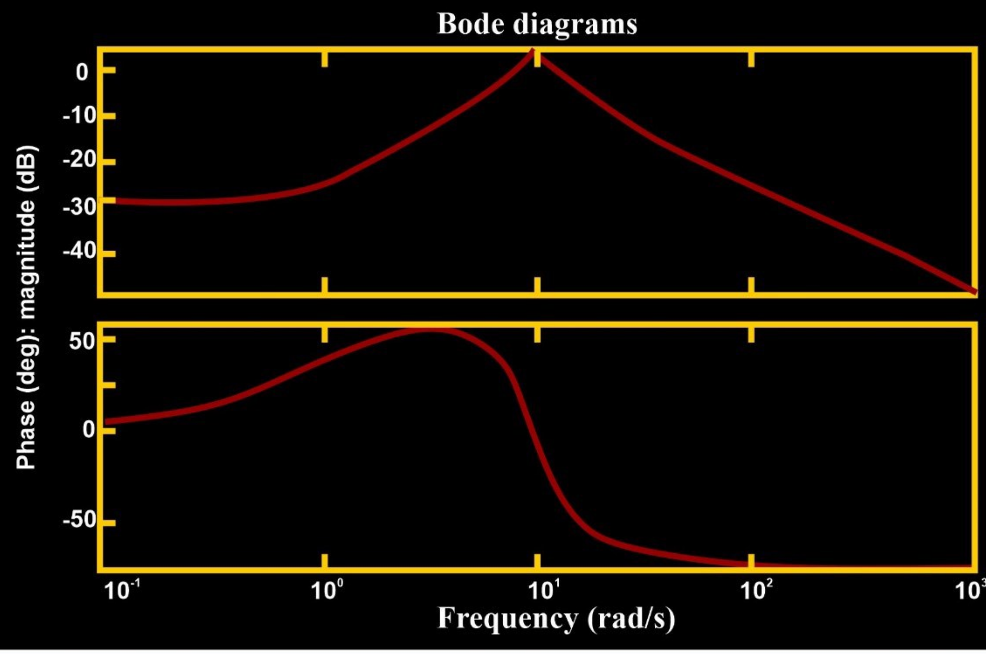
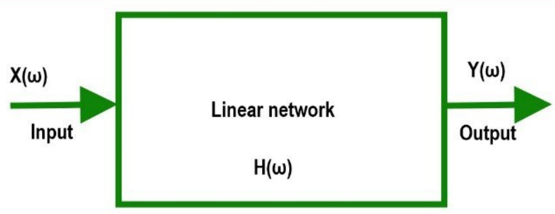
CIRCUIT THEORY TEXTBOOK




# CIRCUIT THEORY
# TEXTBOOK

*with Application for Undergraduate students*



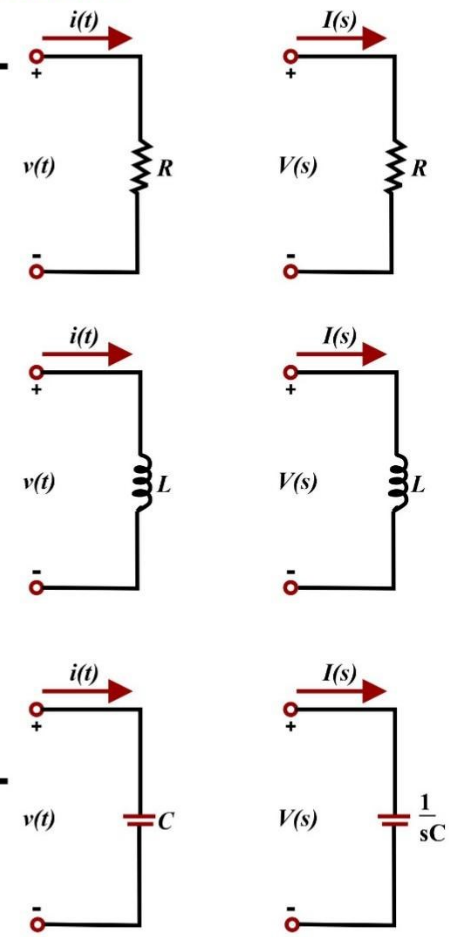

KENNETH UGO UDEZE

AFROPOLITAN PUBLICATIONS

# CURCUIT THEORY TEXTBOOK

## with Application for Undergraduate students

- $i_C = C \frac{dv(t)}{dt}$
- $V_L = L \frac{di(t)}{dt}$
- $i_L = \frac{1}{L}\int_o^t v(t)d(t)$
- $V_C = \frac{1}{C}\int_o^t i(t)d(t)$
- $\tau = RC$
- $\tau = \frac{L}{R}$




[redacted]

# CIRCUIT THEORY TEXTBOOK

(With Application for Undergraduate Students)

By

**Kenneth Ugo Udeze**

**Circuit Theory Textbook (with Application for Undergraduate Students)**



Published by Afropolitan Publications Int'l

200132, Institute of African Studies,
University of Ibadan, Nigeria.
https://books.afropolitanjournals.com/index.php/pub

The online version of this publication is permanently available at Afropolitan Books, The e-Book platform of Afropolitan Publications, https://books.afropolitanjournals.com/index.php/pub (Open Access).



**ISBN:** 978-978-768-818-2

**CIP Data:**
NATIONAL LIBRARY OF NIGERIA CATALOGUING-IN-PUBLICATION DATA
CIRCUIT THEORY TEXTBOOK: With Application for Undergraduate Students
UDEZE, Kenneth Ugo, 1975-
1. Electric circuits
2. Electrical engineering
I. Title

| | | |
|---|---|---|
| TK3001.U19 | 2024 | 621.3192 |
| ISBN: 978-978- 768-818-2 | (pbk) | AACR2 |

# Table of Contents

# Preface

This book originates from notes used in teaching Electrical Circuit Theory courses at the third-year level of Electrical and Electronics Engineering Department, Federal Polytechnic, Oko, Anambra State, Nigeria. Along with other materials gathered by the author during his degree and post-degree years of academic pursuit, and over fifteen (15) years of teaching experience in accordance with course curriculum guidelines from the National Board for Technical Education (NBTE), this text, "**CIRCUIT THEORY** with Application for Undergraduate Students", was written.

The content of each chapter was designed to accommodate Higher National Diploma (HND) and Bachelor of Science/Engineering (B.Sc./B.Eng.) undergraduate students as the materials presented were made comprehensive enough to cover both classes of programs at their mid-course levels.

Chapters 1 and 2 cover the basic knowledge of transients in inductive and capacitive circuits in first-order systems as a function of circuit parameters and time constant. Chapters 3 and 4 discuss transients in $RLC$ circuits, in both series and parallel configuration. Damping factors in damping conditions are also investigated.

Chapter 5 covers two-port networks in impedance-, admittance- and transmission-parameters ($z$-, $y$- and $t$-parameters) as functions of the proverbial "black box". Also discussed are the image impedance and insertion loss of various networks.

Chapters 6, 7 and 8 cover, respectively, pole-zero *constellations*, Bode plots and Filters.

Chapter 9 discusses in detail, small-signal transmission lines, with primary and secondary constants.

At the end of the chapters are enough review problems designed to help the students exercise their level of comprehension of the treated matters, and by so doing internalize the underlying principles of the lessons taught.

# About the Author

**Udeze Kenneth Ugo** hails from Onicha Ugbo in Aniocha North Local Government Area in Delta State, Nigeria. He attended his primary school at Aniemeke Primary School Onicha Ugbo, Delta State, Nigeria and attended his secondary education at Model Secondary School Maitama, Abuja, FCT, Nigeria where he obtained his Senior School Certificate in 2003.

Between 2005 and 2010, he obtained his National Diploma (ND) and Higher National Diploma (HND) in Electrical and Electronics Engineering (Telecommunication and Electronics Options) with a CGPA of 3.68/4 i.e., Distinction Honors from Federal Polytechnic Oko, Anambra State, Nigeria. He also obtained his first degree in Electrical and Electronics Engineering (Power, Telecommunication and Electronics Options), in 2013 from the prestigious University of Ibadan, Oyo State, Nigeria with a CGPA of 5.8/7 i.e., a Second-Class Upper Division (2.1).

After his one year mandatory National Youth Service Corp (NYSC) in Electrical and Electronics Engineering Department in Federal Polytechnic Oko, Anambra State, Nigeria in 2015, he proceeded to obtain his Masters degree in Offshore Engineering in 2016, majored in Offshore Design and installations, Subsea umbilical cables designed, Installation and maintenance of offshore facilities, Submarine power cable design and maintenance, Subsea instrumentation and control system (E&I) from Offshore Technology Institute, School of Advance Engineering, University of Port-Harcourt, Rivers State, Nigeria. Then a second Masters degree in Electrical and Electronics Engineering and majored in Power System Engineering, from University of Lagos, Lagos State, Nigeria in 2023. He graduated with a CGPA of 4.7/5 i.e., Distinction Honors.

He is currently a staff of Federal Polytechnic Oko, Anambra State, Nigeria attached to Electrical and Electronics Engineering Department. He teaches Mathematics and Electrical Engineering courses.

He is presently prospecting for PhD admission overseas for researches in Renewable Energy.

# CHAPTER 1

# TRANSIENT ANALYSIS

## 1.0 Introduction

By "transient" is implied *transitory* with respect to processes or conditions that do not last, that is, phenomena that are temporary by their nature or circumstances. So, *Transient Analysis* has to do with the response of an (electrical) circuit between two distinct steady-state conditions. We usually require the behaviour of voltage or current signal during the transition that takes place between two steady states the first of which can be, for instance, opening or closing of a switch in a typical circuit that has, prior to the switching action, been closed or open (respectively as the case might have been) for a *longtime*. The "long time" here simply means long enough for the circuit to have settled down, "settling down" which in turn means that all the energy-storage elements in the circuit (typically capacitors and inductors) have been energized (that is, if previously de-energized prior to the switching action) or fully de-energized (if previously energized before witching action). This might take just a few ticks of seconds-hand of the clock, or even subdivisions of this!

After the switch is thrown (i.e., opened or closed), the circuit is "disturbed", some process takes place, and thereafter the circuit once again settles down to a new steady state. In the absence of a driving source (voltage or current) the circuit is thereafter "deadened" and reverts to its state prior to the original energizing action. If, however, a driving source is present, its effect is then the sole result that remains after all the others have reverted to their original, de-energized state prior to the energizing action.

The task before us is to investigate what takes place during the aforementioned transition between the two distinct steady states. When no driving source (this is known in calculus as *forcing function*) is present, then the response is entirely dependent on the initial conditions of the energy storage elements, and this is known as the "natural" (in mathematics, *homogeneous*) response. If, however, a driving source is present, the response would have two components namely:

1. a homogeneous solution (explained above) that depends, as mentioned earlier, entirely on the initial conditions of the circuit elements (parameters);
2. a complementary solution that replicates the nature of the forcing function.

By the last statement is meant, for instance, if the forcing function is a constant quantity, then the forced-response component of the total response would also be a (possibly scaled up/down) constant quantity. If a dc (direct current) potential difference (voltage source) is applied at the input, then a dc voltage (or current) would result at the output terminals. [Applying a dc current source would likewise give rise to a dc voltage (or current) response.] If, on the other hand, an ac (alternating current) signal is applied, then an ac forced response is obtained, again with a possibly scaled-up/down amplitude but strictly of the same frequency (and perhaps – in fact, usually – a different phase angle) as the input signal. [While we're at it, it's pertinent to point out here, as implicitly elucidated in the foregoing statements, as learnt in basic electrical circuit course, that a voltage source at the input can entirely result in current (does not have to be just voltage) at the output, and *vice versa*. There's no hard and fast rule, it all depends on the designer's objective. However, the article of faith is that, the frequency of the forced part must stay true to its origin regardless of the nature of the inputting signal].

We usually require a current response through an inductor or a voltage response across a capacitor, the obvious reason being, as learnt in an earlier course, that these quantities cannot change in zero time (i.e., instantaneously) through or across, these respective elements. (Law of energy conservation would not permit the above to be violated since that would imply the ability to come up with infinite amount or unlimited supply of energy!)

Circuits with one energy storage element (capacitor or inductor) are known as first-order circuits, whereas those with two energy storage elements [including both capacitor(s) and inductor(s)] are called second order-circuits. A typical first order circuit has a voltage or current source in series or parallel with a resistor and a capacitor or an inductor, and possibly switch(es). Regardless of any number of resistors, switches and capacitors or inductors that are involved, it's still a first-order circuit so long as it does not mix capacitor(s) and inductor(s). In this case, only one initial condition is required to determine the response. A second-order circuit would, on the other hand, contain both capacitor(s) and inductor(s) along with voltage and/or current sources(s), possibly in addition to other passive network elements. Any number of resistors involved, as was the case for a first-order circuit, is immaterial since a resistor only dissipates energy and cannot store.

## 1.1 First-Order System

Any system with the ability to store energy in one form or to dissipate energy stored, is a first-order system; so is any system with a single energy storage element [capacitor(s) or inductor(s)] and a combination of sources and resistors (and possibly switches). The three factors that uniquely determine the response of a first-order system are;

1. initial conditions,
2. steady-state solution,
3. the time constant.

The response can be with respect to the variation of either voltage or current. In this aspect it is convenient to consider the voltage across a capacitor, or current through an inductor since these quantities, as explained earlier, respectively do not charge in zero time (instantaneously) in these elements. Recall the relationship between voltage across, and current through, an inductor; and (dually) current through, and voltage across, a capacitor:

$$v_L(t) = L\frac{di_L(t)}{dt}, \qquad i_L(t) = \frac{1}{L}\int v_L(t)dt\,; \qquad 1.1$$

$$i_C(t) = C\frac{dv_C(t)}{dt}, \qquad v_C(t) = \frac{1}{C}\int i_C(t)dt \qquad 1.1a$$

Where $v_L$ , $i_L$ are voltage across, and current through, an inductor; $i_C$, $v_C$ being current through, and voltage across, a capacitor, respectively. (Note the **duality** between these two pairs of relationships).

From elementary science, it was taught that if current through an inductor were to change instantaneously, this would require an infinite supply of energy since $\frac{\mathrm{d}i_L(t)}{dt}$ would then be infinite. Infinite voltage means infinite driven current which in turn means infinite energy $\left(W = \frac{1}{2}Li^2\right)$, a physical impossibility that violates law that energy cannot be created (or destroyed, but can only be transformed from one form to another)! The duality of the foregoing applies to voltage across a capacitor. Instantaneous change *vis-à-vis* voltage implies infinite current which in turn means infinite voltage resulting in unlimited supply of energy $\left(W = \frac{1}{2}Cv^2\right)$, thereby again violating the law of energy conservation.

What all the foregoing means is that the value of a typical voltage across a capacitor just prior to throwing a circuit switch (opening or closing), $v_C(0^-)$, in the same just after the switch is thrown, $v_C(0^+)$. So,

$$v_C(0^+) = v_C(0^-) = v_C(0) \qquad 1.2$$

The above applies to current with respect to an inductor:

$$i_L(0^+) = i_L(0^-) = i_L(0) \qquad 1.2a$$

Voltage and current in a circuit are due to the superposition of two effects, namely, (i) The presence of stored energy (which can either decay, or further accumulate if a source is present) (ii) The action of external source(s) [forcing function(s) as they are called in mathematics]. The response is in two parts: a. Natural response – consider stored energy; b. Forced response – consider external sources.

Complete response is the sum of the two. We bear in mind that, depending on the values and/or configuration of circuit elements, either of the two components of the total response might entirely predominate, thereby necessitating that the other ought to be ignored for all practical purposes!

### 1.1.1 Source-Free *RL* Circuit

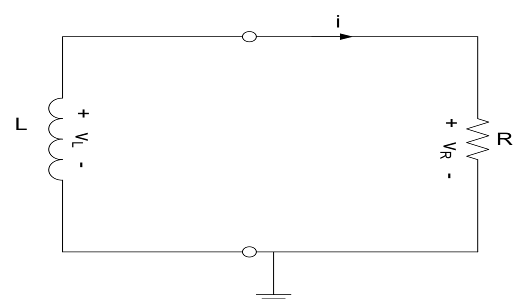



**Figure 1.1 a source-free inductive circuit**

Consider the series connection of a resistor and an inductor, as shown in Fig. 1.1. Our goal is to determine the circuit response, which we'll assume to be the current $i(t)$ through the inductor (also obviously through the resistor). We select the inductor current as the response in order to take advantage of the idea that the inductor current cannot change instantaneously. At $t = 0$, we assume that the inductor has an initial current $I_0$, or

$i(0) = I_0$

Applying KVL we have that

$$Ri + v_L = 0 \qquad 1.3$$

$$Ri + L\frac{di}{dt} = 0 \qquad 1.4$$

$$\frac{di}{dt} + \frac{R}{L}i = 0$$

$$\frac{di}{dt} = -\frac{R}{L}i$$

$$\frac{di}{i} = \frac{-R}{L}dt \qquad 1.5$$

At $t = 0$ the current is $I_o$

Integrating Eq 1.5, we have Eq 1.6

$$\int_{I_o}^{i(t)} \frac{di}{i} = \int_0^t \frac{-R}{L}dt$$

$$\ln i\Big|_{I_o}^{i(t)} = \frac{-R\,t}{L}\Big|_o^t$$

$$\ln i(t) - \ln I_o = \frac{-R}{L}(t - 0)$$

$$\ln\frac{i(t)}{I_o} = \frac{-Rt}{L}$$

$$\frac{i(t)}{I_o} = e^{-Rt/L}$$

$$i(t) = I_o e^{-Rt/L} \qquad 1.6$$

Thus, the natural response of the *RL* circuit is an exponentially decaying result of the initial current. The current response is shown in Fig. 1.2. It is evident from Eq 1.6 that the time constant for the *RL* circuit is

$$\boxed{\tau = \frac{L}{R}} \qquad 1.7$$

with $\tau$ again having the unit of seconds. Thus, Eq 1.6 may be written as

$$i(t) = I_0 e^{-t/\tau} \qquad 1.8$$

With the current in Eq 1.8, we can find the voltage across the resistor as

$$v_R(t) = iR = I_0 R e^{-t/\tau} \qquad 1.9$$

The power dissipated in the resistor is

$$p = v_R i = I_0^2 R e^{-2t/\tau} \qquad 1.10$$

The energy absorbed by the resistor is

$$w_R(t) = \int_0^t p\, dt = \int_0^t I_0^2\, R e^{-\frac{2t}{\tau}} dt = -\frac{1}{2}\tau\, I_0^2 R e^{-\frac{2t}{\tau}}\Big|_0^t = -\frac{1}{2}\tau\, I_0^2 R(e^{-\frac{2t}{\tau}} - 1) \text{ joules;}$$

$$\tau = \frac{L}{R}$$

or

$$W_R(t) = \frac{1}{2} L I_0^2 \left(1 - e^{-2t/\tau}\right) \text{J}$$

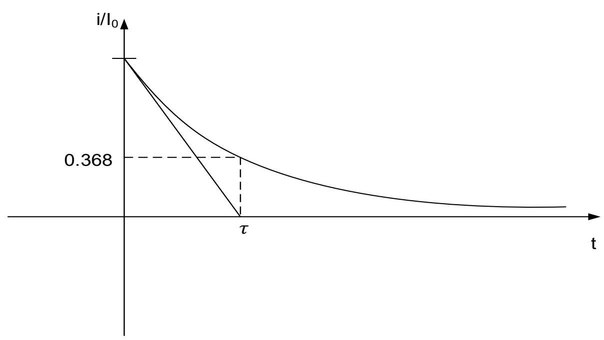



**Figure 1.2 The Current response of *RL* circuit (decaying inductive current curve)**

At $\quad t = \tau$

$$\frac{i(\tau)}{I_o} = e^{-1} = 0.3679 \;\; \Longrightarrow \;\; i(\tau) = 0.3679 I_o$$

Check for $t = 2\tau,\ 3\tau$

At, for instance, $t$ = 5$\tau$, $i(\tau) = (I_o)e^{-5} \approx 0.007 I_o$, that's less than one percent of the original magnitude of the signal of interest (i.e., it's equally applicable to voltage). What it entails is that, although a decaying signal theoretically never get to be exactly zero except at time $'t'$ tends to infinity), yet for all practical purposes it's virtually that, as far as an engineering designer is concerned!

**Example 1.1:** For the circuit of Fig 1.3, determine the voltage of the $40\ \Omega$ resistor as a function of time and the circuit parameters.

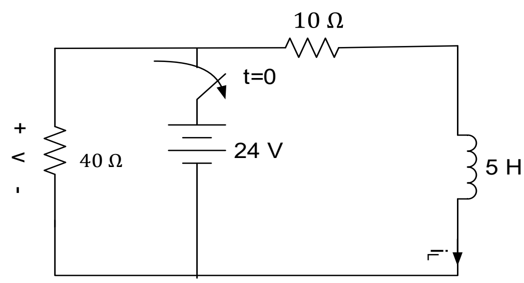



**Figure 1.3**

Solution:

The assumption is always made that circuit conditions are "long" enough to have settled down the circuit so that a steady state condition has been reached. In the above circuit, prior to opening the switch it is assumed that it had been closed long enough for the inductor (energy storage element) to have been fully energized (by the presence of the dc battery). The switching action then takes the battery entirely out of the circuit so that it becomes sources-free. The response $v(t)$ thus is entirely a natural ("transient") one that depends on the initial conditions of the circuit elements, and whose duration depends on the time constant of this particular circuit (*$L_{eq}$/$R_{eq}$*).

$KVL\ at\ t = 0^+$: $$-v + 10\, i_L + 5\frac{di}{dt} = 0$$

Ohm's law:

$$i_L = -\frac{v}{40}$$

$$\Rightarrow \qquad -v - \frac{10v}{40} + 5\frac{d}{dt}\left(-\frac{v}{40}\right) = 0$$

Rationalizing, $$-40v - 10v - 5\frac{dv}{dt} = 0$$

$\Longrightarrow$ $\frac{dv}{dt} + 10v = 0$, leading to the characteristic equation: $s + 10 = 0$, producing the single root at $s = -10$.

$\Longrightarrow$ $v(t) = Ke^{-10t}$ with K as yet-to-be determined constant.

From the initial conditions, $i_L(0^-) = \frac{24v}{10\Omega} = 2.4\text{ A}$ since the inductor presents a short circuit to the dc battery.

But $i_L(0^+) = i_L(0^-) = i_L(0) = 2.4\text{ A}$ for the reason that has been severally adduced [current does not change instantaneously (finite change in zero time) through an inductor].

$\Longrightarrow$ $v(0^+) = i(0^+) \times 40\ \Omega$, by ohm's law

$$= (-2.4\text{ A}) \times 40\ \Omega = -96\text{ V} = Ke^0 = K$$

$\Longrightarrow$ $v(t) = -96e^{-10t}\text{ V}$

Taking cognizance of the time constant, which is easily determined in this case of first order circuit by inspection, the nature of the answer could have been readily written down:

$$\tau = \frac{L(eq)}{R(eq)} = \frac{5\text{ H}}{(40 + 10)\ \Omega}$$

after removing the battery. So, time constant

$$\tau = \frac{5}{50} = 0.1\text{ s}$$

$\Longrightarrow$ $i_L(t) = Ke^{-\frac{t}{0.1}} = Ke^{-10t}$, and the rest of the procedure is as the foregoing.

and the rest of the procedure is as the foregoing.

To determine the voltage across the $40\ \Omega$ resistor (which, incidentally, was the original requirement),

$$v_{40\Omega}(t) = 40 \times (-2.4\, e^{-10t}) = -96e^{-10t}\text{ V}$$

as before when it was derived directly. At, say, $t = 100\text{ ms}$,

$$v(100\text{ ms}) = -96e^{-10(100\times10^{-3})} = -96e^{-1} = -35.32\text{ V}$$

The negative result indicates a "wrong" orientation of the battery, meaning that a "positive" voltage would have been obtained simply by reversing the polarity of any of the network *active* elements.

The foregoing analysis has to do with a source-free circuit, that is, a circuit with no driving source (forcing function). The sole purpose of the dc source (i.e., the battery) in the just concluded example above, was to initially energize the elements (that are "energize-able"), in this case just the series inductor, and thereafter it's taken entirely out of the circuit.

