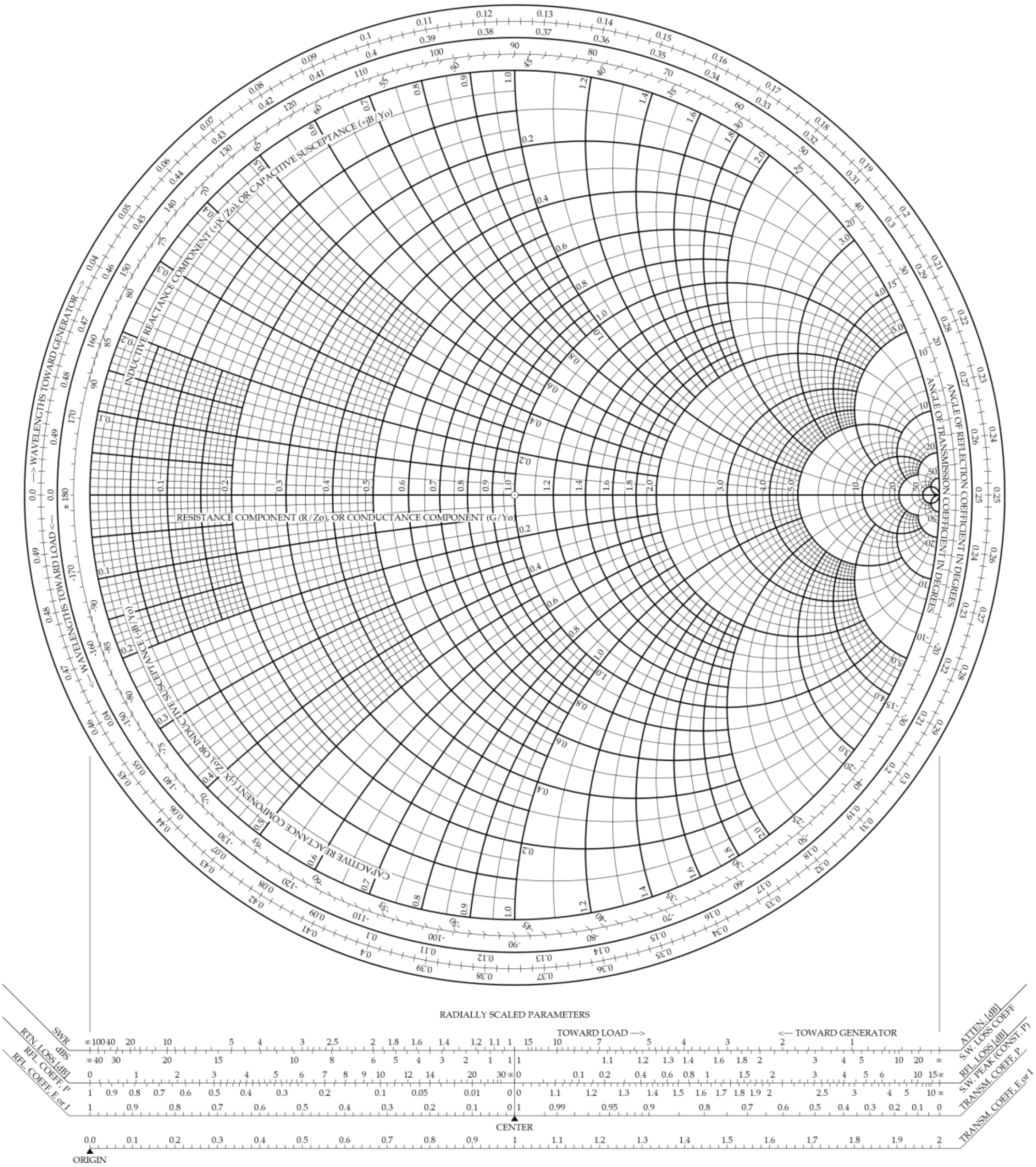
Given that $u = 0.8c$ on the line, $c = 3 \times 10^8\ m/s$ determine by the use of smith chart:

(i) Load admittance
(ii) Voltage reflection coefficient
(iii) VSWR
(iv) z_{in} , z_{max}, z_{min} **Use the smith chart in Fig. Q_1 to answer question 3**



WAVELENGTHS TOWARD GENERATOR —>
<— WAVELENGTHS TOWARD LOAD <—
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
ANGLE OF TRANSMISSION COEFFICIENT IN DEGREES
RADIALLY SCALED PARAMETERS
TOWARD LOAD —>
<— TOWARD GENERATOR
SWR
dBS
RTN. LOSS [dB]
RFL. COEFF, P
RFL. COEFF, E or I
ATTEN. [dB]
S.W. LOSS COEFF
RFL. LOSS [dB]
S.W. PEAK (CONST. P)
TRANSM. COEFF, P
TRANSM. COEFF, E or I
CENTER
ORIGIN