**Example 1.2:** In the circuit shown in Fig. 1.4, the switch has been open for a long time and then suddenly closed at $t = 0$. Determine $i_L(t)$. Hence at $t = 0.15$ s find the values of (a) $i_L$ (b) $i_1$ (c) $i_2$

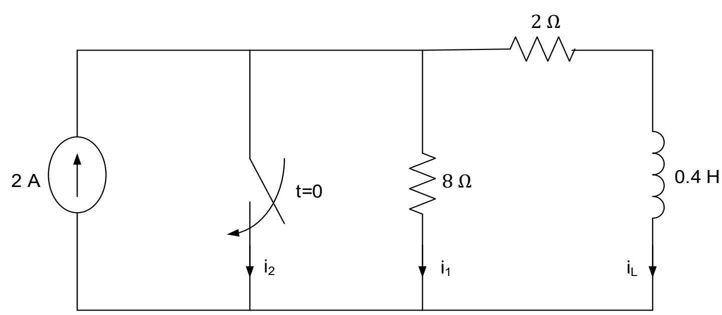



**Figure 1.4**

Solution:

By current division rule,

$$i_L(0) = I_0 = \frac{8}{2+8} \times 2 = 1.6 \text{ A}$$

$$i_L(t) = I_0 e^{-\left(\frac{2}{0.4}\right)t} = 1.6\mathrm{e}^{-5\mathrm{t}}\mathrm{A} \text{ At t} = 0.15 \text{ s,}$$

(a) $i_L(0.15) = 1.6e^{-(5\times0.15)} = 0.756$ A

(b) $i_1 = 0$ A (Short circuit)

(c) $i_2 = 2 - 0.756 = 1.244$ A

[Note: the current source plays no role after *t*= 0, so the response is entirely a **natural** one.]

**Example 1.3**: The switch in the RL circuit shown in Fig. 1.5 is moved from position 1 to position 2 at $t = 0$. Obtain

(a) $v_R$ and $v_L$ with polarities as indicated.

(b) The powers dissipated $P_R$ and $P_L$

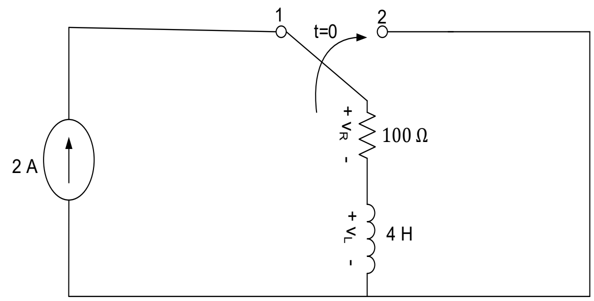



**Figure 1.5**

Solution:

(ai) For $V_R$

At position 1 (i.e. $t < 0$): $i(0^-) = i(0^+) = 2\text{ A}$

At $t > 0$ position 2

$$\tau = \frac{4}{100} = \frac{1}{25}$$

$$i(t) = i(0^+)e^{-t/\tau} = 2e^{-25t}\text{ A}$$

Since the same current passes through the resistor,

$$V_R = iR = (2e^{-25t})(100) = 200e^{-25t}\text{ V}$$

(aii) For $V_L$

$$V_L = L\frac{di}{dt} = 4\frac{d(2e^{-25t})}{dt} = -200e^{-25t}\text{ V}$$

(b) For $P_R$ and $P_L$

Recall $P_R = IV_R = (2e^{-25t})(200e^{-25t})$ $\quad P_R = 400e^{-25t-25t} = 400e^{-50t}\text{ W}$

$P_L = IV_L$

But, $V_L = 4\frac{d}{dt}(2e^{-25t}) = -200e^{-25t}\text{V}$ $\quad P_L = -200e^{-25t} \times 2e^{-25t} = -400e^{-50t}\text{ W}$

### 1.1.2 The Source-Free *RC* Circuit

A source-free *RC* circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.

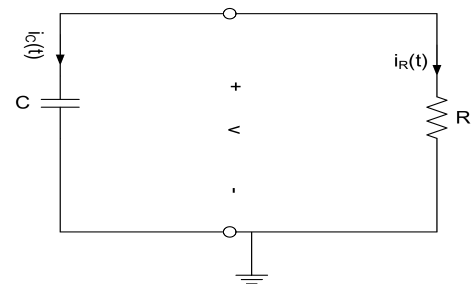



**Figure 1.6: Source-free *RC* circuit**

Consider a series combination of a resistor and an initially charged capacitor, as shown in Fig. 1.6. (The resistor and capacitor may be the equivalent resistance and equivalent capacitance of combinations of resistors and capacitors.) Our objective is to determine the circuit response, which, for pedagogical reasons, we assume to be the voltage $v(t)$ across the capacitor. Since the capacitor was initially charged, we can assume that at time $t = 0$, the initial voltage is

$$v(0) = V_0 \qquad 1.11$$

with the corresponding value of the energy stored as

$$w(0) = \frac{1}{2}CV_0^2 \qquad 1.12$$

Applying KCL at the top node of the circuit in Fig. 1.6 yields

$$i_C + i_R = 0 \qquad 1.13$$

$$\text{From } i_c = C\frac{dv}{dt} \text{ and } i_R = \frac{v}{R},$$

$$C\frac{dv}{dt} + \frac{v}{R} = 0 \qquad 1.14a$$

$$\frac{dv}{dt} + \frac{v}{RC} = 0 \qquad 1.14b$$

This is a first-order differential equation, since only the first derivative of $\upsilon$ is involved. Rearranging,

$$\frac{dv}{v} = -\frac{1}{RC}dt \qquad 1.15$$

$$\ln v = -\frac{t}{RC} + \ln \mathrm{A}$$

with $ln$ A as the integration constant of the indefinite integral. (What do you think would have resulted if we chose simply A, instead of in A, as the constant of integration? Let the reader discern!) Thus,

$$ln\frac{v}{A} = -\frac{t}{RC} \qquad 1.16$$

Taking power of exponential $e$ [so-called "natural number" (find out its value by raising same to the power of 1 using your pocket calculator!), not to be confused with natural response as their technical relationship is merely mathematically coincidental!] produces

$$v(t) = Ae^{-t/RC}$$

But from the initial conditions, $v(0) = A = V_0$.

Hence,
$$v(t) = V_0e^{-t/RC} \qquad 1.17$$

So, the voltage response of the *RC* circuit is an exponentially decaying result of the initial voltage. Since the response is due to the initial energy stored and the physical characteristics of the circuit, and not due to some external voltage or current source, it's called the **natural response** of the circuit.

*The natural response of a circuit refers to the response (in terms of voltages and currents) of circuit itself, with no external sources of excitation.*

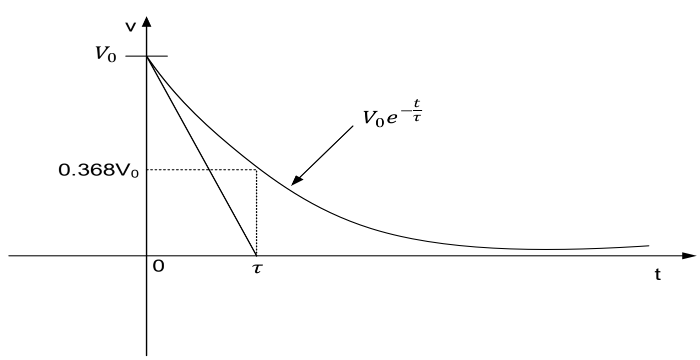



**Figure 1.7 The voltage response of *RC* circuit**

The natural response is illustrated graphically in Fig. 1.7. Note that at $t = 0$, we have the correct initial condition as in Eq (1.11). As $t$ increases, the voltage decreases toward zero. The rapidity with which the voltage decreases is expressed in terms of the time constant, denoted by $\tau$, the lowercase Greek letter *tau*. The time constant of a circuit is the time required for the response to decay to a factor of $\frac{1}{e}$, or approximately 36.8 percent, of its initial value.

That is, at $t = \tau$, Eq (1.17) becomes

$$V_0 e^{-\tau/RC} = V_0 e^{-1} = 0.368V_0$$

or

$$\boxed{\tau = RC} \qquad 1.18$$

In terms of the time constant, Eq (1.17) can be written as

$$\boxed{v(t) = V_0 e^{-t/\tau}} \qquad 1.19$$

With a calculator it is easy to show that the value of $\frac{v(t)}{V_0}$ is as shown in Table 1.1. It is evident from Table 1.1 that the voltage $v(t)$ is less than I percent of $V_0$ after $5\tau$ (five time-constants). Thus, it is customary to assume that the capacitor is fully discharged (or charged) after five-time constants. In other words, it takes approximately $5\tau$ for the circuit to reach its final state or steady state when no changes take place with time (i.e., in the absence of a forcing function). Notice that for every time interval of $\tau$, the voltage is reduced by 36.8 percent of its previous value: $v(t + \tau) = \frac{v(t)}{e} = 0.368v(t)$, regardless of the value of t.

**Table 1.1**

Values of $\frac{v(t)}{V_o} = e^{-t/\tau}$

| $t$ | $\frac{v(t)}{V_o}$ |
|---|---|
| $\tau$ | 0.36788 |
| $2\tau$ | 0.13534 |
| $3\tau$ | 0.04979 |
| $4\tau$ | 0.01832 |

| | |
|---|---|
| $5\tau$ | $0.00674$ |

Observe from Eq (1.18) that the smaller the time constant, the more rapidly the voltage decreases, that is, the faster the response.

This is illustrated in Fig. 1.9. A circuit with a small-time constant gives a comparatively fast response in that it reaches the steady state (i.e., final value) due to quick dissipation of energy stored, whereas a circuit with a large time constant by comparison, gives a slow response because it takes longer to reach steady state. Whether time constant is small or large, however, the circuit reaches steady state in five time-constants.

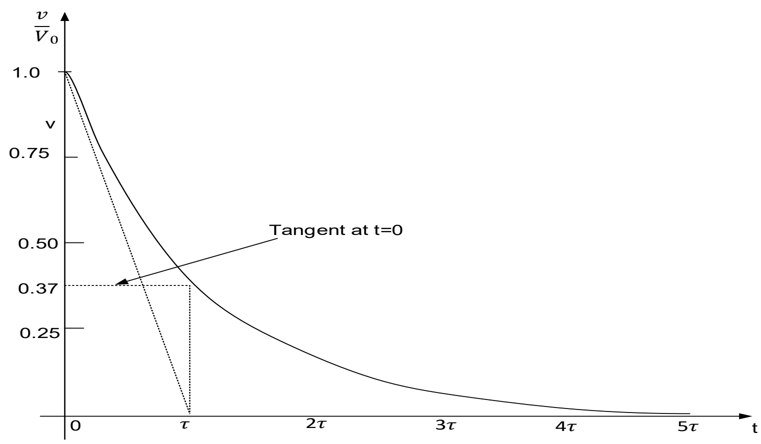



**Figure 1.8 Graphical determination of the time constant $\tau$ from the response curve**

With the voltage $v(t)$ in Eq (1.19), we can find the current $i_R(t)$,

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau} \qquad 1.20$$

The time constant may be viewed from another perspective. Evaluating the derivative of $v(t)$, in Eq (1.17) at $t = 0$ we obtain

$$\frac{d}{dt}\left(\frac{v}{V_0}\right)\bigg|_{t=0} = -\frac{1}{RC} e^{-t/RC}\bigg|_{t=0} = -\frac{1}{RC} = -\frac{1}{\tau}$$

(Employ *Duality Principle* to predict that for *RL* circuit.) Thus, the (negative) reciprocal of the time constant is the initial rate of decay, or the time taken for $\frac{v}{V_0}$ to decay from unity to zero, assuming a constant rate of decay. So, viewed from another perspective, time constant is the time taken by a signal to decay to zero, if it were to theoretically keep decaying at the same initial rate. But we know that, in reality, the rate of decay is not constant but rather keeps changing and getting (in this particular case) less negative (i.e., increasing) as its approaches zero.

This initial slope interpretation is used in the laboratory to find $\tau$ graphically from the response of an oscilloscope. To find $\tau$ from the response curve, draw the tangent to the curve at $t = 0$, as shown in Fig. 1.8. The tangent intercepts with the time axis at $t = \tau$.

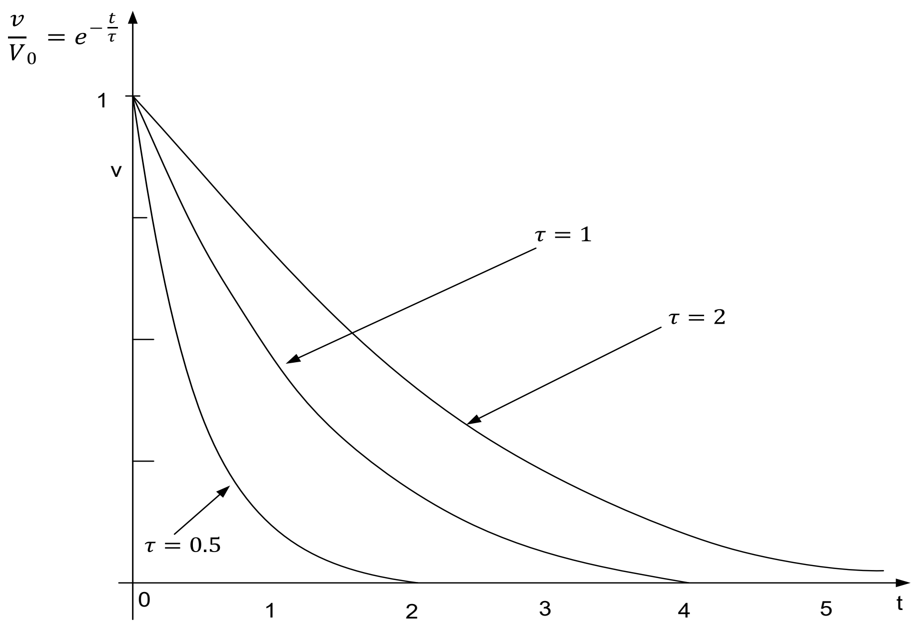



**Figure 1.9 Plot of $\frac{v}{V_0} = e^{-t/\tau}$ for various values of the time constant**

The power dissipated in the resistor is

$$p(t) = vi_R = \frac{V_0^2}{R} e^{-2t/\tau} \qquad 1.21$$

The energy absorbed by the resistor up to time t is

$$w_C(t) = \int \frac{V_0^2}{R} e^{-2t/RC} dt = -\left(\frac{RC}{2}\right)\frac{V_0^2}{R} e^{-\frac{2t}{RC}} + \mathrm{k} = -\frac{1}{2} C V_0^2 e^{-\frac{2t}{RC}} + \mathrm{k} \qquad 1.22$$

With k, the constant of the indefinite integral, as the initial (internal) energy stored in the system. k is ideally nonzero for this particular circuit [a **practical capacitor** is modeled as an ideal capacitor in parallel with some (internal) resistance]. Ignoring k, however, and the minus sign since energy is a scalar quantity (or it can be viewed as capacitor dissipating energy to, instead of absorbing from, the resistor), capacitor energy is expressed as

$$\boxed{w_C(t) = \frac{1}{2}CV_0^2 e^{-2t/RC} \text{ joules (J)}} \qquad 1.23$$

**Note**: If the capacitor is replaced with a **practical inductor** – modeled as an ideal inductor *in series* with some (internal) resistance – then the required response would be current, meaning that duality principle would tell us that the corresponding energy expression is

$$\boxed{w_L(t) = \frac{1}{2}LV_0^2 e^{-2tR/L} \text{ J}} \qquad 1.24$$

Notice that as $t \to 0$, $w_R(0) \to \frac{1}{2}CV_0^2$ which is the same as $w_C(0)$, the energy "initially" stored in the capacitor. The energy that was initially stored in the capacitor is eventually dissipated in the resistor. In summary:

The keys to working with a source-free *RC* circuit are:

(1) The initial voltage $v(0) = V_0$ across the capacitor
(2) The time constant $\tau$

With these two items sorted out from the particular circuit configuration, we readily obtain the response of the capacitor voltage $v_C(t) = v(t) = V_0e^{-t/\tau}$ With this, other variables (capacitor current $i_C$, resistor voltage $v_R$, and resistor current $i_R$) can be determined. In finding the time constant; $\tau = RC$, $R$ is often the Thevenin equivalent resistance at the terminals of the capacitor; that is, we take out the capacitor $C$ and find $R = R_{\text{Th}}$ at its terminals.

**Example 1.4:** In Fig. 1.10, let $v_C(0) = 15$ V. Find $v_C, v_x$and $i_x$ for $t > 0$.

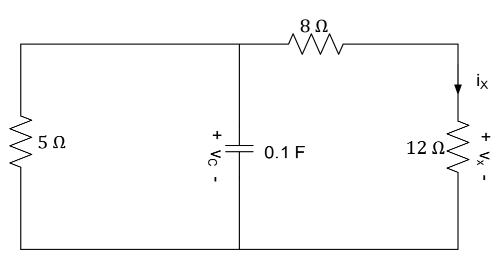



**Figure 1.10**

Solution:

We first need to make the circuit in Fig 1.10 conform to the standard *RC* circuit in Fig 1.6. We find the equivalent resistance or the Thevenin resistance at the capacitor terminals. Our objective is always to first obtain capacitor voltage $v_C$. From this, we can determine $v_x$ and $i_x$. The 8 Ω and 12 Ω resistors in series can be combined to give a 20 Ω resistor. This 20 Ω resistor in parallel with the 5 Ω resistor can be combined so that the equivalent resistance is

$$R_{eq} = \frac{20 \times 5}{20 + 5} = 4\ \Omega$$

Hence, the equivalent circuit is as shown in Fig. 1.10a, which is analogous to Fig.1.6. The time constant is

$$\tau = R_{eq}C = 4(0.1) = 0.4\ \text{s}$$

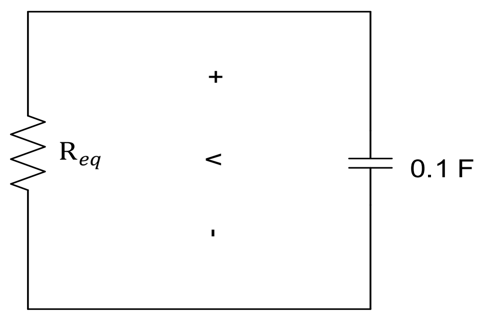



**Figure 1.10a Equivalent circuit for the circuit in Fig. 1.10**

Thus,

$$v = v(0)e^{-t/\tau} = 15e^{-t/0.4}\ \text{V}, v_C = v = 15e^{-2.5t}\ \text{V}$$

From Fig. 1.10 we can use voltage division to get $v_x$:

$$v_x = \frac{12}{12 + 8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t}\ \text{V}$$

Finally,

$$i_x = \frac{v_x}{12} = 0.75e^{-2.5t}\ \text{A}$$

**Example 1.5**: The switch in the circuit in Fig.1.11 has been closed for a long time, and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.

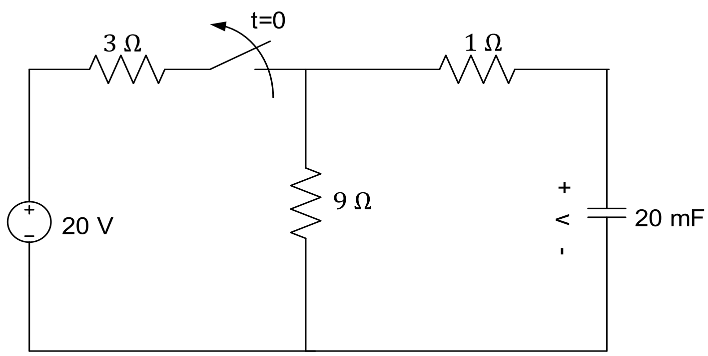



**Figure 1.11**

Solution:

For $t < 0$ the switch is closed; the capacitor is an open circuit to dc, as represented in Fig.1.11.1(a). Using voltage division,

$$v_C(t) = \frac{9}{9+3}(20) = 15 \text{ V}, \qquad t < 0$$

Since the voltage across the capacitor cannot change instantaneously, the voltage across the capacitor at $t = 0^-$ is the same at $t = 0^+$ (or 0), or

$$v_C(0) = V_0 = 15 \text{ V}$$

For $t > 0$, the switch is opened, and we have the *RC* circuit shown in Fig. 1.11.1(b). [Notice that the RC circuit in Fig. 1.11.1(b) is source-free; the independent source in Fig. 1.11 is needed to provide the initial energy in the capacitor.] The 1 Ω and 9 Ω resistors in series give

$$R_{eq} = 1 + 9 = 10 \, \Omega$$

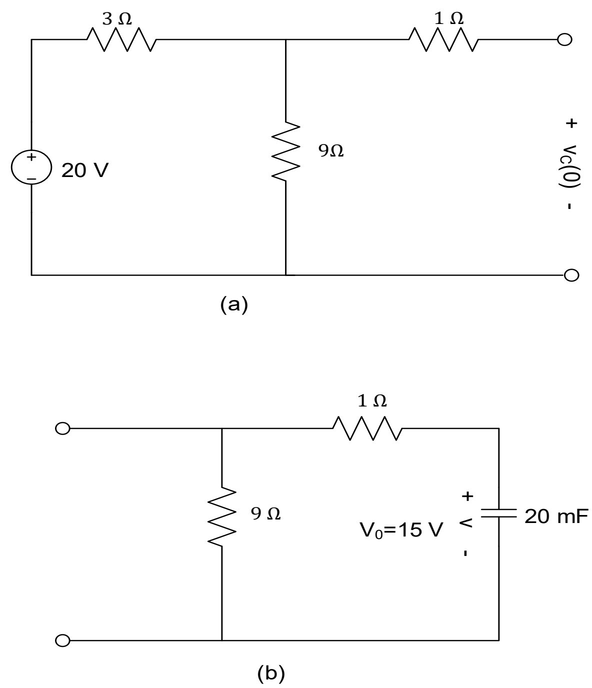



**Figure 1.11.1** $(\mathbf{a})\ \boldsymbol{t<0}, (\boldsymbol{b})\ \boldsymbol{t>0}$

The time constant is

$$\tau = R_{eq}C = 10 \times 20 \times 10^{-3} = 0.2 \text{ s}$$

Thus, the voltage across the capacitor for $t \geq 0$ is

$$v(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0.2} \text{ V}$$

or$v(t) = 15e^{-5t}$ V

The initial energy stored in the capacitor is

$$w_C(0) = \frac{1}{2}Cv_C^2(0) = \frac{1}{2} \times 20 \times 10^{-3} \times 15^2 = 2.25 \text{ J}$$

**Example 1.6**: At $t = 0$, the switch in Fig.1.12 is moved from position 1 to 2. Solve for $i(t)$; determine voltage across each 250 kΩ resistor of the circuit at $t = 1$ s

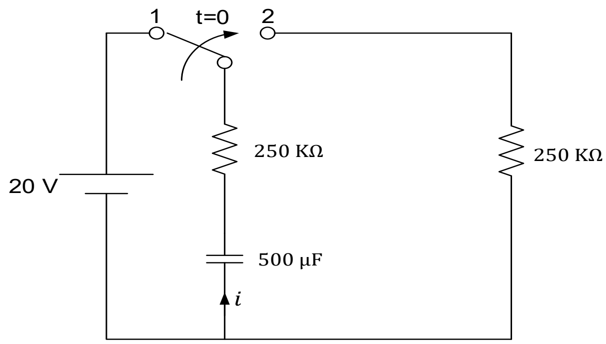



**Figure 1.12**

At $t < 0$ $v_C(0) = 20$ V

At $t > 0 \tau = R_{eq}.C$

$$\tau = (500 \times 10^3)(500 \times 10^{-6})$$

$\tau = 250$ s

$$v(t) = 20e^{-\frac{t}{250}}\,\text{V} = 20e^{-0.004t}\,\text{V}$$

$$i_c(t) = C\,\frac{dv}{dt} = 500 \times 10^{-6}\,\frac{d(20e^{-0.004t})}{dt}$$

$$i_c(t) = 500 \times 10^{-6} \times 20 \times (-0.004)e^{-0.004t}$$

$i_c(t) = -4 \times 10^{-5}e^{-0.004t}$

$$= -40e^{-0.004t}\ \mu\text{A}$$

The voltage across resistor is $v_R(t) = i_c(t)R$

$v_R(t) = (-40 \times 10^{-6}e^{-0.004t})\,\text{V} \times 250\ \text{k}\Omega$ At t = 1 s

$$v_R(1) = -40 \times 10^{-6}e^{-0.004(1)} \times 250 \times 10^3$$

$$v_R(1) = -10e^{-0.004} = -9.96\ \text{V}$$

**Example 1.6:** For the circuit in Fig. 1.13, the switch has been open for a long time. Determine $I$ at $t = 0^+$

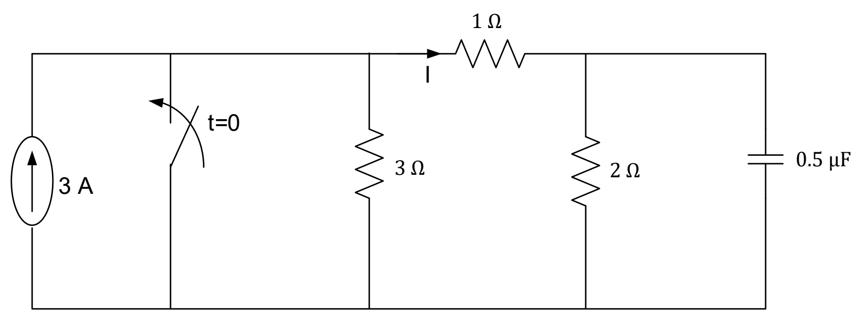



**Figure 1.13**

Solution:

$$I(0^-) = 3 \times \left(\frac{3}{3+1+2}\right) = 1.5\ \text{A}$$

$$\Longrightarrow \quad V_{2\Omega}(0^-) = 1.5 \times 2 = 3\ \text{V}$$

$$V_{2\Omega}(0^+) = 3\ \text{V}$$

$$\Longrightarrow I(0^+) = -\left(\frac{3\ \text{V}}{1\ \Omega}\right) = -3\ \text{A}$$

**Example 1.7**: For the circuit Fig.1.14, given that $V = 10\ \text{V}$, determine $\frac{dv(t)}{dt}$ at the given instant in time

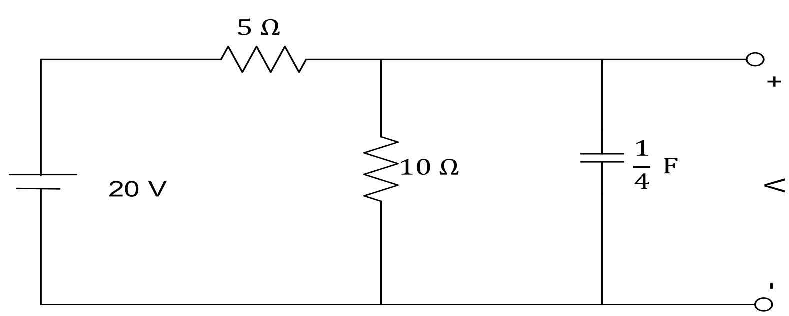



**Figure 1.14**

Solution:

$$i_{10\Omega} = \frac{10\ \text{V}}{10\ \Omega} = 1\ \text{A}$$

$$i_{5\Omega} = \frac{(20-10)}{5} = 2\ \text{A}$$

$$i_c = C\frac{dv_c}{dt} = i_{5\Omega} - i_{10\Omega}$$

$$i_c = \frac{1}{4}\frac{dv_c}{dt} = 2\text{ A} - 1\text{ A} = 1\text{ A}$$

$\Longrightarrow \frac{dv}{dt} = 4 \times 1\text{ A} = 4\text{ V/s}$

Alternatively, applying KVL to the loop

$$-20 + 5i_1 + 10(i_1 - i_2) = 0 \quad \Longrightarrow \quad 15i_1 - 10i_2 = 20 \qquad *$$

$$V + 10(i_2 - i_1) = 0$$

$$-10i_1 + 10i_2 = -10$$

$$\text{for} \quad V = 10\text{ V} \qquad **$$

Solving Eqs ** and * simultaneously,

$$i_2 = i_C = 1\text{ A} \;\;\&\;\; i_1 = 2\text{ A} = C\frac{dv_c(t)}{dt}$$

$$\frac{dv_c(t)}{dt} = \frac{i_C}{C} = \frac{1}{0.25} = 4\text{ V/s}$$

## 1.2 Exercises

**1.** (a) For the circuit Fig. 1, express the voltage of the $50\ \Omega$ resistor as a function of time and circuit parameters. (b) What are the initial currents through each of the two resistors and the inductors in Fig. 1?

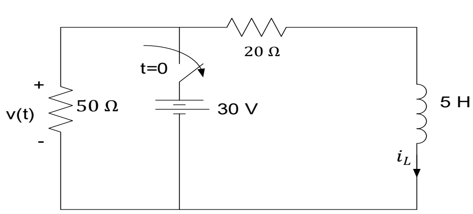



**Figure. 1**

**2.** For the circuit Fig. 2 the switch has been open for a long time. Determine I at $t = 0^+$.

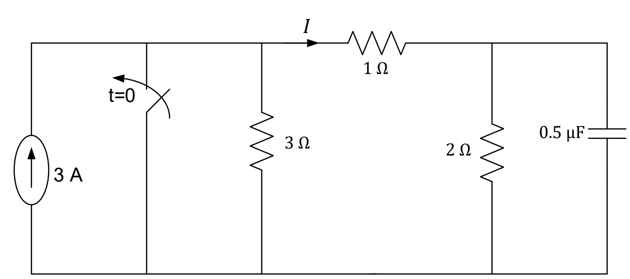



**Figure. 2**

**3.** For the circuit of Fig. 3, switch is closed for a long time, then opened at

$t = 0$. Determine current through the $1\mu\text{F}$ capacitor at $t = 0^+$.

**6.** In Fig. 4, identify the element/parts A, B, C, D and E; j, k, l, m and n. Then match each electrical component with its mechanical counterpart.

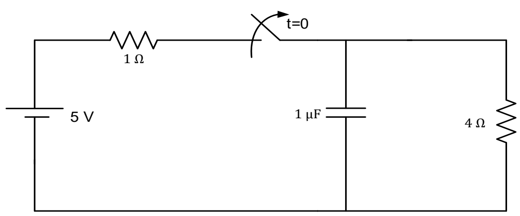



**Figure. 3**

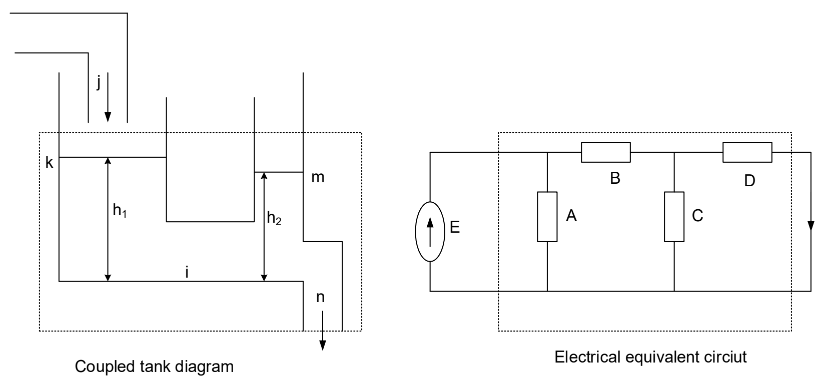



**Figure. 4**

**4.** Given the voltage signal $v(t) = 5\sin(0.5\pi + \pi/3)\,\text{V}$ determine (i) radian frequency (ii) phase (iii) frequency in hertz (iv) the period

**5.** Why is it usually required, as a transient response, either current through an inductor, or voltage across a capacitor?

**7.** Given $R = 2\text{ M}\Omega$ in *RC* circuit of the form of Fig. 5, $C$ if we want a time constant of 10 s

**Answer:** $\boldsymbol{C = 5 \times 10^{-6}\text{F} = 5\ \mu\text{F}}$

**8.** To what voltage $V_o$ of the capacitor of Fig. 5 decay over a period of one time constant?

**Answer:** $\boldsymbol{V_c(\tau) = 0.368V_o}$

# CHAPTER 2

# DRIVEN R-L AND R-C CIRCUITS

## 2.0 *R-C* Circuit with Step Response

We now consider a simple *R-C* series circuit shown Fig. 2.1:

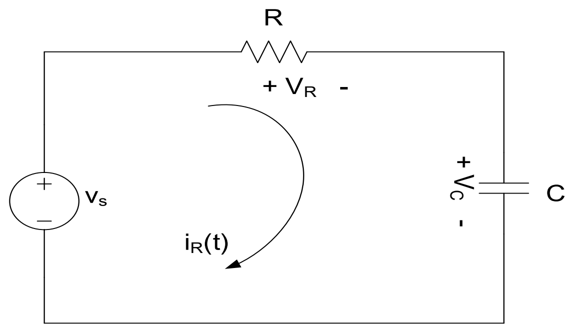



**Figure 2.1 *RC* Driven Circuit**

Applying Kirchhoff's Voltage Law (KVL) to the single loop gives:

$$-v_s(t) + i_R(t)R + \frac{1}{C}\int_{-\infty}^{t} i_R(t)\,dt = 0 \qquad 2.1$$

Rearranging, and noting that the derivative of an integral expression would produce the original (*zeroth* derivative) function, we then differentiate across to obtain:

$$R\frac{di_R(t)}{dt} + \frac{1}{C}i_R(t) = \frac{dv_s(t)}{dt} \qquad 2.2$$

When the dc source of an *RC* circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as a **step response**.

The step response of a circuit is its behaviour when the excitation is the step function, which may be a voltage or a current source. The step response is the response of the circuit due to a sudden application of a dc voltage or current source.

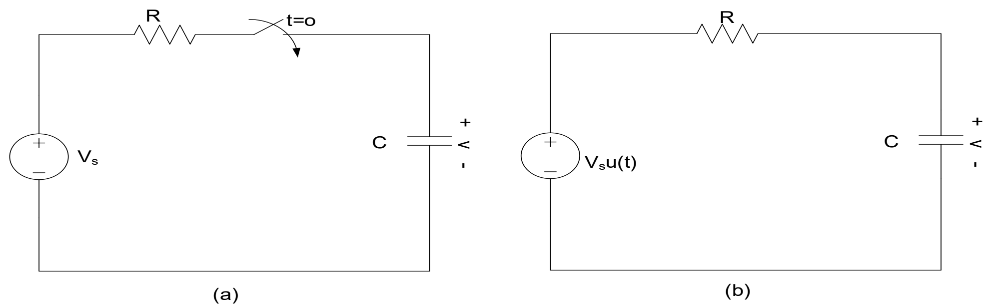



**Figure 2.1.1(a) An *RC* circuit with voltage step input; (b) an equivalent circuit**

Consider the *RC* circuit in Fig. 2.1.1(a) which can be replaced by the circuit in Fig. 2.1.1(b), where $V_s$ is a constant dc voltage source. Again, we select the capacitor voltage as the circuit response to be determined. We assume an initial voltage $V_o$ on the capacitor, although this is not necessary for the step response. Since the voltage of a capacitor cannot change instantaneously,

$$v(0^-) = v(0^+) = V_o \qquad 2.3$$

Where $v(0^-)$ is the voltage across the capacitor just before switching and $v(0^+)$is its voltage immediately after switching. Applying KCL, we have

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$$

Or

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} \qquad 2.4$$

where $v$ is the voltage across the capacitor. Rearranging terms, Eq (2.4) becomes, for $t > 0$,

$$-\frac{dv}{dt} = \frac{v - V_s}{RC} \qquad 2.5$$

or

$$\frac{dv}{v - V_s} = -\frac{dt}{RC} \qquad 2.6$$

Integrating both sides and introducing the initial conditions,

$$\ln(v - V_s)\Big|_{V_o}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$\ln[v(t) - V_s\,] - \ln\,(V_o - V_s) = -\frac{t}{RC} + 0$$

Or

$$\text{In}\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC} \qquad 2.7$$

Taking the exponential of both sides

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \ \ \tau = RC$$

$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

Or

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, t > 0 \qquad 2.8$$

Thus,

$$\boxed{v(t) = \begin{cases} V_0 & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau} & t > 0 \end{cases}} \qquad 2.9\text{a}$$

or, more compactly,

$$\boxed{v(t) = V_0 u(-t) + [V_s + (V_0 - V_s)e^{-t/\tau}]u(t) \text{ V}} \qquad 2.9\text{b}$$

[***NB**: A quick review of the unit-step function might be helpful here!]

This is known as the complete response (or total response) of the *RC* circuit to a sudden application of a dc voltage source, assuming the capacitor is initially charged. The reason for the term "complete" will become evident a little later. Assuming that $V_s > V_0$, a plot of $v(t)$ is shown in Fig. 2.2. [**Food for thought**: What would the Fig 2.2 look like for the case $V_s < V_0$!]

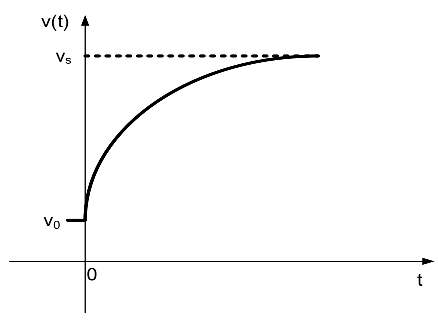



**Figure 2.2 Response of an *RC* with initially charged capacitor**

If the capacitor is uncharged initially, then $V_0 = 0$ in Eq (2.9) so that

$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases} \qquad 2.10$$

which can be written more compactly as

$$v(t) = V_s\left(1 - e^{-t/\tau}\right)u(t) \text{ V} \qquad 2.11$$

This is the complete step response of the *RC* circuit when the capacitor is initially uncharged. The current through the capacitor is obtained from Eq (2.11) using current-voltage relation for capacitor:

$$i(t) = C\frac{dv}{dt} = C\left(0 - -\frac{1}{\tau}\right)V_s e^{-t/\tau}u(t),$$

With $\tau = RC$,

$$i(t) = \frac{V_s}{R}e^{-t/RC}u(t) \text{ A}$$

Eq 2.2 can be seen as a typical first-order differential equation, whose general form is:

$$a_1\frac{dx\,(t)}{dt} + a_0x(t) = f(t) \qquad 2.12$$

where $x(t)$ represents either the capacitor voltage or inductor current, and $f(t)$ may be a voltage or current source (called *forcing function* in calculus), the reason being, as has been severally adduced, that these quantities cannot change in zero time (i.e., instantaneously) across and through, the respective circuit elements. It is called a linear first-order ordinary constant-coefficient differential equation. It's linear because terms like $[x(t)]^2$ or $\left[\frac{dx(t)}{dt}\right]^2$, do not appear in the equation. It is first-order because the highest derivative is once; and ordinary because no partial derivatives are involved, that is, the function is differentiated just with respect to time, and not with respect to any other variable(s) or network parameter(s).

The coefficients $(a_n, a_{n-1}, \dots a_1, a_0)$ are **constants** because they are not functions of time (i.e., time-independent).

### 2.0.1 Solving First-Order Differential Equation

As earlier mentioned, the response is of two parts, namely, (1) the natural response, determined by setting the driving source (*forcing function*) equal to zero, and (2) forced response (called *particular solution* in mathematics).

The latter is a response to a (particular) forcing function, say, voltage or current source, without regard to the initial conditions of the circuit elements, and this component of the total response remains after the transient portion of the response (solution) must have died off. Thereafter the solution (response) depends entirely on the nature of the driving source. By the last statement is meant that, for instance, if the driving source is a dc source, then the forced response would also be a dc type; on the other hand, if the driving source is an ac source, voltage or current, then the resulting forced response would also be ac in nature with the same frequency, only differing, perhaps, in its amplitude and phase.

For the typical first-order differential equation stated earlier in Eq 2.2, to solve *for* the ***natural response***, we make the equation homogeneous by setting the forcing function $f(t)$ equal to zero:

$$a_1 \frac{dx(t)}{dt} + a_0 x(t) = 0 \qquad 2.13$$

Because the parts are separable, a straightforward rearranging and integration would yield:

$$\int_{x_0}^{x} \frac{dx'}{x} = -\frac{a_0}{a_1} \int_0^t dt'$$

where $x'$ is typically understood to be a function of time [i.e.,$x'(t)$] despite the $(t)$ omission, and $x_0$ is its initial value. ($x'$, $t'$are so-called “dummy variables”.)

$$\Longrightarrow \qquad \ln x'|_{x_0}^{x} = -\left(\frac{a_0}{a_1}\right) t'\Big|_0^t = -a_0 \frac{t}{a_1}$$

$$\ln x - \ln x_0 = \ln\left(\frac{x}{x_0}\right) = -a_0 \frac{t}{a_1}$$

$$\Longrightarrow \frac{x}{x_0} = e^{-\frac{a_0}{a_1}t}$$

Finally,

$$x = \boxed{x(t) = x_0 e^{-\frac{a_0}{a_1}t}} \qquad 2.14$$

An alternative approach to determining the natural response is to look for a function that replicates itself upon differentiation (and, therefore, integration). The only function with this peculiar behaviour in all of mathematics, is the exponential function $a^n$ in general, and in particular the *natural* number-based exponential $e^n$. (The trigonometric functions sine and cosine do not satisfy this requirement because they only replicate their own negatives, and even then, after differentiating or integrating twice).

So, we assume a solution in the form of: $x(t) = Ke^{pt}$, where K and p are yet-to-be determined constants. Substituting this in the given first-order differential equation, we obtain:

$$a_1 \frac{d}{dt}(Ke^{pt}) + a_0(Ke^{pt}) = 0 \Longrightarrow a_1 Kpe^{pt} + a_0 Ke^{pt} = Ke^{pt}(a_1 p + a_0) = 0$$

Three possibilities are:

(1) K equals zero — unacceptable because it results in the triviality of $0 = 0$, indicating an identically zero response!
(2) $e^{pt}$ is zero — again unacceptable since this can only be true if time tends to negative infinity for a positive value of $p$,or positive infinity for a negative value of $p$, on account of the fact that the operating exponential factor must of necessity – left on its own as this is the case here – depreciate in order to avoid violating the law of energy conservation.
(3) $a_1 p + a_0 = 0 \Rightarrow p = -\frac{a_0}{a_1}$, which is most reasonably acceptable!

So,$x(t) = Ke^{-a_0 t/a_1}$

It remains to determine the value of the unknown constant K. This is accomplished by resorting to just one initial condition of the network quantity. For a first-order system, just one initial (boundary, as called in mathematics) condition, as the name implies, is adequate, whereas in yet-to-be-looked-into *second-order* system, we'd need an additional boundary (extreme, *viz*. terminal) condition in order to be able to evaluate the values of two resultant unknown constants.

Therefore, given that $x(0)$ [or more precisely, $x(0^+)$] = 0, $x(0) = Ke^0 = K = x_0$. Once again, finally,

$$\boxed{x(t) = x_0 e^{-a_0 t/a_1}}$$

Be reminded, once more, that the above expression strictly applies to just the natural response (called *homogeneous* response in mathematics) and does not consider any forced portion that's generally a component of the total response. Here, the response depends entirely on the initially energized state(s) of the circuit element(s), or lack thereof.

For the circuit of Fig. 2.1, *R* corresponds to $a_1$, and 1/*C* (please, **not** *C*!) corresponds to $a_0$. So, $i(t) = i(0)e^{-\left(\frac{1}{c}\right)\frac{t}{R}} = I_0 e^{-t/RC}$, with $I_0$ understood to be the initial value of the (series) current. Since the power (index) of the exponential term must necessarily be a numeric ("unitless"), *RC* therefore must have the dimension of time. To show this to be consistent by other means is left as an exercise.

In summary, (at the risk of repetition) the above analysis applies strictly to source-free circuits, so $i(t)$ ought to be more properly written as $i_N(t)$, with the subscript $N$ indicating "natural" (response).

A further investigation of "*RC*": Setting *t = RC*,

$i(t) = I_0 e^{-1} \Longrightarrow \frac{i(t)}{I_0} = e^{-1} = \frac{1}{e^1} \approx 0.37$ So, at time $t = RC$ (or $R_{eq}C_{eq}$ for multiple *R's* and/or *C's*) for a typical *R*-*C* circuit, a responding signal (be it current or voltage) would have decayed to approximately 37% of its initial value. Viewed from an alternative perspective, this is the time required for an exponentially rising signal to reach approximately 63% (why?) of its final value.

In the foregoing case (and as met previously), *RC* is a product known as the **time constant**, designated by the Greek letter $\tau$ (*tau*).

Fig.2.2.1 shows the plots of capacitor voltage $v(t)$ and capacitor current $i(t)$.

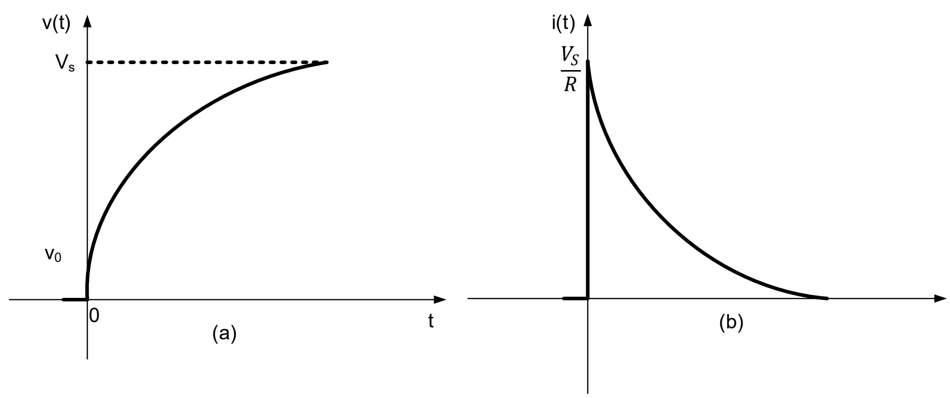



**Figure 2.2.1 Source-free *RC* circuit: (a) voltage response (b) current response**

Rather than going through the derivations above, there's a systematic approach – or rather, a short-cut method – for finding the step response of an *RC* or *RL* circuit. Let's re-examine Eq 2.6 which is more general than Eq 2.11. It is evident that v(t) has two components. Classically there are two ways of decomposing this into two components.

The first is to break it into a "natural response and a forced response" and the second is to break it into a "transient response and a steady-state response." Starting with the natural response and forced response, we write the total or complete response as

| Complete response = natural response + forced response<br>(stored energy) |
|---|

Or $$v = v_N + v_F \qquad 2.15$$

where $$v_N = V_0 e^{-t/\tau}$$

and $$v_F = V_s\left(1 - e^{-t/\tau}\right)$$

We are already familiar with the natural response $v_N$ of the circuit, as discussed in chapter 1. $v_F$ is known as the forced response because it is produced by the circuit when an external "force" (a voltage source in this case) is applied. It represents what the circuit is *forced* to do by the input excitation. The natural response eventually dies out along with the transient component of the forced response, leaving only the steady-state component of the forced response.

Another way of looking at the complete response is to break it into two components, one temporary and the other permanent:

$$Complete\ Response\ =\ Transient\ Response\ +\ Steady-state\ Response$$

⟹Temporary part + Permanent part

Or $$v = v_T + v_{SS} \qquad 2.16$$

where $$v_T = (V_0 - V_s)e^{-t/\tau} \qquad 2.17a$$

and $$v_{SS} = V_s \qquad 2.17b$$

The transient response $v_T$ is temporary; it is the portion of the complete response that decays to zero as time approaches infinity. Thus, the transient response is the circuit's temporary response that will die out with time.

The steady-state response $v_{SS}$ is the portion of the complete response that remains after the transient response has died out. Thus, the steady-state response is the behaviour of the circuit a "long time" after an external excitation is applied.

The first decomposition of the complete response is in terms of the source of the responses, while the second decomposition is in terms of the permanency of the responses. Under certain conditions, the natural response and transient response are the same. The same can be said about the forced response and steady-state response. Whichever way we look at it, the complete response in Eq (2.8) may be written as

$$\boxed{v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}} \qquad 2.18$$

where $v(0)$ is the initial voltage at $t = 0^{+}$and $v(\infty)$ is the final or steady-state value. Thus, finding the step response of an *RC* circuit requires three things:

1. the initial capacitor voltage $v(0)$
2. the final capacitor voltage $v(\infty)$
3. the time constant $\tau$

We obtain item 1 from the given circuit for $t < 0$ and items 2 and 3 from the circuit for$t > 0$. Once these items are determined, we obtain the response using Eq (2.18). This technique equally applies to *RL* circuits as we shall see in the next section.
Note that if the switch changes position at time $t = t_0$instead of $t = 0$, there is a time delay in the response, so that Eq 2.18) is adjusted to

$$v(t) = v(\infty) + [v(t_0) - v(\infty)]e^{-(t-t_0)/\tau} \qquad 2.19$$

where $v(t_0)$ is the initial valueat $t = t_0$ (not necessarily 0). Keeping in mind that Eq (2.18) or (2.19) applies only to step responses, that is, when the input excitation is constant (i.e., a dc source).

[If the term "transient" is at all used in describing natural response, then it properly belongs inside quotation marks to indicate just its fleeting nature and not its actual, purely technical characteristics. A purely *natural* response is not one and the same as the *transient* component, despite the erroneous habit of some texts that tend to use the two interchangeably. When there's a case where the two are equal, then it's strictly coincidental and nothing more! By the same token, forced response is not identical with steady-state response; the two components approach equality only as time tends to infinity. This is because, whereas steady-state response is a constant value, forced response has a transient portion in addition to a constant part (strictly note here that I didn't say "component", but "part"!) within its whole, the latter of which remains following the disappearance of the transient portion as time tends to "infinity". It's this remaining portion that equates to *steady-state* response.]

We shall illustrate these differences with an example:

**Example 2.1**: In the circuit on the Fig. 2.3, it's required to determine (a) $v_c(t)$ as the output being the capacitor voltage; (b) The series current $i(t)$.