Figure Q_1



WAVELENGTHS TOWARD GENERATOR —>
<— WAVELENGTHS TOWARD LOAD
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo) OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
ANGLE OF TRANSMISSION COEFFICIENT IN DEGREES
RADIALLY SCALED PARAMETERS
TOWARD LOAD —>
<— TOWARD GENERATOR
SWR
dBS
RTN. LOSS [dB]
RFL. COEFF, P
RFL. COEFF, E or I
ATTEN. [dB]
S.W. LOSS COEFF
RFL. LOSS [dB]
S.W. PEAK (CONST. P)
TRANSM. COEFF, P
TRANSM. COEFF, E or I
CENTER
ORIGIN

Figure Q$_2$

Name: ……………………………………………………………………………..

Matric Number: ……………………………………………………………….

Date ……………………………

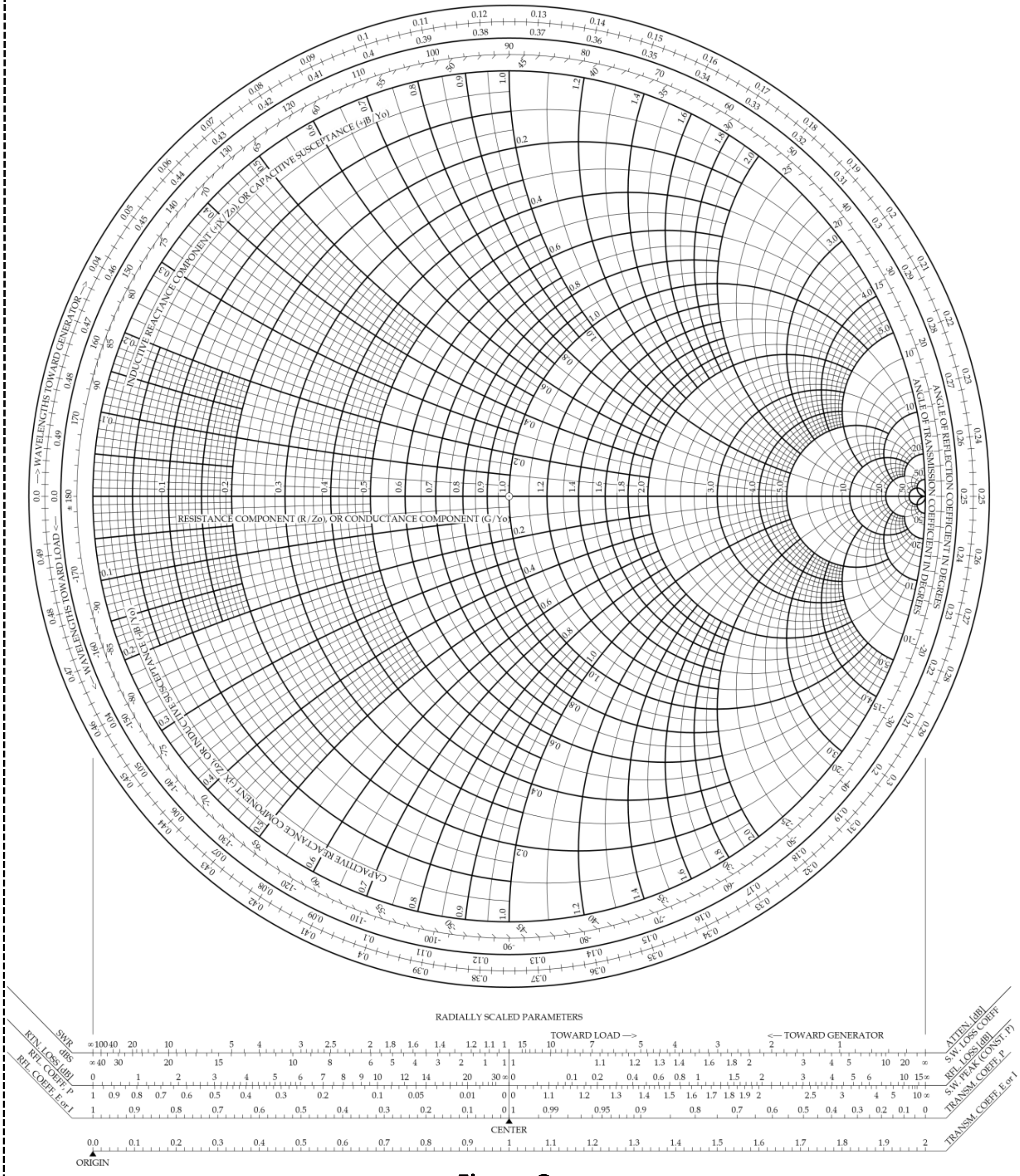


Figure Q_3

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