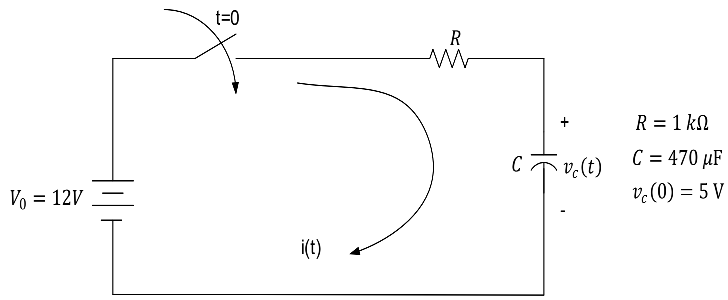



**Figure 2.3**

Solution:

Either $i(t)$ or $v_c(t)$ can be determined, and the other then derived from the relationship:

$$i_C(t) = C\frac{dv\ (t)}{dt} \Rightarrow\ v_C(t) = \frac{1}{C}\int i_C(t)dt$$

To determine $i(t)$ directly,

$$-V_0 + i(t)R + \frac{1}{C}\int_0^t i(t)dt = 0$$

$$-V_0 + i(t)R + v_C(t) = 0 = -V_0 + \left[C\frac{dv(t)}{dt}\right]R + v_C(t)$$

For determining $v_c(t)$ as the response [note that $i(t)$ is here necessarily $i_C(t)$!]:

From (a), differentiating across to clear the integral

$$R\frac{di(t)}{dt} + \frac{1}{C}\ i(t) = 0$$

$$\frac{di(t)}{dt} + \frac{1}{RC}i(t) = 0 \Leftrightarrow s + \frac{1}{RC} = 0 \qquad \text{(s is the differential operator)}$$

with a single root at

$$s = -\frac{1}{RC} = -\frac{1}{10^3 \times 470 \times 10^{-6}}$$

The time constant $\tau = RC = 10^3 \times 470 \times 10^{-6} = 0.47$ s

$\Longrightarrow i(t) = Ke^{-t/0.47}$

But

$$i(0^+) = \frac{[12 - v(0^+)]\ \text{V}}{1\ \text{k}\Omega} = \frac{[12 - v(0^-)]}{10^3} = \frac{(12-5)}{1000} = 0.007\ \text{A}$$

$$\Longrightarrow \quad i(t) = 0.007e^{-\frac{t}{0.47}} = 7e^{-\frac{t}{0.47}}\ \text{mA}$$

$$v_c(t) = \frac{1}{C}\int i(t)dt = \frac{10^6}{470}\int 0.007e^{-t/0.47}dt$$

$$v_c(t) = \frac{10^6}{470}(0.007)\left(-\frac{0.47}{1}\right)e^{-t/0.47} + K = -7e^{-t/0.47} + K$$

$$v_c(0^+) = v_c(0^-) = 5 = -7e^0 + K \Longrightarrow K = 12$$

Finally,

$$v_C(t) = \left(12 - 7e^{-t/0.47}\right)u(t)\ \text{V}$$

Alternatively, from equation (b)

$$RC\ \frac{dv_C(t)}{dt} + v_C(t) = V_0 \Longrightarrow \frac{dv_C(t)}{dt} + \frac{1}{Rc}v_C(t) = \frac{V_0}{RC}$$

Transient response $v_{CT}(t) = Ke^{-t/RC} = Ke^{-t/0.47}$

Steady-state response $v_{css} = 12$ V (as the capacitor is now an open circuit)

$$v_C(t) = v_{CSS} + v_{CT}(t) = 12 + Ke^{-t/0.47}$$

$$v_C(0^+) = v_C(0^-) = 5 = 12 + K \Longrightarrow K = -7$$

$$v_C(t) = \left(12 - 7e^{-t/0.47}\right)u(t)\ \text{V, as before}$$

$$i(t) = i_C(t) = C\frac{dv_C(t)}{dt} = 470 \times 10^{-6}\frac{d}{dt}\left(12 - 7e^{-t/0.47}\right)$$

$$= 470 \times 10^{-6}\left[0 - 7\left(-\frac{1}{0.47}e^{-t/0.47}\right)\right]$$

$$= 0.007e^{-t/0.47}\ \text{A}$$

as previously determined directly above. The natural response $v_{CN}(t)$ is evaluated by ignoring the source $V_0$:

$$RC\frac{dv_{CN}(t)}{dt} + v_{CN}(t) = 0, \text{which leads to:}$$

$$v_{CN}(t) = Ae^{-t/0.47}, \text{with A a different constant from K above.}$$

$$v_{CN}(0) = 5 \Longrightarrow v_{CN}(t) = 5e^{-t/0.47}\text{A}$$

Total response is the sum of either: ⟹ **natural** response plus **forced** response or ⟹ **transient** response plus **steady-state** response
the second component of the latter of which is necessarily constant, i.e., time-independent.

For the example above: Transient response: $-7e^{-\frac{t}{0.47}}$ V

Steady-state response: 12 V Natural response: $5e^{-t/0.47}$V To evaluate the forced response $v_{CF}(t)$, we equate the two sums:

$$12 - 7e^{-t/0.47} = 5e^{-t/0.47} + v_{CF}(t)$$

$$\Longrightarrow v_{CF}(t) = [12 - 7e^{-t/0.47}] - 5e^{-t/0.47} = 12 - 12e^{-t/0.47} = 12(1 - e^{-t/0.47})$$

So, $v_C(t)$ (complete response) $= v_{CN}(t) + v_{CF}(t) = [5e^{-t/0.47} + 12(1 - e^{-t/0.47})]u(t)$ V

For a quick comparison:

$$v_C(t) = \begin{cases} & (12 - 7e^{-t/0.47})u(t)\text{V} \\ \text{or} & \\ & [5e^{-t/0.47} + 12(1 - e^{-t/0.47})]u(t)\text{ V} \end{cases}$$

Forced response, $12(1 - e^{-t/0.47})$, therefore, has a "*transient*" (as in *temporary* and not in the technical sense) portion within it $(-12e^{-t/0.47})$, and approaches equality with the steady-state value of 12 V *only* after that particular *transient* portion has decayed toward zero as time tends to infinity. (Here's once again upbraiding those texts that erroneously equate the two, as well as implying that natural and transient responses are one and the same and proceed to use them interchangeably. If, in the latter case, they happen to be equal, then it's purely coincidental as they can sometimes – depending on the choice of network parameters thereby affecting the particular time constant – even be each other's negative!) Two graphs below give a quick overview of these two different routes of getting to the *same* destination.

To check for the correctness of the results and the legitimacy of Fig. 2.4, boundary (i.e., extreme) conditions have to be considered. Knowing the values of the signal at the time equal to zero (initial value), and then at the time tending to infinity (final value), as well as at the times at which the values become (if applicable as done in *calculus*), respectively, maximum and minimum, helps in quickly appraising the graphical behaviour of the signal response in each case, and hence a sketch of same.

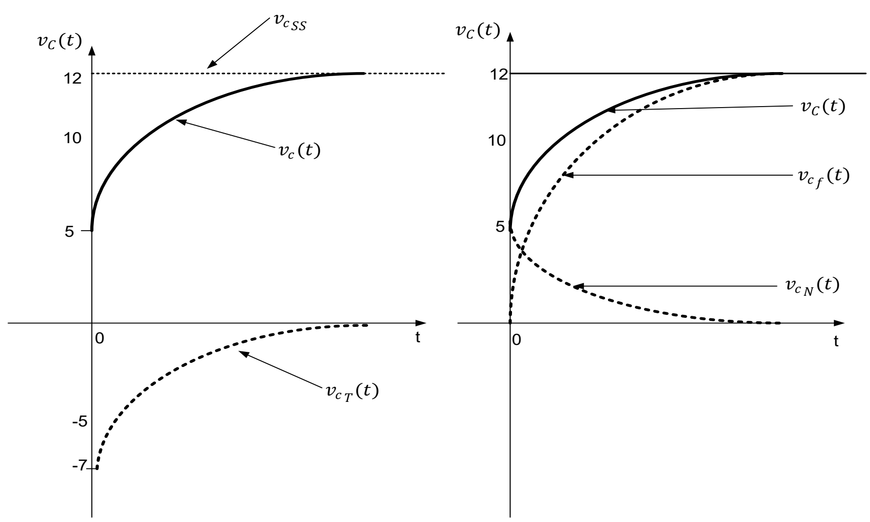



**Figure 2.4** $[v_C(t) = v_{CT}(t) + V_{CSS}]$; $[v_C(t) = v_{CN}(t) + v_{CF}(t)]$

## 2.1 Step Response of an *R-L* Circuit

Consider the *RL* circuit in Fig. 2.5(a), which may be replaced by the circuit in Fig. 2.5(b). Again, our goal is to find the inductor current $i$ as the circuit response. Rather than apply Kirchhoff's laws, we will use the simple technique in Eqs. (2.15) through (2.19). Let the (total) response be the sum of the transient and the steady-state responses:

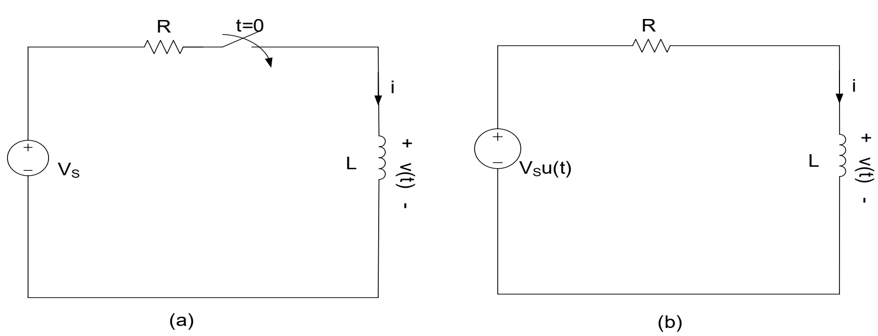



**Figure 2.5 an *RL* circuit with a step input voltage**

**Method 1**

$$i = i_T + i_{SS} \tag{2.20}$$

We know that the transient response is always a decaying exponential, that is,

$$i_T = Ke^{-t/\tau}, \qquad \tau = \frac{L}{R} \tag{2.21}$$

where K is a constant to be determined.

The steady-state response is the value of the current a *longtime* after the switch in Fig. 2.5(a) is closed. We know, furthermore, that the transient response essentially, practically dies out after about five-time constants. At that time, the inductor approximates a short circuit, and the voltage across it is, therefore zero. The entire source voltage $V_s$ appears across R. Thus, the steady-state response is

$$i_{ss} = \frac{V_s}{R} \tag{2.22}$$

Substituting Eqs (2.21) and (2.22) into Eq. (2.20) gives

$$i = Ke^{-t/\tau} + \frac{V_s}{R} \tag{2.23}$$

We now determine the constant $K$ from the initial value of $i$. Let $I_0$ be the initial current through the inductor, which may come from a source other than $V_s$. Since the current through an inductor cannot change instantaneously,

$$i(0^+) = i(0^-) = I_0 \tag{2.24}$$

Thus, at $t = 0$, Eq (2.23) becomes

$$I_0 = K + \frac{V_s}{R} \Longrightarrow K = I_0 - \frac{V_s}{R}$$

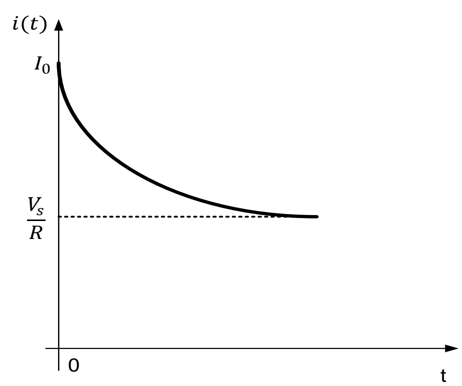



**Figure 2.6 Total response of the RL circuit with initial inductor current $I_0$**

Substituting for $K$ in Eq. (2.23), we get

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right) e^{-Rt/L} u(t) \text{ A} \qquad 2.25$$

This is the complete response of the *RL* circuit. It is illustrated in Fig. 2.6. The response in Eq. (2.25) may be written as

$$\boxed{i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}} \qquad 2.26$$

where $i(0)$ and $i(\infty)$ are the initial and final values of $i$ respectively. Thus, finding the step response of an *RL* circuit requires three things:

i. The initial inductor current $i(0)$ at $t = 0$
ii. The final inductor current $i(\infty)$
iii. The time constant $\tau$

We obtain item 1 from the given circuit for $t < 0$, and items 2 and 3 from the circuit for $t > 0$. Once these items are determined, we can then readily put down the expression for the complete response using Eq. (2.26). Keep in mind that this technique applies only for step responses.

Again, if the switching takes place at time $t = t_0$ instead of $t = 0$, Eq. (2.26) becomes

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-R(t-t_0)/L} \qquad 2.27$$

If $i_0 = 0$, then

$$i(t) = \begin{cases} 0, & t < 0 \\ \frac{V_s}{R}\left(1 - e^{-Rt/L}\right)\text{A}, & t > 0 \end{cases} \qquad 2.28a$$

Or more compactly,

$$i(t) = \frac{V_s}{R}\left(1 - e^{-Rt/L}\right)u(t) \text{ A} \qquad 2.28b$$

This is the step response of the *RL* circuit with no initial inductor current. The voltage across the inductor is obtained from Eq. (2.28b) using $v = L\, di/dt$. We get

$$v(t) = L\frac{di}{dt} = V_s \frac{L}{\tau R} e^{-Rt/L}, \qquad \tau = \frac{L}{R}, \quad t > 0$$

or

$$v(t) = V_s e^{-Rt/L} u(t) \text{ V} \qquad 2.29$$

Fig.2.7 shows the step responses in Eqs (2.28) and (2.29)

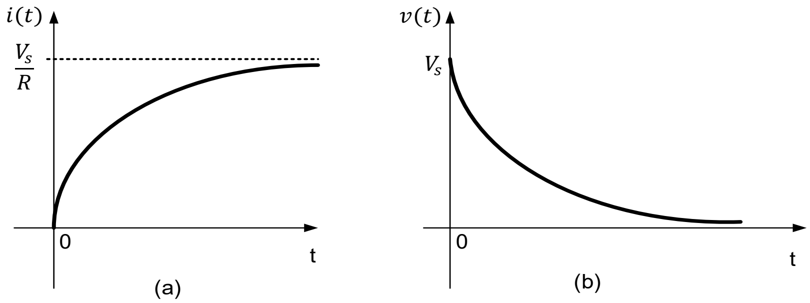



**Figure 2.7 Step responses of an RL circuit with no initial inductor current: (a) current response (b) Voltage response**.

**Method 2**

Refer to the *RL* circuit Fig. 2.5:

$$\text{(KVL):} \qquad V_0 = i(t)R + L\frac{di(t)}{dt} \qquad 2.30$$

Separation of variables allows us to write:

$$\frac{Ldi(t)}{V_0 - i(t)R} = dt \Rightarrow L\int_{i(0)}^{i(t)} \frac{di(t)}{V_0 - i(t)R} = \int_0^t dt \qquad 2.31$$

A simple change of variables $[x = V_0 - i(t)R \Rightarrow dx = -Rdi(t)]$ leads to:

$$-\frac{L}{R}\text{In}\,[V_0 - i(t)R]|_{i(0)}^{i(t)} = t \qquad 2.32$$

$-\frac{L}{R}\{\text{In}\,[V_0 - i(t)R] - \text{In}(V_0 - i(0)R)\} = t$ (*s* is the differential operator)

$$\Rightarrow \text{In}\left[\frac{V_0 - i(t)R}{V_0 - i(0)R}\right] = -\frac{Rt}{L}$$

$$\Rightarrow \left[\frac{0 - i(t)R}{V_s - i(0)R}\right] = e^{-Rt/L}$$

$$\Rightarrow \quad i(t)R = 0 - (0 - i(0)R)e^{-\frac{Rt}{L}} \qquad 2.33$$

But $i(0^+) = i(0^-) = i(0) = 0$, because, prior to the closing of the switch, no current was flowing, and still no current flows immediately after closing the switch because, as has been severally pointed out, current does not change instantaneously through an inductor.

$$\Longrightarrow i(t) = \frac{V_0}{R} - \frac{V_0}{R}e^{-Rt/L} = \boxed{\frac{V_0}{R}\left(1 - e^{-Rt/L}\right)u(t)\ \mathrm{A}} \qquad 2.34$$

**Method 3**

Making the left-hand side of Eq. (2.31) a definite integral while the right side is indefinite, we have:

$$-\frac{L}{R}\mathrm{In}\,[V_0 - i(t)R] = t + K \qquad 2.35$$

$$\Longrightarrow \qquad V_0 - i(t)R = e^{-R(t+K)/L}$$

$$i(0^+) = i(0^-) = i(0) = 0 \Longrightarrow V_0 = e^{-RK/L}$$

$$\Longrightarrow \quad -\frac{RK}{L} = \mathrm{In}V_0 \Longrightarrow K = -\frac{L}{R}\mathrm{In}V_0 \qquad 2.36$$

$$-\frac{L}{R}\mathrm{In}[V_0 - i(t)R] = t - \frac{L}{R}\mathrm{In}V_0$$

$$\frac{L}{R}\{\mathrm{In}[V_0 - i(t)R] - \mathrm{In}V_0\} = -t$$

$$\frac{L}{R}\mathrm{In}\left[\frac{V_0 - i(t)R}{V_0}\right] = -t$$

$$\Longrightarrow \qquad \mathrm{In}\left[\frac{V_0 - i(t)R}{V_0}\right] = -\frac{Rt}{L}$$

$$\Longrightarrow \qquad \frac{V_0 - i(t)R}{V_0} = e^{-Rt/L}$$

$$\Longrightarrow \qquad i(t)R = V_0 - V_0e^{-Rt/L}$$

$$i(t) = \frac{V_0}{R}\left(1 - e^{-Rt/L}\right)u(t)\ \mathrm{A} \qquad 2.37$$

"Reading" the circuit, before closing the switch, no current flows, and immediately after the closure, still no current flows because of the presence of the inductor. Substituting zero for $t$ in the above expression for $i(t)$, results in

$$i(0) = \frac{V_0}{R}(1 - e^0) = 0$$

At the (terminal) steady state, i.e., as $t \to \infty$, transient portion has died off, leaving just $\frac{V_s}{R}$ as the (steady-state) response. Reading from the circuit, the inductor appears as a short circuit in the steady state, and a simple application of Ohm's law results in $\frac{V_s}{R}$ as the current response.

At two-time constants, a signal must have decayed, therefore, to $(0.37)^2 = 0.1369$, approximately 14 of its initial value, at three-time constants, $(0.37)^3 = 0.0506$, approximately 5% of the original value. At five time-constants, a typical signal has gone down to $(0.37)^5 = 0.007$, approximately 0.7% of its initial value. In essence, even though, theoretically, a decaying exponential signal (function) gets to be zero only at time infinity, yet for all practical purposes, the time "infinity" might just be a few ticks of the seconds hand of the clock representing *five*-time constants! It is worthy of note to be mindful that the "second" above might in some circumstantial configuration, be replaced by "micro-seconds"!

So, by inspection, time constant for a first order circuit is the quotient of the coefficient of the derivative term and that of the (zeroth derivative) function. Note that $x(t)$ can be expressed in a lightly varying, more tell-tale form as:

$x(t) = x(0)e^{-t/(\frac{a_1}{a_0})}$ Thereby immediately discerning the time constant term $\tau = \frac{a_1}{a_0}$. Time constant, therefore, depends on the values of the network parameters (elements), and in the foregoing case these are resistance and capacitance. Investigate that for *RL* circuit, the equivalent time constant would be $\tau =$*L/R* (the *dual* of *RC*). Also investigate (as an exercise) that *L/R* (as well as *RC*) also has the dimension of time, that is, seconds. So, for series *RL* circuit (actually *parallel* with *current* source, if viewed strictly from its **dual** perspective!),

$$i(t) = i(0)e^{-\frac{Rt}{L}}$$

$$\tau = \frac{L}{R}$$

In the typical first-order homogeneous differential equation, $a_1\frac{dx}{dt} + a_0x = 0$, where we ended up with $a_1p + a_0 = 0$ as the characteristic equation in determining the response, a cut-to-the-chase determination of the root can be made simply by replacing the "$\frac{dx}{dt}$" with $p$ (differential operator) — [and hence $x$ by unity (1)]— or an "$s$" as is commonly done to account for the general complex nature of the root(s) of the equation.

### 2.1.1 Driven *RL, RC, RLC* Circuits

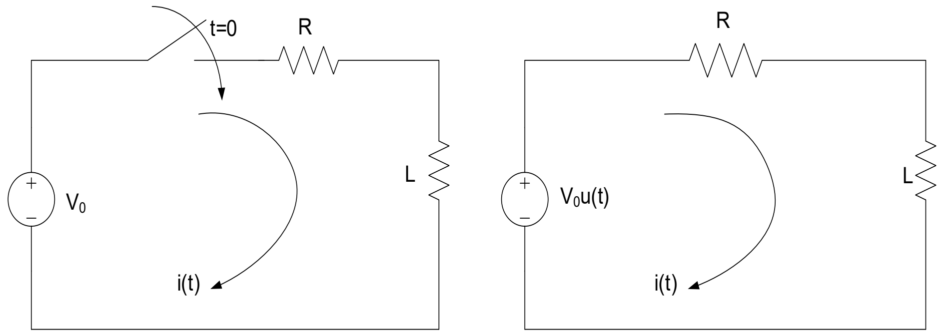



**Figure 2.8**

The equivalence of the two circuits in Fig. 2.8 can be established by noting a quick review of the unit step function:

$$u(t-a) = \begin{cases} 0, t < a \\ 1, t \geq a \end{cases} \qquad 2.38$$

The value of $u(t-a)$ at exactly $t = a$ is, strictly speaking, not determinate (undefined, or "infinity"), although it's generally, practically assumed that $u(t-a)$ starts to take on the value of unity from $t = a$.

In the above circuits, the switch is turned on at $t = 0$, resulting in a driving source voltage of $V_0$ volts which is a dc (i.e., constant) potential. It may be safely assumed that the inductor was initially fully energized, and its reactive nature would induce an exponentially decaying response. The response of the current through the inductor, are of two parts (components):

1. A natural response ("transient") due to the initial (energy) condition(s) of the network element(s) that would eventually die off at time "infinity", leaving …
2. … a forced response, that replicates the very nature of the forcing function (driving source). By this is meant that, a dc forcing function would induce a dc forced response, whereas an ac forcing function would likewise give rise to an ac forced response. So, the input and output signals would possess similar frequency (and therefore similar basic character), only, possibly, differing in their amplitudes and phase angles.

The total response is the sum of the two component responses (natural and forced):

$i(t) = i_N(t) + i_F(t)$ with both term general being time-dependent.

**2.1.2 A More General Approach**

For a general equation with a forcing function (i.e., source, as we're now beyond age of homogeneity!):

$$\frac{dy}{dt} + py = Q \qquad 2.39$$

where $y$ [understood to be actually $y(t)$ since it's in general time-dependent] stands for either current or voltage signal. Rearranging Eq. 2.39,

$$dy + pydt = Qdt \qquad 2.40$$

with the forcing function $Q = Q(t)$, since it's in general, again, time-dependent. It's further assumed that $p$ is a *positive* constant, and momentarily and for simplicity that $Q$ is also a constant. Multiplying Eq. 2.40 across by the *integrating factor*

$$e^{\int pdt} = e^{pt}\colon e^{pt}dy + ype^{pt} = Qe^{pt}dt \qquad 2.41$$

Using the rule for the differentiation of a product (chain rule) on Eq. 2.41:

$$d(ye^{pt}) = e^{pt}dy + ype^{pt}dt = Qe^{pt}dt \qquad 2.42$$

Integrating both sides of Eq. 2.42: $\int d(ye^{pt}) = \int Qe^{pt}dt$

$$\Longrightarrow \qquad ye^{pt} = \int Qe^{pt}dt\,(+K) \qquad 2.43$$

$$y = e^{-pt}\int Qe^{pt}\,dt + Ke^{-pt} \qquad 2.44$$

where $K$ is the constant of the indefinite integration.

For the natural response (no forcing function),

$$Q = 0 \Longrightarrow y_N(t) = Ae^{-pt} = Ke^{-pt} \qquad 2.45$$

and $p$ is never negative for any circuit with only resistors, inductors and capacitors, and that depends only on the passive circuit elements, meaning, therefore, that we're here concerned only with forward-looking time.

For the steady-state response, assuming momentarily that $Q$ is constant (dc forcing function):

$$y_{ss} = y(t \to \infty) = e^{-pt} Q \int e^{pt} dt + K e^{-pt}(t \to \infty) = e^{-pt} \frac{Q e^{pt}}{p} = {}^{Q}/_{p}$$

$$\Longrightarrow \qquad y(t) = y_{ss} + y_T = {}^{Q}/_{p} + K e^{-pt} \qquad 2.46$$

For the circuit of Fig. 2.8 $y_{ss} = {}^{Q}/_{p} = {}^{V_o}/_{R}\,(inductor\ is\ short)$ $\quad p = \frac{1}{\tau} \Longrightarrow \tau = 1/p$ Applying initial condition$y(0) = Y_0$, $\quad y(0) = {}^{Q}/_{p} + K \Longrightarrow K = Y_o - {}^{Q}/_{p}$

Finally,

$$y(t) = {}^{Q}/_{p} + \left(Y_0 - {}^{Q}/_{p}\right) e^{-pt} = y_{ss} + y_T(t) \qquad 2.47a$$

Rearranging,

$$y(t) = \left({}^{Q}/_{p}\right)(1 - e^{-pt}) + Y_0 e^{-pt} = y_F(t) + y_N(t) \qquad 2.27b$$

i.e., two different summations of the same result.

So, given a circuit with a driving source (mathematically known as *forcing function*), these steps need to be taken in order to be able to readily put down the expression for the complete response, either way:

1. Determine the time constant $\tau = (R_{eq})(C_{eq})\ or\ \frac{(L_{eq})}{(R_{eq})}$ ;
2. determine the initial value ($Y_o$ in the foregoing case);
3. Evaluate the steady state-response, which is achieved by zeroing all the reactive elements in the circuit (shunting all the inductors and open-circuiting all the capacitors).

Then, any or all of the fundamental laws of electrical circuits can be applied to determine the above. (Remember that the three laws that govern any circuit, namely, KVL, KCL and Ohm's law, are applicable at any point in time and space, i.e., instantaneously or at a steady state.

Having done all of these, then:

$$\boldsymbol{Complete\ response} = (\boldsymbol{Steady - state\ response}) + (\boldsymbol{Initial\ value\ minus\ Steady - state\ value})\boldsymbol{e^{-pt}}$$

$$or$$

$$= (Steady - state\ response)(1 - e^{-pt}) + (Initial\ value)e^{-pt}$$

Comparing and contrasting the two different expressions for the complete response shown in Eq. 2.47, would enable one to quickly appreciate what the writer have previously emphasized: steady-state response $Q/p$ and the forced response $\left(Q/p\right)(1 - e^{-pt})$, are not, as made obvious here, one and the same! They become equal only after the transient portion of the forced response $\left(-Q/p\right)(e^{-pt})$, has died off as time tends to infinity. Similarly, natural response $(Y_o e^{-pt})$ and the transient response $\left(Y_0 - Q/p\right) e^{-pt}$, are obviously not interchangeable (as some texts erroneously assumed based on a situation that may be peculiar to a particular circuit and therefore offer a special case). If, for instance, steady-state value $(Q/p)$ equates to zero – as in a dc current through a capacitor, or – *dually* – a dc voltage across an inductor– then and only then will the natural and transient responses be each other's equal. Furthermore, if perchance the steady-state value is twice the initial value, then natural and transient responses actually become each other's negative! (This of course would depend on the choice of elemental circuit parameters.) So, *touché*! – for some texts that tend to use the two interchangeably. (Analogously, it's exhibiting mathematical dishonesty, for instance, to employ a rectangle, or even more grievously a square, when required to prove a property of a four-sided figure! Upholding of scientific integrity dictates that we should use, by way of generalising, a figure with four *unequal* sides, because every rectangle is a four-sided figure but not vice versa.)

**Example 2.2**: In the circuit of Fig. 2.9, determine $i_L(t)$ for all time $(-\infty < t < \infty)$

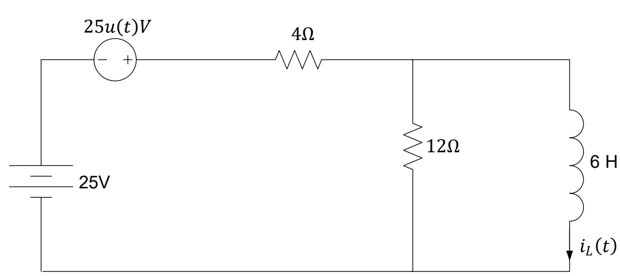



**Figure 2.9**

Solution:

To get the time constant, $\tau = \frac{L_{eq}}{R_{eq}}$, when all the sources (both voltage sources) are zeroed by short circuiting them, and their Thevenin equivalent resistance $R_{eq}$ is thereby determined:

$$R_{eq} = 4||12 = \frac{(4)(12)}{(4+12)} = \frac{48}{16} = 3\ \Omega$$

$$L_{eq} = 6\ \mathrm{H} \Longrightarrow \tau = \frac{6\ \mathrm{H}}{3\ \Omega} = 2\ \mathrm{s}$$

Transient response:

$$i_T(t) = Ke^{-t/\tau} = Ke^{-\frac{t}{2}}$$

To evaluate the steady-state response, we note that the inductor presents a short circuit to the dc (total) voltage, the $12\ \Omega$ resistor is therefore shorted out leaving only the $4\ \Omega$ "seen" by $25 + 25 = 50\ \mathrm{V}$. So,

$$i_{Lss} = \frac{50\ \mathrm{V}}{4\ \Omega} = 12.5\ \mathrm{A}$$

Total response:

$$i_L(t) = i_{Lss} + i_{LT}(t) = 12.5 + Ke^{-\frac{t}{2}}$$

To evaluate the constant K (note carefully that the K here is different from the constant attached with respect to the *natural response*), the initial value of the current prior to $t = 0^-$ must be determined. Only the 25 V supply is operating as the step voltage is zero prior to the time $t = 0$. In this first steady-state situation the $6\ \Omega$ resistor is shorted out by the inductor seen as a shunt by the dc battery resulting in

$$i_L(0^-) = \frac{25\ \mathrm{V}}{4\ \Omega} = 6.25\ \mathrm{A}$$

$$i_L(0^+) = i_L(0^-) = 6.25 = 12.5 + K \Longrightarrow K = -6.25$$

Finally,

$$i_L(t) = 12.5 - 6.25e^{-\frac{t}{2}}, t > 0$$

That is,

$$i_L(t) = \begin{cases} 6.25\ \mathrm{A}, & t < 0 \\ 12.5 - 6.25e^{-\frac{t}{2}}\ \mathrm{A}, & t \geq 0 \end{cases}$$

Combining these two expressions, $i_L(t) = 6.25 + \left(6.25 - 6.25e^{-\frac{t}{2}}\right) u(t)$ A

$$i_L(t) = 6.25 + 6.25\left(1 - e^{-\frac{t}{2}}\right) u(t) \text{ A}$$

The validity of the above expressions can be ascertained by applying "special" times at $t = 0$ and $t \to \infty$ to determine the initial and steady-state responses, respectively.

**Notes on natural and forced responses**: As some texts erroneously equate natural and transient responses, and forced and steady-state responses, and proceed to same use interchangeably, let's use the just-concluded example to clearly show the respective differences:

⟹ Natural response is evaluated "independently" by equating $Ae^{-t/2} = 6.25$ at $t = 0^+$ (note that $A$ is a different constant from $K$) giving $A = 6.25$, whereas the total response was summed up before evaluating the plain $K$

So, $i_{LN}(t) = 6.25e^{-t/2}$ (for inductor current),

$$i_{LF}(t) = i_{Lss}\left(1 - e^{-t/2}\right) = 12.5\left(1 - e^{-t/2}\right) \text{ A}, t > 0$$

For $t > 0$, $i_{ss} = 12.5$ A (constant, i.e., time-independent)

$$i_{LT}(t) = [i_L(0) - i_{Lss}]e^{-t/2} = (6.25 - 12.5)e^{-t/2} = -6.25e^{-t/2} \text{ A}$$

Thus $i_{Lss}$ becomes equal to $i_{LF}(t)$ only as $t \to \infty$, and $i_{LT}(t)$ and $i_{LN}(t)$ are actually each other's negative! This is because, as I pointed out previously, the steady-state response (12.5 A) is twice the initial value of 6.25 A [evaluated by substituting $t = 0$ in the expression for (total) $i_L(t)$]

**Example 2.3**: In the circuit of Fig. 2.10, the coil has a $10\,\Omega$ resistance and a 6 H inductance. If $R = 14\,\Omega$, $V = 24$ V and the switch is opened at $t = 0$, determine:(a) $i_R(t)$ (b) the voltage across coil of the circuit at $t$ = 0.1 s

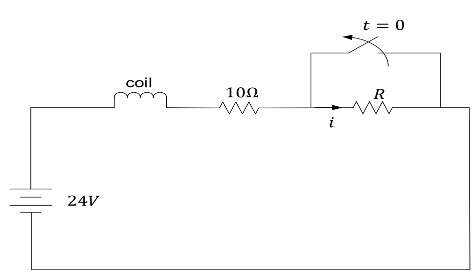



Figure 2.10

Solution:

(a) $t < 0$ (Switch is closed):

$$i(0^+) = i_R(0^-) = \frac{24}{10} = 2.4\text{ A}$$

$t > 0$ (Switch is open):

$$i_{ss} = \frac{24}{14+10} = 24/24 = 1\text{ A}$$

$i_T = Ke^{-t/\tau}$, where $\tau = 6\text{ H}/(10+14)\ \Omega = 0.25\text{ s}$

$$i_R(t) = i_{ss} + i_T(t)$$

$$i_R(t) = 1 + Ke^{-4t}$$

$$i_R(0) = 1 + Ke^{s0}$$

$$2.4 = 1 + K$$

$$K = 1.4$$

$$i_R(t) = 1 + 1.4e^{-4t}\text{ A}$$

(b) $$i(0.1) = 1 + 1.4e^{-4(0.1)} = 1.94\text{ A}$$

Thus,

$$V_{10\Omega} + V_R = i_R(10+14) = 1.94 \times 24 = 46.52\text{ V}$$

But

$$V_{coil} = V - (V_{10\Omega} + V_R) = 24 - 46.52$$

$$V_{coil} = -22.52\text{ V}$$

**Example 2.4:** For the circuit shown in the Fig 2.11 below, the switch has been open for a long time and is then suddenly closed at t = 0. Calculate:

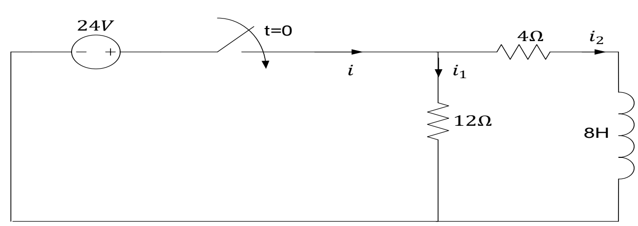



**Figure 2.11**

(a) $i_2$ at $t = 0.3$ s
(b) the steady state power supplied by the source of the circuit
(c) energy stored in the inductor

Solution:
(a) $t < 0$ (switch is open): $i_2(0^-) = 0 = [i_2(0^+)] = i_2(0)$

$t \to \infty$ (Switch is closed): $i_2(\infty) = 24\ \text{V}/4\ \Omega = 6\ \text{A}$

$$i_2(t) = i_2(\infty) + [i_2(0) - i_2(\infty)]e^{-t/\tau},$$

where $\tau = \frac{L}{R_{eq}} = \frac{8}{4} = 2\ \text{s}$

$i_2(t) = 6 + (0 - 6)e^{-0.5t} = 6 - 6e^{-0.5t} \Longrightarrow \qquad i_2(t) = 6(1 - e^{-0.5t})\ \text{A}$

At $t = 0.3$ s: $i_2(0.3) = 6(1 - e^{-0.5\times 0.3}) \Longrightarrow \qquad i_2(0.3) = 0.836\ \text{A}$

(b) $P = (I_1 + I_2)\,V$

$$P = \left(\frac{24}{12} + \frac{24}{4}\right)\text{A} \times 24\ \text{V} = 8 \times 24 = 192\ \text{W}$$

(c) $W_L = \frac{1}{2}Li_L^2$

$$W_L(t) = \frac{1}{2} \times 8 \times [6(1 - e^{-0.5t})]^2$$

$$W_L(t) = 144(1 - 2e^{-0.5t} + e^{-t})\ \text{J}$$

**Example 2.5**: The circuit of Fig. 2.12 is under steady state with the switch at position 1. At $t = 0$, the switch is moved to position 2. Find $i(t)$

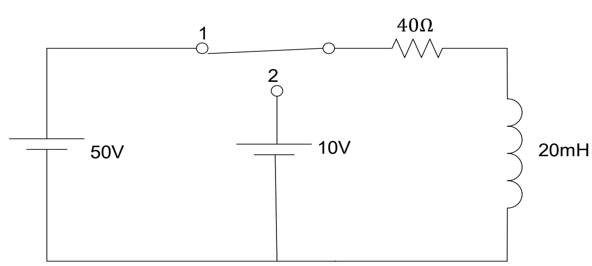



**Figure 2.12**

At $t < 0$ (at point 1)

$$i(0) = \frac{50}{40} = 1.25 \text{ A}$$

At $t = \infty$ (at point 2)

$$i_{(\infty)} = \frac{10}{40} = 0.25 \text{ A}$$

$$\tau = \frac{20 \times 10^{-3}}{40} = \frac{1}{2000}$$

$$i(t) = 0.25 + (1.25 - 0.25)e^{-2000t}$$

$$i(t) = 0.25 + e^{-2000t} \text{ A}$$

**Example 2.6:** The circuit of Fig.2.13 is under steady state with the switch at position 1. At $t = 0$, the switch is moved to position 2. Determine

(i) $v_c$
(ii) The current in the circuit $i_c$
(iii) The energy stored in the capacitor and resistor

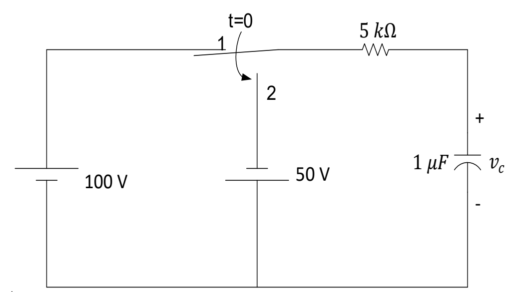



**Figure 2.13**

Solution:

At $t < 0$ (at point 1)

$$v_c(0^-) = v_c(0^+) = 100 \text{ V}$$

At $t = \infty$ (at point 2)

$$v_c(\infty) = -50 \text{ V}$$

$$\tau = RC = (5 \times 10^3)(1 \times 10^{-6})$$

$$\tau = \frac{1}{200}$$

$$v_c(t) = -50 + (100 + 50)e^{-200t}$$

$$v_c(t) = -50 + 150e^{-200t} \text{ V}$$

(ii) The current in the circuit at $t > 0$

$$i = i_c = C\frac{dv_c}{dt}$$

$$i_c = 1 \times 10^{-6}\frac{d}{dt}\{150e^{-200t} - 50\}$$

$$= 1 \times 10^{-6} \times 150 - 200e^{-200t}$$

$$i_c = -0.03e^{-200t} \text{ A}$$

$$i_c = 30e^{-200t} \text{ mA}$$

(iv) Energy in capacitor $w_c = \frac{1}{2}Cv_c{}^2$

$$w_c = \frac{1}{2} \times 1 \times 10^{-6}(-50 + 150e^{-200t})^2$$

$$w_c = \frac{1}{2 \times 1 \times 10^{-6}}[50(3e^{-200t} - 1)]^2$$

$$w_c = 5 \times 10^{-7} \times 2500\ (3e^{-200t} - 1)^2$$

$$w_c = 1.25 \times 10^{-3}(3e^{-200t} - 1)^2$$

$$w_c = 1.25\ (9e^{-400t} - 6e^{-200t} + 1) \text{ mJ}$$

Energy in resistor

$$w_R = \int_0^t \frac{v_R{}^2}{R}dt$$

But $v_R = i_R = (-0.03e^{-200t})(5 \times 10^3)$

$$v_R = -150e^{-200t} \text{ V}$$

$$w_R = \int_0^t \frac{(-150e^{-200t})^2}{5000}$$

$$w_R = \frac{22500}{500}\int_0^t e^{-400t}\,dt$$

$$w_R = \left.\frac{4.5e^{-400t}}{-400}\right|_0^t$$

Evaluating the boundaries

$$w_R = -0.01125\,e^{-400t} + 0.01125e^0$$

$$w_R = 0.01125 - 0.1125\,e^{-400t}\text{J}$$

$$w_R = 0.01125\,(1 - e^{-400t})\text{ J}$$

$$w_R = 11.25(1 - e^{-400t})\text{ mJ}$$

**Example 2.7**: For the circuit of Fig. 2.14, determine the

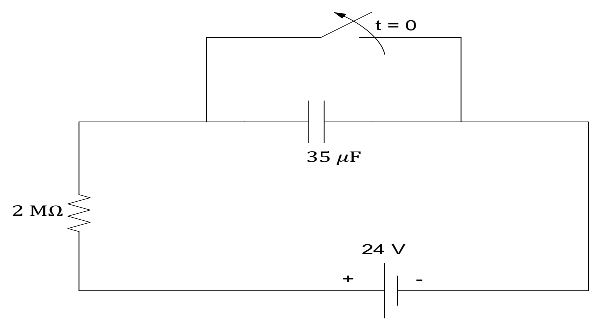



**Figure 2.14**

(i) Current through the and voltage across the capacitor at $t = 0^+ and\ t = 0^-$
(ii) The current in the circuit at $t = 70$ s

Solution:

(i) At $t < 0$

$$v(0^-) = v(0^+) = 0\text{ V}\quad i(0) = 0\text{ A}$$

At $t = 0^+$ [immediately the switch's is close]

$$i(0^+) = i_c(0^+) = \frac{V}{R} = \frac{24}{2\times 10^6}$$

$$i(0^+) = i_c(0^+) = 1.2\times 10^{-6}\text{ A}$$

$$i(0^+) = i_c(0^+) = 12\ \mu\text{A}$$

(ii) $v(0) = 0 \text{ V}$

$$v(\infty) = 24 \text{ V}$$

$$v(t) = V_0 + (V_0 - V_\infty)e^{-\frac{t}{\tau}}$$

Where $\tau = RC = (35 \times 10^{-6})(2 \times 10^{6})$

$$\tau = 70$$

$$v_c(t) = 24 - 24e^{-\frac{1}{70t}}\, V$$

$$i_c(t) = \frac{Cdv}{dt} = 35 \times 10^{-5} \frac{d}{dt}\left(24 - 24e^{-\frac{t}{70}}\right)$$

$$i_c(t) = 35 \times 10^{-6} \times -24 \times -\frac{1}{70} e^{-\frac{t}{70}}$$

$$i_c(t) = 1.2 \times 10^{-5} e^{-\frac{t}{70}} \text{ A}$$

At $t = 70s$

$$i_c(70s) = 1.2 \times 10^{-5} e^{-1}$$

$$i_c(70s) = 4.415 \times 10^{-6} \text{ A}$$

$$i_c(70s) = 4.415 \text{ μA}$$

**Example 2.8**: The circuit of Fig. 2.15 is under steady state with the switch at position 1. At $t = 0$, the switch is moved to position 2. Determine

(i) $i(t)$

(ii) How much energy is dissipated in the 2 Ω resistor at $t = 2$ s

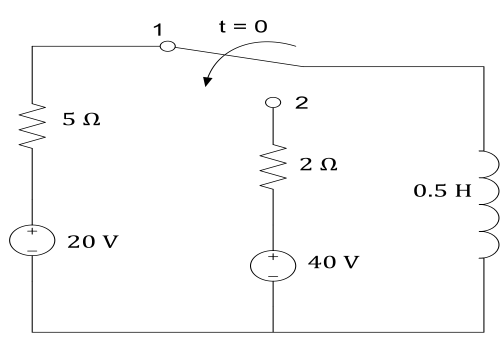



**Figure 2.15**

Solution:

(i) At point 1 (i.e., $t < 0$)

$$i(0^-) = i(0^+) = \frac{20}{5} = 4 \text{ A}$$

At point 2 (i.e., $t = \infty$)

$$i(\infty) = \frac{40}{2} = 20 \text{ A}$$

$$\tau = \frac{0.5}{2} = \frac{1}{4}$$

$$i(t) = 20 + (4 - 20)e^{-4t}$$

$$i(t) = 20 - 16e^{-4t} \text{ A}$$

$$i(t) = 20 - 16e^{-4t} \text{ A}$$

(ii) Energy at 2 Ω resistor at $t = 0.25$ s

$$w_R = \int_0^t i^2 \; Rdt$$

$$w_R = \int_0^t 2 \times (20 - 16e^{-4t})^2 dt$$

$$= \int_0^t 2 \times (400 - 640e^{-4t} + 256e^{-8t})\, dt$$

$$= 2\left[400t + \frac{640e^{-4t}}{4} - \frac{256e^{-8t}}{8}\right]_0^t$$

$$= 2[400t + 160e^{-4t} - 32e^{-8t}]|_0^t$$

Evaluating the boundaries

$$w_R = 800t + 320e^{-4t} - 64e^{-8t} - 320 + 64$$

At $t = 0.25$ s

$$w_R = 53.0599 \text{ J}$$

**Example 2.9**: The switch in the circuit of Fig.2.16 is closed at $t = 0$, at which moment the capacitor has charge $Q_0 = 500\ \mu\text{C}$, with the polarity indicated. Determine $q$ and $i$ at $t > 0$

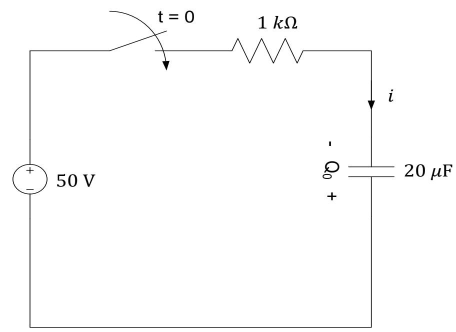



**Figure 2.16**

Solution:

Given $Q_0 = 500\ \mu\text{C}$ at $t < 0$

Recall $\quad Q_0 = C v_{c(0)}$

$$v_{c(0)} = \frac{Q_0}{C} = \frac{500 \times 10^{-6}}{20 \times 10^{-6}}$$

$$V_{c(0)} = -25\ \text{V}$$

At $\quad t = \infty$

$$v(t) = V_\infty + (V_0 - V_\infty)e^{-t/\tau}$$

$$\tau = 1 \times 10^3 \times 20 \times 10^{-6} = \frac{1}{50}$$

$$v(t) = 50 + (25 - 50)e^{-50t}$$

$$v(t) = 50 - 75e^{-50t}\ \text{V}$$

But $Q_\infty = C v_\infty$

$$Q_\infty = (20 \times 10^{-6})(50 - 75e^{-50t})$$

$$= 0.001 - 0.0015e^{-50t}\ \text{C}$$

$$Q_\infty = 1000 - 1500e^{-50t}\ \mu\text{C}$$

$$i_c = C\frac{dV}{dt} = 20 \times 10^{-6}\frac{d[50 - 75e^{-50t}]}{dt}$$

$$i_c = 20 \times 10^{-6} \times -75 \times -50e^{-50t}$$

$$i_c = 0.075e^{-50t} \text{ A}$$

$$= 70e^{-50t} \text{ mA}$$

$$i_c = 0.5 \times 10^{-6} \times 52.64 \times -4000e^{-4000t}$$

$$i_c = -0.10528e^{-4000t}$$

$$i_c = -105.28e^{-4000t} \text{ mA}$$

**Example 2.10**: Obtain the current $i(t)$, for all values of $t$, in the circuit shown in the Fig.2.17.

To obtain $i(t)$ below for all time

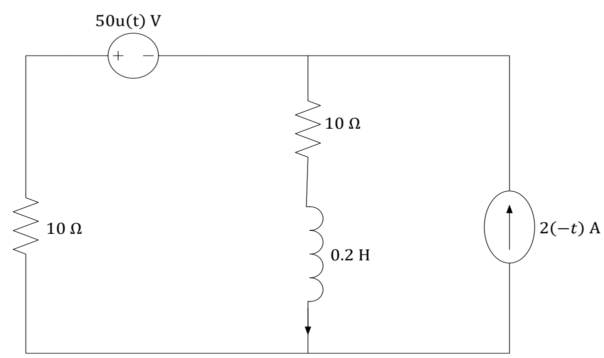



**Figure 2.17**

At $t < 0$

$$i_{(0)} = \frac{10}{10+10} \times 2 = \frac{20}{20} = 1 \text{ A}$$

$$i(0^+) = i(0^-) = 1 \text{ A}$$

At $t > 0$

$$-V = i\, R_{eq}$$

$$i_\infty = \frac{-V}{R_{eq}} = \frac{-50}{10+10} = \frac{-50}{20}$$

$$i_\infty = -2.5 \text{ A}$$

For time $t$

$$i(t) = i_\infty + (i_0 - i_\infty)e^{-t/\tau}$$

$$\tau = \frac{L}{R_{eq}} = \frac{0.2}{10+10} = \frac{0.2}{20} = \frac{1}{100}$$

$$i(t) = -2.5 + (1+2.5)e^{-100t}$$

$$i(t) = -2.5 + 3.5e^{-100t} \text{ A}$$

## 2.2 Exercise

**1.** For Fig. 1: (a) Determine the time constant of the network

(b) With C changed to L, determine the time constant

(c) With C again changed to 2L, determine $\tau$

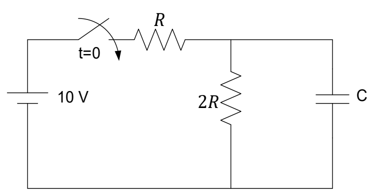



**Figure 1**

**2.** For the circuit of Fig. 2, given that $\text{V} = 10\text{ V}$ determine $\frac{dv}{dt}$ at the given instant in time.

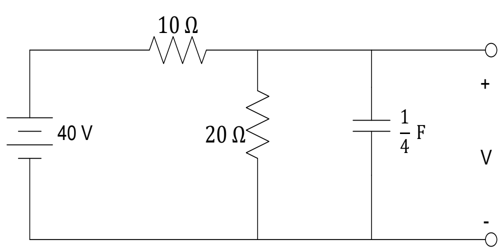



**Figure 2**

**3.** For the circuit shown in Fig. 3, determine $v(t)$ when the switch has been opened for a long time and then suddenly closed at t = $0^+$.

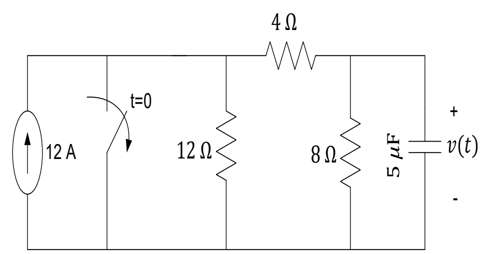



**Figure 3**

**4.** The circuit of Fig. 4 was under steady state before the switch was opened. If $R_1 = 1\ \Omega, \text{R}_2 = 2\ \Omega, \text{C} = 0.167\text{ F}$, and the battery voltage is 24 V, determine $v_c(0^-)$ and $v_c(0^+)$. Also find $i(0^+)$

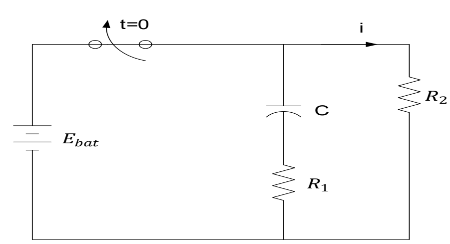



**Figure 4**

**Answer:** $\boldsymbol{i(0^+) = -8\ \mathrm{A}}$

**5.** Determine $i$ of **Exercise (4)** $1s$ after the switch is opened.

**Answer:** $\boldsymbol{i = -1.08\ \mathrm{A}}$

**6.** The initial current in the inductor L of the circuit of Fig. 5 with $S$ open $I_0$. Determine the current after $S$ is closed

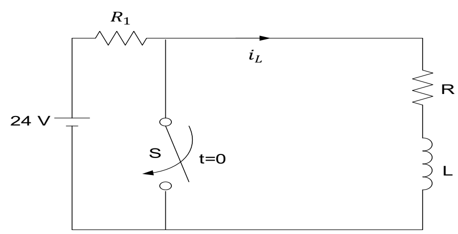



**Figure 5**

**Answer:** $\boldsymbol{i_L = I_o e^{-(R/L)t}}$

**7.** In the circuit of Fig. 6, the switch is closed at $t = 0$ when the 2 H inductor has a current $I_o = 10$ A. Find the voltage across the resistor.

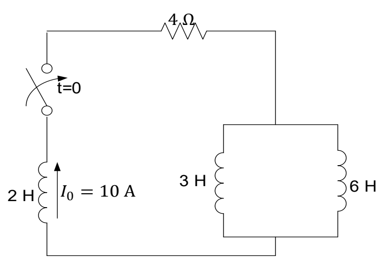



**Figure 6**

**Answer:** $\boldsymbol{V_R(t) = 40e^{-t}\ \mathrm{V}}$

**8.** A 240 V dc generator supplies current to a parallel circuit consisting of a resistor and a coil as shown in Fig.7. The system is in a steady state. Determine the current in the coil one second after the breaker is tripped.

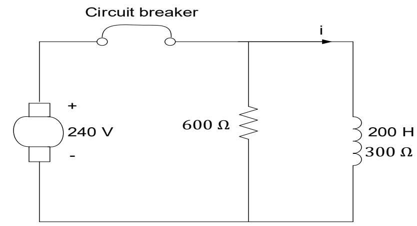



**Figure 7**

**Answer:** $\boldsymbol{i = 0.0089\ \mathrm{A}}$

**9.** What is the voltage induced in the coil and the voltage across the coil $1s$ after the breaker is tripped in the circuit of Fig.7

**Answer:** $\boldsymbol{v_{coil} = -5.34\ \mathrm{V}}$

**10.** The circuit of Fig 8 is under steady state with the switch at position 1. At $t = 0$, the switch is moved to position 2. Find $i$

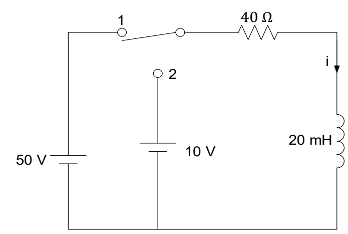



**Figure 8**

**Answer:** $\boldsymbol{V_c = -50 + 150e^{-200t}\ \mathrm{V}}$

**11.** Find $t > 0$ in the circuit of Fig. 9. Assume the switch has been open for a long time and is closed at $t = 0$. Calculate $v(t)$ at $t = 0.5$ s.

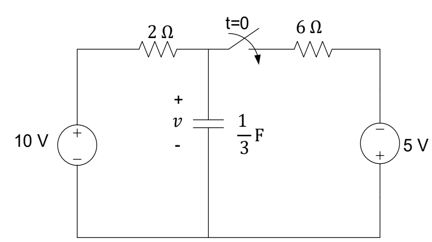



**Figure 9**

# CHAPTER 3

# TRANSIENT IN *R-L-C* SOURCE-FREE CIRCUIT

## 3.0 Introduction

In the previous chapter we considered circuits with a single storage element (a capacitor or an inductor). Such circuits are first-order because the differential equations describing them are first-order. In this chapter we will consider circuits containing two energy-storage elements. These are known as second-order circuits because their responses are described by differential equations that contain second derivatives.

Typical examples of second-order circuits are *RLC* circuits in which two or more passive elements are present. Examples of such circuits are shown in Figs. 3.1 and 3.5

## 3.1 Source-Free (Parallel) *R-L-C* Circuit

Areas of application that have to do with the understanding of the natural behavior of the parallel R-L-C circuit include filter designs, communication networks etc.

Consider the parallel *RLC* circuit shown in Fig 3.1. Assume initial inductor current $I_0$ and initial capacitor voltage $V_0$:

$$i(0) = I_0 = \frac{1}{L}\int_{-\infty}^{0} v(t)dt \qquad 3.0a$$

$$v(0) = V_0 \qquad 3.0b$$

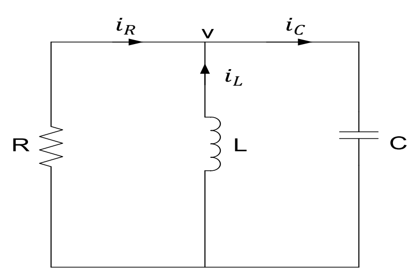



**Figure 3.1**

For the parallel-connected circuit shown above, an assumption can be made that the resistor is used to “practicalize” the “ideal” capacitor*. [*Recall: A practical capacitor is modelled as an ideal (i.e., zero internal resistance) capacitor in parallel with some resistance, and a practical inductor is, on the other hand, modelled as an ideal inductor in series with some (internal) resistance]. Therefore, the implication of the circuit in Fig. 3.1 is that a practical capacitor is in parallel with an ideal inductor. Obviously, it would be best to determine the voltage as the response since this is common to all three network elements, and thereafter, if need be, each of the branch currents can then be evaluated. Were we to set up any of the branch currents as the object for determination, the intermediate step would still involve determining the voltage, which then exposes the redundancy of the initial step!

To obtain the integral-differential equation that describes the response of the above network (either current or voltage), we take a single KCL involving nodal equation:

$$\frac{v}{R} + \frac{1}{L}\int_{t_o}^{t} v\,dt - i(t_o) + C\frac{dv}{dt} = 0 \qquad 3.1$$

With the assumed direction of the current at time $t_o$ shown. Differentiating Eq. 3.1 across to clear the integral term:

$$\frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v + C\frac{d^2v}{dt^2} = 0 \qquad 3.2$$

with $i(t_o)$ “zapped” out, being a constant term.

Rearranging Eq. 3.2,

$$C\,\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0 \qquad 3.3$$

The characteristic eq. 3.3 is

$$Cs^2 + \left(\frac{1}{R}\right)s + \left(\frac{1}{L}\right) = 0 \qquad 3.4$$

Applying the “almighty formula” for $ax^2 + bx + c = 0$ to determine the roots of $x$ (in our cases, representing complex frequency $s = (\alpha + j\omega)$

$$x = \frac{-b\ \pm\sqrt{b^2 - 4ac}}{2a}\begin{Bmatrix} a \Rightarrow C \\ b \Rightarrow \frac{1}{R} \\ c \Rightarrow \frac{1}{L} \end{Bmatrix} \qquad 3.5$$

So,

$$s_1, s_2 = \frac{-\frac{1}{R} \pm \sqrt{\left(\frac{1}{R}\right)^2 - \frac{4C}{L}}}{2C} \qquad 3.6a$$

Rearranging,

$$s_1, s_2 = \frac{-1}{2RC} \pm \sqrt{\frac{\left(\frac{1}{R}\right)^2 - \frac{4C}{L}}{(2C)^2}} \qquad 3.6b$$

$$s_1, s_2 = \frac{-1}{2RC} \pm \sqrt{\frac{L - 4CR^2}{(2CR)^2 L}} \qquad 3.6c$$

$$\Rightarrow \quad \boxed{s_1, s_2 = \frac{-1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}} \qquad 3.7$$

for a ***parallel*** connection.

Keeping faith on expression of the response, for example voltage in the case above, we write the response:

$$v(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \qquad 3.8$$

Where;

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \qquad 3.9a$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \qquad 3.9b$$

Each of the terms for $v(t)$ in Eq. 3.8 is a solution, therefore, their linear combination is also a solution.

For simplicity and ease of reference, let's designate the expressions in Eqs. 3.9a and 3.9b involving the network parameters *R*, *L*and *C* with terms:

$$\frac{1}{2RC} = \alpha \qquad 3.10a$$

where $\alpha$(Greek letter alpha) is the neper frequency, also called exponential damping coefficient

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad 3.10b$$

where $\omega_0$ (omega zero) is the resonant frequency already met in an earlier course. (Complex frequency $s = \alpha + j\omega$)

So, $$s_1 = -\alpha + \sqrt{\alpha^2 - {\omega_0}^2};\ s_2 = -\alpha - \sqrt{\alpha^2 - {\omega_0}^2} \qquad 3.11a, b$$

With the expressions Eq. 3.11 for the two roots $s_1$ and $s_2$, three different possibilities exist of the types of responses depending on the relationship of $\alpha$ and $\omega_o$:

1. $\alpha > \omega_o \Longrightarrow$ two real and unequal roots, and the responding signal is called **over-damped** circuit signal;
2. $\alpha = \omega_0 \Longrightarrow$ two real and equal roots, leading to **critically damped** circuit signal;
3. $\alpha < \omega_0 \Longrightarrow$ two complex conjugates (and therefore, unequal) roots, leading to a signal called **under-damped** sinusoid.

### 3.1.1 For Over-damped Condition

**Example 3.1**: Assigning values to the circuit of Fig.3.1 to be $R = 4\ \Omega,\ L = 5\ \text{H},\ C = \frac{1}{20}\,\text{F}$, with initial current $i_L(0) = 5\ \text{A}$, and initial voltage $v(0) = 0$, $\alpha = \frac{1}{2RC} = \frac{20}{2\times 4} = 2.5$ $\omega_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{5}{20}}} = \sqrt{4} = 2\ \text{rad/s}$

$$s_1 = -\alpha + \sqrt{\alpha^2 - {\omega_0}^2} = -2.5 + \sqrt{2.5^2 - 2^2} = -2.5 + 1.5 = -1$$

$$s_2 = -2.5 - 1.5 = -4$$

$$v(t) = K_1 e^{-t} + K_2 e^{-4t}$$

$$v(0) = 0 = K_1 + K_2 \Longrightarrow K_1 = -K_2$$

$$\frac{dv}{dt} = -K_1 e^{-t} - 4K_2 e^{-4t}$$

as this, being a second-order differential equation, needs two initial conditions to compute the response. We make use of $i_c(t) = C\frac{dv_c(t)}{dt}$ and evaluate at $t = 0^+$:

$$C\left.\frac{dv_c(t)}{dt}\right|_{t=0^+} = i_c(0^+)$$

$$= C\frac{dv(t)}{dt}\bigg|_{t=0^+} = \left(\frac{1}{20}\right)(-K_1 - 4K_2)$$

$$i_c(0^+) = i_R(0^+) + i_L(0^+)$$

$$i_c(0^+) = \frac{v(0)}{4} + 5 = 0 + 5 = 5 \Longrightarrow -K_1 - 4K_2 = -K_1 + 4K_1$$

$$20i_c(0^+) = 20 \times 5 = 100 = 3K_1$$

$$K_1 = \frac{100}{3}, \qquad K_2 = -\frac{100}{3}$$

Finally, $$v(t) = \left(\frac{100}{3}\right)e^{-t} - \left(\frac{100}{3}\right)e^{-4t}\ \mathrm{V}$$

$$v(t) = \frac{100}{3}(e^{-t} - e^{-4t})\ \mathrm{V}$$

To sketch the response in Fig. 3.2, we identify the critical points, namely,

1. The initial value of the sum (i.e., difference, actually, of the two terms);
2. Final value of the signal (zero, since both are decaying exponential terms);
3. The time at which the maximum value occurs; and…
4. The maximum value itself. $v(0) = 0$ obviously, and $v(\infty) = 0$ as the two terms have both damped out.

For time at maximum:

$$\frac{dv(t)}{dt} = \frac{100}{3}(-e^{-t} + 4e^{-4t})$$

At maximum point,

$$\frac{dv(t)}{dt} = 0$$

$$\frac{100}{3}(-e^{-t_{max}} + 4e^{-4t_{max}}) = 0 \Longrightarrow -e^{-t_{max}} + 4e^{-4t_{max}} = 0$$

$$\Longrightarrow e^{-t_{max}} = 4e^{-4t_{max}} \Longrightarrow 1 = 4e^{-4t_{max}}/e^{-t_{max}}$$

$$1 = 4e^{-3t_{max}} \Longrightarrow e^{-3t_{max}} = \frac{1}{4}$$

$$\Rightarrow -3t_{max} = \text{In}\frac{1}{4}$$

$$3t_{max} = \text{In } 4$$

$$t_{max} = \frac{\ln 4}{3} = 0.46 \text{ s}$$

where $t_{max}$ stands for time at which maximum voltage response occurs.

$$v(0.46) = \frac{\frac{100}{3}}{\frac{1}{e^{0.46}} - \frac{1}{e^{4\times 0.46}}}$$

$$= (100/3)(0.63 - 0.16) = 15.67 \text{ V}$$

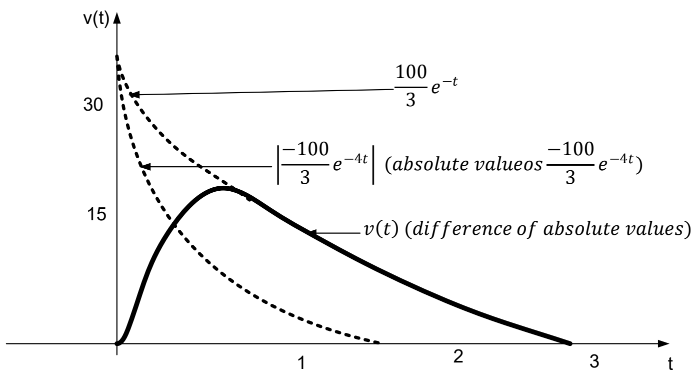



**Figure 3.2 The graph of Over-Damped Response**

### 3.1.2 Critically-Damped Response (Parallel Circuit)

For critical damping, the term inside the square root sign, $\alpha^2 - {\omega_0}^2$ is theoretically set equal to zero, making the two roots (value of $-\alpha$) equal to each other. We say, theoretically, because it's physically impossible to construct a parallel *R*-*L*-*C* circuit whereby the neper frequency $\alpha$ is exactly equal to the radian frequency $\omega_o$.

So, $$s_1 = s_2 = -\alpha = -\omega_0 \qquad 3.12$$

$$\Rightarrow \qquad \frac{1}{2RC} = \frac{1}{\sqrt{LC}}$$

$$\frac{1}{4R^2C^2} = \frac{1}{LC}$$

$$L = 4R^2C \qquad 3.13$$

**Example 3.2**: From **Example 3.1**, in order to make $L = 4R^2C$ (thereby effecting $\alpha = \omega_0$ for **critical damping**), we alter the values of $R$, leaving $L$ and $C$ (thus $\omega_o$) unchanged until $\alpha$ and $\omega_o$ become equal. Using the same initial conditions as in **Example 3.1**:

$$L = 4R^2C \Longrightarrow R = \frac{1}{2}\sqrt{\frac{L}{C}} = \frac{1}{2}\sqrt{\frac{5}{(\frac{1}{20})}} = \frac{1}{2}\sqrt{100} = 5\ \Omega$$

$$\text{Check: } \alpha = \frac{1}{2RC} = \left[\frac{1}{2} \times \frac{1}{5} \times \frac{20}{1}\right] = \frac{20}{2 \times 5} = 2$$

$$\omega_o = \frac{1}{\sqrt{LC}} == \sqrt{\frac{20}{5}} = \sqrt{4} = 2\ \text{rad/s}$$

Refer to Eqs. 3.2 and 3.3

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L} = 0 \qquad 3.14$$

$$\Longrightarrow \qquad \frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

$$\Longrightarrow \qquad \frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \alpha^2 v = 0 \qquad 3.15$$

where $\alpha = \omega_0 = \frac{1}{\sqrt{LC}}$

The general solution for critically damped case is:

$$v = e^{-\alpha t}(K_1 t + K_2) = e^{-2t}(K_1 t + K_2) \qquad 3.16$$

$$v(0) = 0 \Longrightarrow e^{o}[K_1(0) + K_2] = 0 \Longrightarrow K_2 = 0$$

So, $v(t)$ is simply: $\qquad v(t) = K_1 t e^{-2t}$

$$\frac{dv(t)}{dt} = -2K_1\, te^{-2t} + K_1 e^{-2t}$$

$$C\frac{dv(t)}{dt}\bigg|_{t=0} = \frac{1}{20}(K_1) = i_C(0^+)$$

$$\frac{K_1}{20} = i_C(0^+) = i_R(0^+) + i_L(0^+)$$

$$= \frac{v_R(0^+)}{4} + 5 = 0 + 5 = 5$$

$K_1 = 20 \times 5 = 100$, with the same initial conditions imposed on $v, i_L$.

Finally, $v(t) = 100te^{-2t}$ V

To sketch the response

$\frac{dv(t)}{dt} = 100\,(-2te^{-2t} + e^{-2t}) = 0$ at maximum value.

$$t_{max} = \frac{e^{-2t}}{2e^{-2t}} = 0.5 \text{ s}$$

$$v(0.5) = 100 \times \frac{0.5}{e^{2\times0.5}} = \frac{50}{e^1} = 18.39 \text{ V}$$

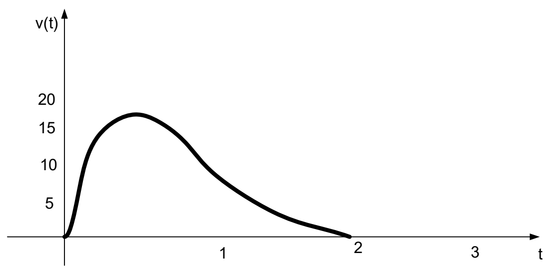



**Figure 3.3 Graph of Critically-Damped Response**

### 3.1.3 Under-damped Parallel Circuit

Here, $\alpha < \omega_o$, making the term inside the square root sign negative, signing rise to complex roots for $s_1, s_2$:

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - {\omega_o}^2} = -\alpha \pm \sqrt{(-1)(\omega_o - \alpha^2)}$$

$$= -\alpha \pm \sqrt{-1}\sqrt{{\omega_o}^2 - \alpha^2} = -\alpha \pm j\omega_d \qquad 3.17$$

Where $\omega_d = \sqrt{{\omega_o}^2 - \alpha^2}$ is called the ***natural resonant frequency***.

$$s_1 = -\alpha + j\omega_d;\ s_2 = -\alpha - j\omega_d \qquad 3.18\text{a, b}$$

[The roots are complex conjugates; strictly note that $\omega_d$, the square root term, is itself real, as is always the case with the *imaginary* part of any complex quantity!]

The response can be written in the same manner as for the over-damped case, since the roots here are *different* (even if conjugate):

$$v(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

$$v(t) = K_1 e^{(-\alpha + k\omega_d)t} + K_2 e^{(-\alpha - j\omega_d)t}$$

$$v(t) = e^{-\alpha t}\left(K_1 e^{j\omega_d t} + K_2 e^{-j\omega_d t}\right)$$

$\left(\text{Euler's identity: } e^{ja} = \cos a + j \sin a\right)$

$$v(t) = e^{-\alpha t}[K_1(\cos \omega_d t + j \sin \omega_d t) + K_2(\cos \omega_d t - j \sin \omega_d t)]$$

$$v(t) = e^{-\alpha t}[(K_1 + K_2)\cos \omega_d t + (K_1 - K_2) j \sin \omega_d t] \qquad 3.19a$$

$$v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t) \qquad 3.19b$$

Where $A_1 = K_1 + K_2,\ A_2 = j(K_1 - K_2)$, replace the former constants in Eq. 3.19a.

**Question**: Why is there no complex term in the final expression for $v(t)$ in Eq. 3.19b? **Explanation**: $s_1,\ s_2;\ K_1$ and $K_2$ are generally all complex quantities, and it's entirely possible to obtain real numbers from the *addition* (or multiplication) of complex numbers. In general, $s_1$ and $s_2$; $K_1$ and $K_2$ are respectively one another's complex conjugate; and addition of complex conjugates results in a real number (twice their real parts). (For actual physical system, this can be viewed as being "self-practicalized"!) As in the previous case, the two constants $A_1$ and $A_2$ are evaluated from initial (and maybe also, final) conditions.

**Example 3.3**: Assign the values of $L = 5$ H, $C = \frac{1}{20}$F to the circuit of Fig.3.1, but this time around force $\alpha$ to become less than $\omega_o(\alpha < \omega_o)$ by increasing the value of $R$ to, say 6 Ω.

$$\Rightarrow \alpha = \frac{1}{2RC} = \frac{20}{2 \times 6} = \frac{5}{3}$$

$$\Rightarrow \quad \omega_d = \sqrt{{\omega_o}^2 - \alpha^2}$$

$$= \sqrt{2^2 - 1.667^2} = 1.105 \text{ rad/s}$$

$$v(t) = e^{-1.667t}(A_1 \cos 1.105t + A_2 \sin 1.105t)$$

With the same initial conditions as in **Example 3.1:**

$$v(0) = e^0(A_1 \cos 0 + A_2 \sin 0) = A_1 = 0$$

So, $v(t)$ is simply: $v(t) = A_2 e^{-1.667t} \sin 1.105t$

$$\frac{dv(t)}{dt} = A_2(1.105e^{-1.667t} \cos 1.105t - 1.667e^{-1.667t} \sin 1.105t)$$

$$\left(\frac{1}{20}\right)\frac{dv(t)}{dt}\bigg|_{t=0} = i_C(0^+) = i_R(0^+) + i_L(0^+) = 0 + 5 = 5 \text{ V/s}$$

$$\Longrightarrow \quad (\frac{1}{20})A_2(1.105e^0 \cos 0 - 1.667e^0 \sin 0) = (\frac{1}{20})(1.105A_2) = 5$$

$$A_2 = \frac{20 \times 5}{1.105} = 90.50$$

Finally, $v(t) = 90.50e^{-1.667t} \sin 1.105t \text{ V}$

Notice that the above expression for $v(t)$ involves both a damped part $(e^{-1.667t})$ and asinusoidal factor $(\sin 1.105t)$, resulting in a **damped sinusoid**.

Sketching under-damped signal (see Fig.3.4):
The sinusoidal factor, considered alone,

starts at zero for $t = 0$; is zero again for $t = \frac{n\pi}{1.105}$, where $n$ is any integer;

becomes maxima$(+)$ $at\ t = \frac{n\pi}{2 \times 1.105}, n = 3,7,11, \ldots$

Setting $\frac{dv(t)}{dt} = 0$, and after the requisite manipulation:

$$\tan 1.105t = 1.105/1.667$$

$$\Longrightarrow \text{maximum occurs at} t = \frac{\tan^{-1}\left(\frac{1.105}{1.667}\right)}{1.105} = 0.530 \text{ s}$$

Maxima: $t = 0.530 + n\pi, n$ even   Minima: $t = 0.53 + n\pi, n$ odd

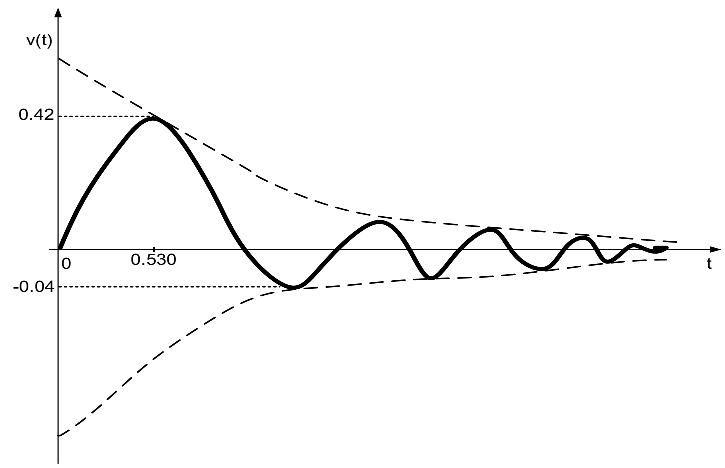



**Figure 3.4 Underdamped Response Curve**

## 3.2 Source Free *R-L-C* Series Circuits

An understanding of the natural response of the series *RLC* circuit is a necessary background for future studies in filter design and communications networks.

Consider the series *RLC* circuit shown in Fig. 3.5. The circuit is being excited by the energy initially stored in the capacitor and inductor. The energy is represented by the initial capacitor voltage $V_0$ and initial inductor current $I_0$. Thus, at $t = 0$,

$$v(0) = \frac{1}{C}\int_{-\infty}^{0} i\, dt = V_0 \qquad 3.20$$

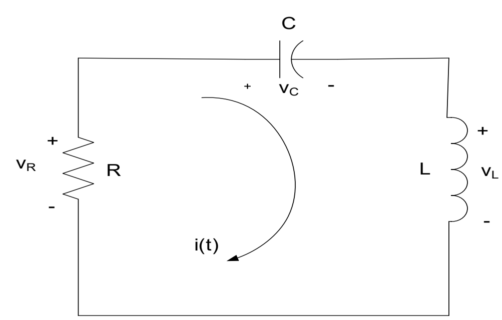



**Figure 3.5**

A single KVL gives:

$$L\frac{di(t)}{dt} + i(t)R + \frac{1}{C}\int_{t_o}^{t} idt - v_c(t) = 0 \qquad 3.21$$

Eq. 3.21 could have been derived directly by taking the *dual* of the expression for the parallel-connected circuit we dealt with previously:

$$C\frac{dv}{dt}+\frac{1}{R}V+\frac{1}{L}\int_{t_o}^{t}v\,dt-i_L(t)=0 \Leftrightarrow L\frac{di}{dt}+Ri+\frac{1}{C}\int_{t_o}^{t}i\,dt-v_c\,(t_o)=0 \quad 3.22$$

$$s_1,s_2=-\frac{1}{2RC}\pm\sqrt{\left(\frac{1}{2RC}\right)^2-\frac{1}{LC}} \Leftrightarrow -\frac{R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{CL}} \quad 3.23$$

Thus, for the above *series* connection:

$$\alpha=\frac{R}{2L};\omega_o=\frac{1}{\sqrt{CL}}=\frac{1}{\sqrt{LC}}(\text{same as for parallel connection})$$

$$i(t)=K_1e^{s_1t}+K_2e^{s_2t}\text{(over-damped)}$$

$$i(t)=K_1te^{s_1t}+K_2e^{s_2t}\text{ (Critically-damped)}$$

$$i(t)=e^{-\alpha t}(A_1\cos\omega_d t+A_2\sin\omega_d t)\text{(under-damped)}$$

with $\omega_d=\sqrt{\omega_0{}^2-\alpha^2}$, as in the case for parallel connection

**Example 3.4:** For the circuit of Fig.3.5, assigning the values for the network:

$R=4\text{ M}\Omega;\text{L}=2\text{ H};\text{C}=\frac{1}{200}\ \mu\text{F};i(0)=4\text{ mA};\ v_c(0)=4\text{ V}$, determine and sketch $i(t)$ for $t>0$.

$$\omega_0=\frac{1}{\sqrt{LC}}=\frac{1}{\sqrt{\left(\frac{2}{200}\right)\times10^{-6}}}=\sqrt{10^8}=10^4\text{ rad/s}$$

$$\alpha=\frac{R}{2L}=\frac{4\times10^6}{2\times2}=10^6$$

$$\omega_d=\sqrt{\omega_0{}^2-\alpha^2}=\sqrt{10^8-10^6}=\sqrt{99\times10^6}=10^3\sqrt{99}\text{ rad/s}$$

$$i(t)=e^{-1000t}\left(A_1\cos10^3\sqrt{99}t+A_2\sin10^3\sqrt{99}t\right)$$

$$i(0)=0.004=A_1$$

$$\frac{di(t)}{dt} = e^{-1000t}\left(-10^3\sqrt{99}A_1 \sin 10^3\sqrt{99}t + 10^3\sqrt{99}A_2 \cos 10^3\sqrt{99}\right) - 1000e^{-1000t}\left(A_1 \cos 10^3\sqrt{99}t + A_2 \sin 10^3\sqrt{99}t\right)$$

$$L\left.\frac{di(t)}{dt}\right|_{t=0} = v_L(0) = 2\left(10^3\sqrt{99}A_2 - 1000A_1\right) = i_R(0)R - v_c(0)$$

$$2\left(10^3\sqrt{99}A_2 - 1000 \times 4 \times 10^{-3}\right) = -4 \times 10^{-3} \times 4 \times 10^6 - 4$$

$$2\left(10^3\sqrt{99}A_2 - 4\right) = -16000 - 4 = -16004$$

$$\Longrightarrow \; A_2 = \frac{\left[\left(-\frac{16004}{2}\right)\right]}{\left(10^3\sqrt{99}\right)} \approx 0.804$$

$$i(t) = e^{-1000t}\left(0.004 \cos 10^3\sqrt{99}\, t - 0.804 \sin 10^3\sqrt{99}\, t\right) \text{ A}$$

**The overdamped, critically damped and underdamped response curves are the same as the parallel case with voltage and current duality**

**Example 3.5:** The current in a circuit is given by a second order equation:

$$\frac{d^2i}{dt^2} + 3\frac{di}{dt} + 2i = 0$$

With initial conditions $i(0^+) = 2$ A, $\frac{di}{dt}(0^+) = 1$ A/s, determine the time $t$ current $i(t)$ takes to reach the maximum value.

Solution:

$$\frac{d^2i}{dt^2} + 3\frac{di}{dt} + 2i = 0$$

Characteristic equation is

$$p^2 + 3p + 2 = 0$$

Where the roots are

$p_1 = -1, p_2 = -2$ [overdamped]

Recall the current response for overdamped response

$$i(t) = A_1e^{s_1t} + A_2e^{s_2t}$$

$$i(t) = A_1e^{-t} + A_2e^{-2t} \qquad \text{(i)}$$

At $\quad t = 0$

$$i(0) = A_1e^{-(0)} + A_2e^{-2(0)}$$

$$2 = A_1 + A_2$$

$\therefore \qquad A_1 + A_2 = 2 \qquad$ (ii)

Taking the derivative of $i(t)$ in equation (i)

$$\frac{di(t)}{dt} = -A_1e^{-t} - 2A_2e^{-2t}$$

$$\left.\frac{di}{dt}\right|_{t=0} = -A_1e^{-(0)} - 2A_2e^{-2(0)}$$

$$1 = -A_1 - 2A_2$$

$\therefore \qquad A_1 + 2A_2 = -1 \qquad$ (iii)

Recall equation (ii) and equation (iii)

$$A_1 + A_2 = 2$$

$$A_1 + 2A_2 = -1$$

Solving (ii) & (iii) simultaneously for the value of $A_1$ and $A_2$

$$A_1 = 5, \qquad A_2 = -3$$

$\therefore \qquad i(t) = 5e^{-t} - 3e^{-2t}A$

For $i_{max}$, $\quad \frac{di(t)}{dt} = 0$

$$\frac{di(t)}{dt} = -5e^{-t} + 6e^{-2t} = 0$$

$$6e^{-2t} = 5e^{-t}$$

$$6e^{-2t+t} = 5$$

$$6e^{-t} = 5$$

$$e^{-t} = \frac{5}{6}$$

∴ $$-t = \ln\left[\frac{5}{6}\right]$$

$$-t = -0.1823$$

∴ $$t = 0.1823 \text{ s} = 182.3 \text{ ms}$$

**Example 3.6**: The circuit of Fig below is under steady state and the switch is open at $t = 0$. Find the frequency of the current $i_L$. Also, determine its magnitude.

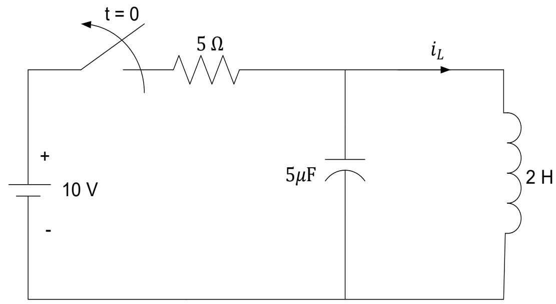



**Figure 3.6**

Solution:

(a) To determine the frequency $f$ and $i_L$

At $t(0^+)$

$$i_L(0^+) = i_L(0^-) = \frac{10}{5} = 2 \text{ A}$$

$$\frac{1}{5 \times 10^{-6}} \int_0^t v\, dt + \frac{2di_L}{dt} = 0 \qquad \textbf{KVL}$$

$$200000 \int V dt + \frac{2di_L}{dt} = 0$$

$$\frac{2d^2 i_L}{dt^2} + 200000 i_L = 0$$

$$2p^2 + 200000 = 0$$

$$p^2 - 100000 = 0$$

Where $i_L(0^+) = \dfrac{10}{5} = 2 \text{ A}$; $\quad \dfrac{di(0^+)}{dt} \leq 0 \text{ A/s}$

$p = \sqrt{-100000}$ **(underdamped)**

$$p = -j316.227$$

$\therefore \quad \omega_d = 316.227 \text{ rad/s}$

Recall $i$ for underdamped response

$$i(t) = e^{\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

Where $\alpha = 0$

$$i(t) = B_1 \cos(316.23t) + B_2 \sin(316.23t)$$

But $\omega_d = 2\pi \mathrm{f_d}$

$$\mathrm{f_d} = \frac{\omega_d}{2\pi} = \frac{316.223}{2 \times 3.142} = 50.3 \text{ Hz}$$

$$\mathrm{f_d} = 50.3 \text{ Hz}$$

$$i_L(0^+) = 2 \therefore B_1 = 2 \; and \; B_2 = 0$$

Hence the circuit instantaneous current is $i(t) = 2 \cos 316.23t$

Where amplitude is $i_L = 2 \text{ A}$

## 3.3 Exercise

**1.** (i) Determine the natural response $v(t)$ for the circuit of Fig. 1. (ii) at what time does the voltage achieve maximum value? (iii) determine this maximum value. Given that $\left(\frac{dv}{dt} = 4 \text{ V/s}\right)$.

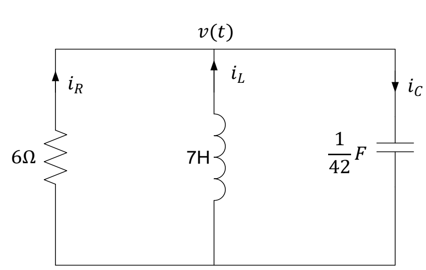



**Figure 1**

3. (i) Determine the natural response $i(t)$ for the circuit Fig. 2. (ii) at what time does the current achieve maximum value? (iii) determine this maximum value. Given that $\left(\frac{di}{dt} = 5 \text{ A/s}\right)$

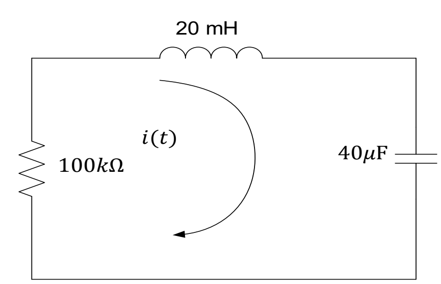



**Figure 2**

**3.** For a parallel RLC circuit with inductance of 100 mH and capacitance of 40 $\mu$F, determine, resistor values that would lead to

(i) over-damped

(ii) under-damped

(iii) critically-damped responses respectively.

**4.** For a parallel RLC circuit containing 50 $\Omega$ resistor with parametric values of $\alpha = 500\ \text{s}^{-1}$ and $\omega_0 = 400\ \text{rad/s}$ determine: $(i)\ C\ (ii)\ L\ (iii)\ s_1\ (iv)\ s_2$

**5.** For a parallel RLC circuit with inductance of 200 mH and capacitance of $800\ \mu\text{F}$ what are the resistor values that would lead to

(i) over-damped

(ii) under-damped

(iii) critically-damped responses

**6.** Given a parallel RLC circuit containing 25 $\Omega$ resistor with parametric values of $\alpha = 250\ \text{s}^{-1}$ and $\omega_0 = 400\ \text{rad/s}$, determine $(i)\ C\ (ii)\ L\ (iii)\ s_1\ (iv)\ s_2$

**7.** For a parallel RLC circuit with inductance of 50 mH and capacitance of $500\ \mu\text{F}$ what are the resistor values that would lead to (i) over-damped (ii) underdamped (iii) critically-damped responses respectively

**8.** For a parallel RLC circuit containing $100\ \Omega$ resistor with parametric values of $\alpha = 1000\ \text{s}^{-1}$ and $\omega_0 = 800\ \text{rad/s}$, determine $(i)\ C\ \ (ii)\ \ L\ (iii)\ s_1\ (iv)\ s_2$

**9.** In the circuit of Fig.3 the parameter of coil and coil 2 are respectively, $1.5\ \text{H}$ and $8\ \Omega$ and $0.5\ \text{H}$ and $4\ \Omega$. If $C = \frac{1}{18}\ \text{F}$ and it is charged to $100\ \text{V}$, determine the current $0.2\ \text{s}$ after the switch is closed.

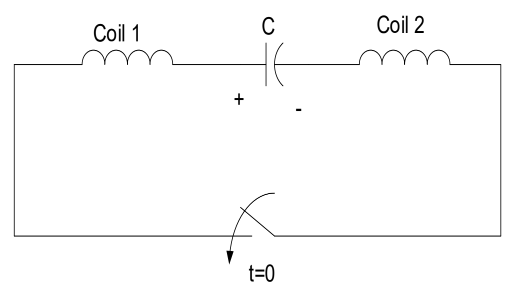



**Figure 3**

**Answer:** $\boldsymbol{i = -5.49\ \text{A}}$

**10.** What is the voltage across coil 2 in the circuit of Fig. 3 at $t = 0.2\ \text{s}$?

**Answer:** $\boldsymbol{v = -27.44\ \text{V}}$

**11.** If the switch in Fig.4 closed at $t = 0$, find $v(t)$ for $t \geq 0$ and $w_C(0)$.

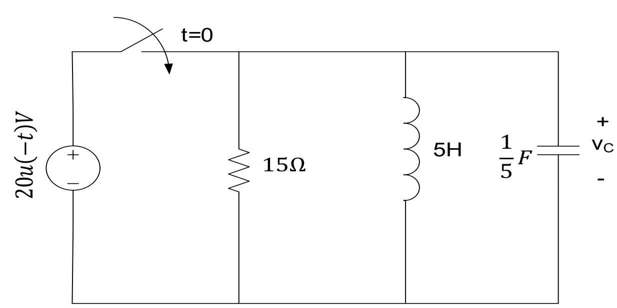



**Figure 4**

**12**. In the circuit of Fig.5, the switch is moved from position 1 to 2 at $t = 0$. Determine $\frac{d^2 i(0^+)}{dt^2}$

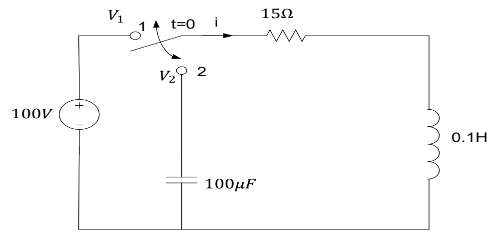



**Figure 5**

**Answer:** $\frac{d^2 i(0^+)}{dt^2} = \mathbf{10^8 A/s^2}$

# CHAPTER 4

# TRANSIENT IN R-L-C CIRCUIT DRIVEN BY A FORCE RESPONSE

## 4.0 R-L-C Circuit with Force Response

As we learned in the preceding chapter, the step response is obtained by the sudden application of a dc source.

### 4.0.1 Step Response of a Series R-L-C Circuit

Consider the series RLC circuit shown in Fig. 4.1. Applying KVL around the loop for $t > 0$,

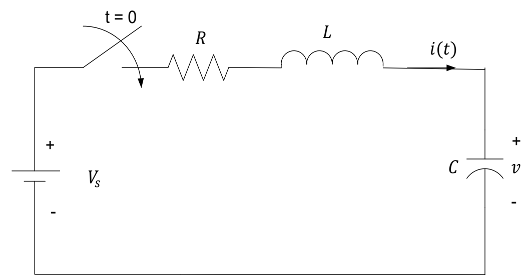



**Figure 4.1 Step response applied to a series RLC circuit**

$$L\frac{di}{dt} + Ri + v = v_s \qquad 4.1$$

But

$$i = C\frac{dv}{dt}$$

Substituting for $i$ in Eq (4.1) and rearranging terms

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC} \qquad 4.2$$

which has the same form as Eq. (3.2). More specifically, the coefficients are the same (and that is important in determining the frequency parameters) but the variable is different. (Likewise, see Eq. (4.9).) Hence, the characteristic equation for the series RLC circuit is not affected by the presence of the dc source.

The solution to Eq. (4.2) has two components: the transient response $v_t(t)$ and the steady-state response $v_{ss}(t)$; that is,

$$v(t) = v_t(t) + v_{ss}(t) \quad 4.3$$

The transient response $v_t(t)$ is the component of the total response that dies out with time. The form of the transient response is the same as the form of the solution obtained in Section 3 for the source-free circuit. Therefore, the transient response $v_t(t)$ for the overdamped, underdamped, and critically damped cases are:

$$v_t(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \text{ (overdamped)} \quad 4.4a$$
$$v_t(t) = (A_1 + A_2 t)e^{-\alpha t} \text{ (Critically damped)} \quad 4.4b$$
$$v_t(t) = (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \text{ (Under damped)} \quad 4.4c$$

The steady-state response is the final value of $v(t)$. In the circuit in Fig. 4.1, the final value of the capacitor voltage is the same as the source voltage $V_s$. Hence,

$$v_{ss}(t) = v(\infty) = V_s \quad 4.5$$

Thus, the complete solutions for the overdamped, underdamped, and critically damped cases are:

$$v(t) = V_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \text{ (overdamped)} \quad 4.6a$$
$$v(t) = V_s + (A_1 + A_2 t)e^{-\alpha t} \text{ (Critically damped)} \quad 4.6b$$
$$v(t) = V_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \text{ (Under damped)} \quad 4.6c$$

The values of the constants $A_1$ and $A_2$ are obtained from the initial conditions: $v(0)$ and $\frac{dv(0)}{dt}$. Keep in mind that $v$ and $i$ are, respectively, the voltage across the capacitor and the current through the inductor. Therefore, Eq. (4.6) only applies for finding $v$. But once the capacitor voltage $v_c = v$ is known, we can determine $i_C = C\frac{dv}{dt}$, which is the same current through the capacitor, inductor, and resistor. Hence, the voltage across the resistor is $v_R = iR$, while the inductor voltage is $v_L = L\frac{di}{dt}$.

Alternatively, the complete response for any variable $x(t)$ can be found directly, because it has the general form

$$x(t) = x_{ss}(t) + x_t(t) \quad 4.7$$

where the $x_{ss} = x(\infty)$ is the final value and $x_t(t)$ is the transient response. The final value is found as in **Section 3.** The transient response has the same form as in Eq. (4.4), and the associated constants are determined from Eq. (4.6) based on the values of $x(0)$ and $\frac{dx(0)}{dt}$.

For a d.c. excitation, the forced response,

$v_f(t) = v_f$, say for a voltage signal

Natural response:

$$v_n(t) = K_1 e^{s_2 t}$$

Complete response $v(t) = v_f(t) + v_n(t)$

$$= V_f + K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

With the unknown $K_1$ and $K_2$ yet to be determined from initial conditions.

The same basic procedure used for R-L and R-C driven circuit, also obtains here the only difference being that, whereas only one energy storage element was involved in the former case, meaning that only one initial condition was required, this time around, two different initial conditions would be required since it now contain two energy-storage elements.

### 4.0.2 Step Response of a Parallel R-L-C Circuit

Consider the parallel RLC circuit shown in Fig. 4.2. We want to find i due to a sudden application of a dc current. Applying KCL at the top node for $t > 0$,

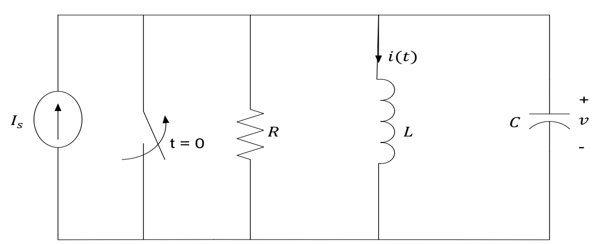



**Figure 4.2 Parallel RLC circuit with an applied current**

$$\frac{v}{R} + i + C\frac{dv}{dt} = I_s \qquad 4.8$$

But

$$v = L\frac{di}{dt}$$

substituting for $v$ in Eq. (4.8) and dividing by LC, we get

$$\frac{d^2i}{dt^2}+\frac{1}{RC}\frac{di}{dt}+\frac{i}{LC}=\frac{I_s}{LC} \qquad 4.9$$

has the same characteristic equation as Eq. (3.4).

The complete solution to Eq. (4.9) consists of the transient response $i_t(t)$ and the steady-state response $i_{ss}$, that is,

$$i(t)=i_t(t)+i_{ss}(t) \qquad 4.10$$

The transient response is the same as what we had in Section 3. The steady-state response is the final value of $i$. In the circuit in Fig. 4.2, the final value of the current through the inductor is the same as the source current $I_s$. Thus

$$\begin{gathered} i(t)=I_s+A_1e^{s_1t}+A_2e^{s_2t}(\text{overdamped}) \\ i(t)=I_s+(A_1+A_2t)e^{-\alpha t}(\text{Critically damped}) \\ i(t)=I_s+(A_1\cos\omega_d t+A_2\sin\omega_d t)e^{-\alpha t}\ (\text{underdamped}) \end{gathered} \qquad 4.11$$

The constants $A_1$ and $A_2$ in each case can be determined from the initial conditions for $i$ and $di/dt$. Again, we should keep in mind that Eq. (4.11) only applies for finding the inductor current $i$. But once the inductor current $i_L=i$ is known, we can find $v=L\frac{di}{dt}$ which is the same voltage across the inductor, capacitor, and resistor. Hence, the current through the resistor is $i_R=\frac{v}{R}$, while the capacitor current is $i_C=C\frac{dv}{dt}$. Alternatively, the complete response for any variable $x(t)$ may be found directly, using

$$x(t)=x_{ss}(t)+x_t(t) \qquad 4.12$$

where $x_{ss}$ and $x_t$ are its final value and transient response, respectively.

## 4.1 General Second Order Circuit

Now that we have mastered series and parallel RLC circuits, we are prepared to apply the ideas to any second-order circuit having one or more independent sources with constant values. Although the series parallel RLC circuits are the second-order circuits of greatest interest, other second-order circuits including op amps are also useful. Given a second-order circuit, we determine its step response $x(t)$ (which may be voltage or current) by taking the following four steps:

1. We first determine the initial conditions $x(0)$ and $\frac{dx(0)}{dt}$ and the final value $x(\infty)$, as discussed in Section 2.

2. We turn off the independent sources and find the form of the transient response $x_t(t)$ by applying KCL and KVL. Once a second-order differential equation is obtained, we determine its characteristic roots. Depending on whether the response is overdamped, critically damped, or underdamped, we obtain $x_t(t)$ with two unknown constants as we did in the previous sections.
3. We obtain the steady-state response as

$$x_{ss}(t) = x(\infty) \qquad 4.13$$

where $x(\infty)$ is the final value of $x$, obtained in step 1.

4. The total response is now found as the sum of the transient response and steady-state response

We finally determine the constants associated with the transient response by imposing the initial conditions $x(0)$ and $\frac{dx(0)}{dt}$, determined in step 1.

We can apply this general procedure to find the step response of any second-order circuit, including those with op amps. The following examples illustrate the four steps.

**Example 4.1**: For the circuit shown in Fig.4.3, determine:

1. $i_L(0^-)$
2. $v_c(0^+)$
3. $v_R(0^+)$
4. $i_L(\infty)$
5. $i$ (0.2 ms), given $i_s = 10u(-t) - 20u(t)$ A

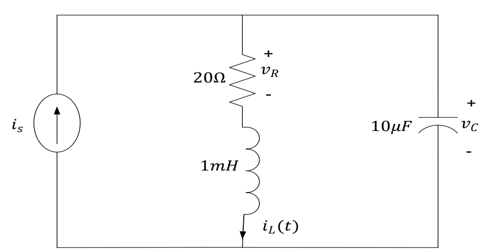



**Figure 4.3**

Solution:

For $t = 0^+$, $i_s = 10u(-t) - 20u(t)\ \mathrm{A} = 10 - 0 = 10\ \mathrm{A}$

The inductor and capacitor are like short and open circuit, respectively, as “seen” by the d.c source

(1) $i_L(0^+) = i_L(0^-) = i_s = 10\text{ A}$

(2) $v_c(0^+) = v_c(0^-) = v_R(0^-) = i_s \times 20\Omega = 10\text{ A } \times 20\ \Omega = 200\text{ V}$

(3) $v_R(0^+) = 20\ \Omega \times i_L(0^+) = 20\ \Omega \times i_L(0^-) = 20 \times 10 = 200\text{ V}$

(4) $i_L(\infty) = i_s(\infty) = 0 - 20 = -20\text{ A}$ (Capacitor is open)

5. To determine the natural response, we set the source equal to zero, resulting in a series R-L-C circuit with

$$\left.\begin{aligned} \alpha &= \frac{R}{2L} = \frac{20}{2 \times 10^{-3}} = 10^4 \\ \omega_0 &= \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 10^{-3} \times 10 \times 10^{-6}}} = 10^4\text{ rad/s} \end{aligned}\right\}\text{critically damping}$$

$$s_1 = -\alpha = -10^4 = s_2$$

$$i_L(t) = i_{L,ss} + i_{L,T} = -20 + e^{-10t}(K_1 t + K_2)$$

$$i_L(0^+) = 10 = -20 + K_2 \Longrightarrow K_2 = 30$$

$$v_L(0^+) = L\frac{di_L}{dt}\bigg|_{t=0^+} = 1 \times 10^{-3} \times \left[e^{-10^4 t}K_1 - 10^4 e^{-10^4 t}(K_1 t + K_2)\right]$$

$$= 1 \times 10^{-3}[K_1 - 10^4 K_2]$$

$$v_L(0^+) = v_o(0^+) - v_R(0^+) = 200 - 200 = 0$$

$$\Longrightarrow K_1 - 10^4 K_2 = 0$$

$$K_1 = 10^4 K_2 = 10^4 \times 30 = 3 \times 10^5$$

$$i_L(t) = -20 + e^{-10^4 t}(3 \times 10^5 t + 30)$$

$$i_L(0.2\text{ ms}) = -20 + e^{-10^4 \times 0.2 \times 10^{-3}}(3 \times 10^5 \times 0.2 \times 10^{-3} + 30)$$

$$= -20 + \frac{(60 + 30)}{e^2} = -20 + \frac{90}{e^2} = -20 + 12.18 = -7.82\text{ A}$$

**Table 4.1 Summary: R-L-C Circuits**

| **Parallel Connection** | **Series Connection** |
|---|---|
| $\alpha = \frac{1}{2RC}$ | $\alpha = \frac{R}{2L}$ |
| $\omega_0 = \frac{1}{\sqrt{LC}}$ | $\omega_0 = \frac{1}{\sqrt{LC}}$ |

$\alpha > \omega_0 \Longrightarrow$ overdamped:

$f_N(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$, with $s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - {\omega_0}^2}$

$\alpha = \omega_0 \Longrightarrow$ critically damped:

$$f_N(t) = e^{-\alpha t}(K_1 t + K_2)$$

$\alpha > \omega_0 \Longrightarrow$ underdamped:

$$f_N(t) = e^{-\alpha t}(K_1 \cos \omega_d t + K_2 \sin \omega_d t),$$

With $\omega_d = \sqrt{{\omega_0}^2 - \alpha^2}$

N.B: in all these cases $f_N(t)$ stands for the natural response of either current through an inductor or voltage across a capacitor. Voltage cross, and current through these respective elements can be found from the relationships

$$v_L = L\frac{di_L(t)}{dt}$$

$$i_C = C\,\frac{dv_c(t)}{dt}$$

And as has been repeatedly noted, these cannot charge instantaneously in these respective elements

Total response:

$$f(t) = f_{ss}(\text{steady} - \text{state}) + f_T(transient)$$

**Example 4.2**: For the circuit shown in Fig.4.4, determine:

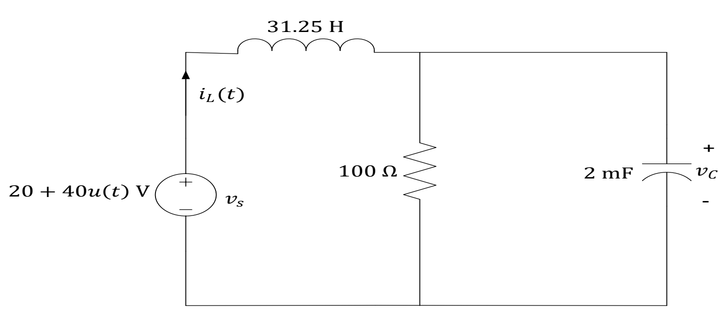



**Figure 4.4**

(i) $i_L(0)$

(ii) $v_c(0)$
(iii) $i_{L_{SS}}$
(iv) Expression for $i_L(t), t > 0$
(v) hence $i_L(0.2\text{ s})$

Solution:

for $t = 0^-$, $v_s$ is simple 20 V, (inductor is short and capacitor is open), therefore,

(i) $i_L(0^-) = \frac{20V}{100\Omega} = 0.2\text{ A} = i_L(0^+) = i_L(0)$
(ii) $v_C(0^-) = v_R(0^-) = 0.2\text{ A} \times 100\ \Omega = 20\text{ V} = v_C(0^+) = v_c(0)$
(iii) $i_{L_{SS}} = \frac{(20+40)V}{100\Omega} = \frac{60}{100} = 0.6\text{ A}$
(iv) $\alpha = \frac{1}{2RC} = \frac{1}{2\times100\times2\times10^{-3}} = \frac{1}{4\times10^{-1}} = 2.5\ s^{-1}$

$$\omega_0 = \frac{1}{\sqrt{31.25 \times 2 \times 10^{-3}}} = 4\ rad/s > \alpha = 2.5$$

$$\omega_d = \sqrt{4^2 - 2.5^2} = 3.122\text{ rad/s}$$

$$i_L(t) = 0.6 + e^{-2.5t}(K_1 \cos 3.122t + K_2 \sin 3.122t)$$

$$v_L(0^+) = 31.25 \frac{di_L(t)}{dt}\bigg|_{t=0^+} = 31.35[-2.5K_1 + 3.122K_2]$$

$$= 31.25[3.122K_2 - 2.5K_1]$$

$$v_L(0^+) + v_s - v_c(0^+) = (20 + 40) - 20 = 40$$

$$i_L(0^+) = 0.2 = 0.6 + K_1 \Longrightarrow K_1 = -0.4$$

$$3.122K_2 - 2.5K_1 = 3.122K_2 - 2.5(-0.4) = \frac{40}{31.25} = 1.28$$

$$K_2 = \frac{(1.28 - 1)}{3.122} = 0.089$$

$$i_L(t) = 0.6 + e^{-2.5t}(-0.4 \cos 3.122t + 0.089 \sin 3.122t)\text{ A}$$

(v) $$i_L(0.2\text{ s}) = 0.6 + \frac{[(-0.4 \cos 3.122 \times 0.2) + (0.089 \sin 3.122 \times 0.2)]}{e^{2.5\times0.2}}$$

$$= 0.6 + \frac{[-0.4 \times 0.81 + 0.089 \times 0.585]}{e^{0.5}}$$

$$= 0.6 + \frac{[-0.324 + 0.0521]}{e^{0.5}} = 0.6 - 0.1649 = 0.4351 \text{ A}$$

**Example 4.3**: Write the equation governing $V_1$ and $V_2$ in the circuit of Fig. 4.5 and determine the initial conditions on these voltages.

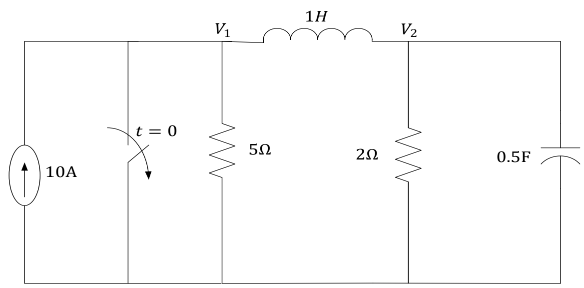



**Figure 4.5**

Using KCL relating node 1&2

For node 1

$$10 = \frac{V_1}{5} + \int V_1 dt - \int V_2 dt$$

$$\frac{V_1}{5} + \int V_1 dt - \int V_2 dt = 10$$

Fore Node 2

$$\int V_2 dt + \frac{V_2}{2} + 0.5\frac{dV_2}{dt} - \int V_1 dt = 0$$

For the initial condition

$$V_1(0^+) = 10 \times 5 = 50 \text{ V} \qquad\qquad V_2(0^+) = 0 \text{ V}$$

The initial conditions above is giving by the first derivatives of the voltage $V_1$ and $V_2$

$$\frac{dv_1(0^+)}{dt} = -5v_1(0^+) = -250 \text{ V/s}$$

When the current source $i = e^{-t}A$

$$e^{-t} = \frac{V_1}{5} + \int V_1\, dt - \int V_2 dt$$

$$\int V_2 dt + \frac{V_2}{2} + 0.5\,\frac{dV_2}{dt} - \int V_1 dt = 0$$

$$v_1(0^+) = 5i(0^+) = 5 \times 1 = 5\text{ V}$$

$$v_2(0^+) = 0$$

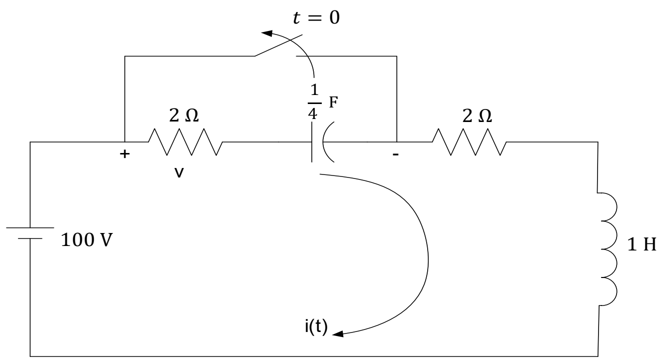



**Figure 4.6**

**Example 4.4**: In the circuit of Fig. 4.6, write a set of integral differential equations to solve for $v$. Find the natural response of the circuit and the time the current takes to reach maximum value.

(a) To solve for V

Applying KVL (for final condition)

$$100 = 2i + 2i + 4\int_0^t i dt + \frac{di}{dt}$$

$$\frac{di}{dt} + 4\int_0^t i dt + 4i = 100$$

With initial condition $v(0) = i_0 R$

$$i(0) = \frac{v(0)}{R} = \frac{100}{2} = 50A$$

$$v(0^+) = \frac{1}{2} \times 100 = 50$$

The natural response of a circuit is

$$\frac{d^2i}{dt^2} + 4\frac{di}{dt} + 4i = 0$$

With the initial conditions $i(0^+) = 2\text{ A}$,

$$\frac{di(0^+)}{dt} = 4\text{ A/s}$$

Solving for $i$ as the natural response,

Recall

$$\frac{d^2i}{dt^2} + \frac{4di}{dt} + 4i = 0$$

The characteristic equation is given by

$$s^2 + 4s + 4 = 0$$

$s_1 = -2,\ s_2 = -2$ (Critically damp)

Recall the response of a critically damped response

$$i(t) = A_1e^{st} + A_2te^{st} = (A_1 + A_2t)e^{st}$$

$$i(t) = A_1e^{-2t} + A_2te^{-2t}$$

At $t = 0$

$$i(0) = A_1e^{-2\times 0} + A_2(0)e^{-2\times 0} \Longrightarrow 2 = A_1$$

$$A_1 = 2$$

Taking the derivative of $i(t)$

$$\frac{di(t)}{dt} = -2A_1e^{-2t} - 2A_2te^{-2t} + A_2e^{-2t}$$

$$\left.\frac{di(t)}{dt}\right|_{t=0} = -2A_1e^{-2(0)} - 2A_2(0)e^{-2(0)} + A_2e^{-2(0)}$$

$$4 = -2(2) + A_2$$

$$A_2 = 4 + 4 = 8$$

$\therefore$ $i(t) = 2e^{-2t} + 8te^{-2t}$

$$i(t) = (2 + 8t)e^{-2t}\ \mathrm{A}$$

**Example 4.5:** A critically damped circuit has the natural response $i = 4te^{-10t}\ \mathrm{A}$.

When does $i$ reach its maximum value?
Solution:

From $\quad i = 4te^{-10t}$

$$\frac{di}{dt} = 4t \times -10e^{-10t} + 4e^{-10t}$$

$$\frac{di}{dt} = -40te^{-10t} + 4e^{-10t}$$

$$i_{max} \quad \Longrightarrow \frac{di}{dt} = 0$$

$$-40te^{-10t} + 4e^{-10t} = 0$$

$$4e^{-10t} = 40te^{-10t}$$

$$1 = 10t$$

$$t = \frac{1}{10} = 0.1\ \mathrm{s}$$

**Example 4.6:** In the two-mesh network of circuit of Fig.4.7, the loop currents are selected as shown. Write the KVL equation and find the current $i_1$ and $i_2$ at $t = 0$, taking $i(0) = 0$

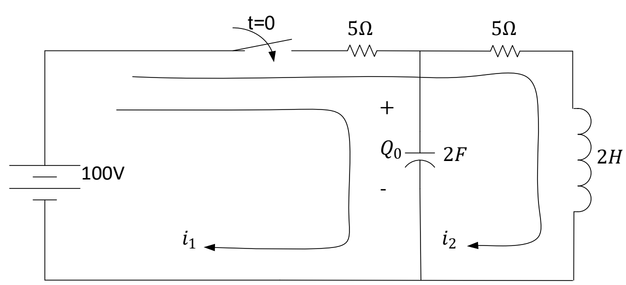



**Figure 4.7**

Solution:

$$5i_1 + \frac{1}{2}\left[Q_o + \int_0^t i_1(t)d(t)\right] + 5\, i_2 = 100 \qquad \text{(i)}$$

$$100 = 10\mathrm{i}_2 + 2\frac{di_2}{dt} + 5i_1 \qquad \text{(ii)}$$

Making $i_1$ the subject of formula in Eq (ii) we have

$$i_1 = \frac{100 - 10i_2 - 2\frac{di_2}{dt}}{5}$$

$$i_1 = 20 - 2i_2 - 0.4\frac{di_2}{dt} \qquad \text{(ii)}$$

Differentiating Eq (i) at $Q_o = 0$ and $i_o = 0$,

$$5\frac{di_1}{dt} + \frac{i_1}{2} + 5\frac{di_2}{dt} = 0 \qquad \text{(iv)}$$

then substituting for $i_1$ in Eq (iii) into Eq (iv)

$$5\frac{d}{dt}\left(20 - 2i_2 - 0.4\frac{di_2}{dt}\right) + \frac{1}{2}\left(20 - 2i_2 - 0.4\frac{di_2}{dt}\right) + \frac{5di_2}{dt} = 0$$

$$5\left(-2\frac{di_2}{dt} - 0.4\frac{d^2i_2}{dt}\right) + \frac{1}{2}\left(20 - 2i_2 - 0.4\frac{di_2}{dt}\right) + 5\frac{di_2}{dt} = 0$$

$$-10\frac{di_2}{dt} - \frac{2d^2i_2}{dt^2} + 10 - i_2 - 0.2\frac{di_2}{dt} + 5\frac{di_2}{dt} = 0$$

$$-\frac{2d^2i_2}{dt^2} - 5.2\frac{di_2}{dt} - i_2 = -10$$

$$\frac{2d^2i_2}{dt^2} + 5.2\frac{di_2}{dt} + i_2 = 10 \qquad \text{(v)}$$

Let $\frac{d}{dt} = D$

$$[2D^2 + 5.2D + 1]_{i_2} = 10 \qquad \text{(vi)}$$

$$2p^2 + 5.2p + 1 = 0 \qquad \textbf{[Characteristic equation]}$$

$$p = \frac{-5.2 \pm \sqrt{5.2^2 - 4 \times 2 \times 1}}{2 \times 2}$$

$$p = \frac{-5.2 \pm \sqrt{27.04 - 8}}{4}$$

$$p = \frac{-5.2 \pm \sqrt{19.04}}{4} = \frac{-5.2 \pm 4.36}{4}$$

$$p = \frac{-5.2 + 4.36}{4} \text{ or } \frac{-5.2 - 4.36}{4}$$

$p = -0.21$ or $p = -2.39$

$$i_{2_n} = Ae^{-0.21t} + Be^{-2.39t}$$

From Eq (vi)

$$i_{2_f} = \frac{10e^{\sigma t}}{2D^2 + 5.2D + 1}$$

For $(D \Longrightarrow \sigma = 0)$

$$i_{2_f} = \frac{10e^{(0)t}}{2(0)^2 + 5.2(0) + 1}$$

$$i_{2_f} = 10 \text{ A}$$

$$i_2 = i_{2_n} + i_{2_f}$$

$$i_2(t) = Ae^{-0.21t} + Be^{-2.39t} + 10 \qquad \text{(vii)}$$

From Eq (vii) substituting for initial conditions

$$i_{2(0)} = A + B + 10 \quad \Longrightarrow \quad A + B = -10 \quad \Longrightarrow \quad A = -10 - B \qquad \text{(viii)}$$

$$\left.\frac{di_2}{dt}\right|_{t=0} = -0.21A - 2.39B = 0$$

$$A = \frac{-2.39B}{0.21} = -11.38B \qquad \text{(ix)}$$

Putting Eq (viii) into (ix)

$$-11.38B = -10 - B$$

$$B(1 - 11.38) = -10$$

$$B = \frac{-10}{-10.38} = 0.98 \;\; \& \; A = -10 - 0.963 = -10.963$$

$$i_2(t) = 0.963e^{-2.39t} - 10.965e^{-0.2t} + 10\text{ A}$$

From Eq (iii)

$$i_1 = 20 - 2i_2 - 0.4\frac{di_2}{dt}$$

$$i_1 = 20 - 2(0.963e^{-2.39t} - 10.963e^{-0.2t} + 10) - 0.4\frac{d}{dt}[0.963e^{-2.39t} - 10.963e^{-0.21t} + 10]$$

$$i_1 = 20 - 1.926e^{-2.39t} + 21.926e^{-0.21t} - 20 + 0.9121e^{-2.39t} - 0.921e^{-0.21t}$$

$$i_1 = 21.005e^{-0.21t} - 1.005e^{-2.39t}\text{ A}$$

**Example 4.7:** In the circuit of Fig.4.8, with $i_1\ and\ i_2$ as shown,

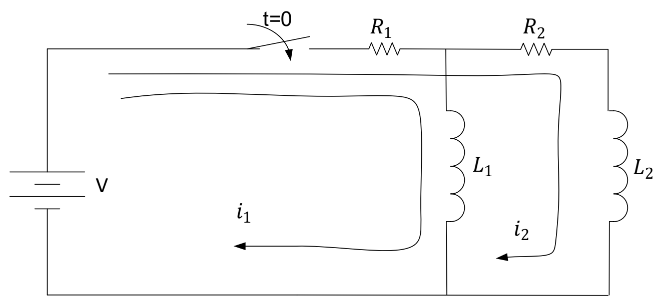



**Figure 4.8**

a) Obtain a differential equation for $i_1$by KVL

b) Obtain the characteristic equation and write the initial conditions

c) For $\text{V} = 240\text{ V},\ \text{L}_1 = 0.1\text{ H},\ \text{L}_2 = 0.2\text{ H},\ \text{R}_1 = 50\ \Omega,\ \text{R}_2 = 100\ \Omega$, obtain the instantaneous values of $i_1\ and\ i_2$

Taking the initial conditions as: $i_1(0^+) = i_1(0^-) = 0,\ \ i_2(0^+) = i_2(0^-) = 0,$ $\frac{di_1(0^+)}{dt} = \frac{V}{L_1}$

Solution:

**a)** $R_1 i_1 + L_1\frac{di_1}{dt} + R_1 i_2 = V$ (i)

$R_1 i_1 + (R_1 + R_2)i_2 + L_2\frac{di_2}{dt} = V$ (ii)

Differentiating equation (i)

$R_1 \frac{di_1}{dt} + L_1 \frac{d^2 i_1}{dt^2} + R_1 \frac{di_2}{dt} = 0$ (iii)

The eliminating $i_2$ and $\frac{di_2}{dt}$ between Eqs (i), (ii) & (iii) we have

$$\frac{d^2 i_1}{dt^2} + \left(\frac{R_1 L_1 + R_2 L_1 + R_1 L_2}{L_1 L_2}\right)\frac{di_1}{dt} + \frac{R_1 R_2}{L_1 L_2} i_1 = \frac{R_2 V}{L_1 L_2}$$

Let $\frac{d}{dt} = D$

$$\left[D^2 + \left(\frac{R_1 L_1 + R_2 L_1 + R_1 L_2}{L_1 L_2}\right)D + \frac{R_1 R_2}{L_1 L_2}\right]_{i_1} = \frac{R_2 V}{L_1 L_2} \quad \Longrightarrow \quad \textbf{ODE} \qquad \text{(iv)}$$

(iv)

**b)** Again, let $D = p$ i.e. the characteristic equation is given by:

$$p^2 + \left(\frac{R_1 L_1 + R_2 L_1 + R_1 L_2}{L_1 L_2}\right)p + \frac{R_1 R_2}{L_1 L_2} = 0 \qquad \text{(v)}$$

**c)** For $L_1 = 0.1\ \text{H},\ L_2 = 0.2\ \text{H},\ R_1 = 50\ \Omega,\ R_2 = 100\ \Omega$

$$\left[D^2 + \left(\frac{50 \times 0.1 + 100 \times 0.1 + 50 \times 0.2}{0.1 \times 0.2}\right)D + \frac{50 \times 100}{0.1 \times 0.2}\right]_{i_1} = \frac{100 \times 240}{0.1 \times 0.2}$$

$$\left[D^2 + \frac{25}{0.02}D + 25000\right]_{i_1} = 1200000$$

$$[D^2 + 1250D + 250000]_{i_1} = 1200000 \qquad \text{(vi)}$$

Let $D = p$: $p^2 + 1250p + 250000 = 0$ ***[Characteristic equation]***

$$p = \frac{-1250 \pm \sqrt{1250^2 - 4 \times 1 \times 250000}}{2 \times 1}$$

$$p_{(1,2)} = \frac{-1250 \pm \sqrt{562500}}{2}$$

$$\Longrightarrow p_1 = \frac{-1250 + 750}{2} \text{ or } p_2 = \frac{-1250 - 750}{2}$$

$p_1 = -250$ *or* $p_2 = -1000$ (Overdamped response)

$$i_{1_n} = Ae^{-250t} + Be^{-1000t} \qquad \text{(vii)}$$

From Eq (iv)

$$[D^2 + 1250D + 250000]_{i_1} = 1200000$$

$$i_{1_f} = \frac{1}{D^2 + 1250D + 250000} \times 1200000e^{\sigma t}$$

For $(D \Longrightarrow \sigma = 0)$

$$i_{1_f} = \frac{1200000e^{0t}}{0^2 + 250(0) + 250000} = \frac{12}{25}$$

$$i_{1_f} = \frac{120}{25} = 4.8 \text{ A}$$

$$i_1(t) = i_{1_n} + i_{1_f} \qquad \text{(viii)}$$

$$i_1(t) = Ae^{-250t} + Be^{-1000t} + 4.8$$

(ix)

At $t = 0$

$$i_{(0)} = Ae^{-250(0)} + Be^{-1000(0)} + 4.8$$

$$0 = A + B + 4.8$$

$$A + B = -4.8 \qquad \text{(x)}$$

Differentiating Eq (ix)

$$\frac{di_1(t)}{dt} = -250Ae^{-250t} - 1000Be^{-100t}$$

$$\left.\frac{di_1(t)}{dt}\right|_{t=0} = -250A - 1000B \quad \Longrightarrow -2400 \qquad \text{(xi)}$$

$$\left.\frac{di(t)}{dt}\right|_{t=0} = -250A - 1000B$$

$$\text{Where } L_1\frac{di_1(t)}{dt} = V_0, \quad \frac{di_1(t)}{dt} = \frac{V_0}{L_1} = \frac{240}{0.1} \qquad [\textbf{initial condition}]$$

$$\frac{di(t)}{dt} = 2400 \text{ A/s}$$

$\therefore \qquad 2400 = -250A - 1000B$

$$250A + 1000B = -2400 \qquad \text{(xii)}$$

From Eq(x)

$A + B = -4.8$

$A = -4.8 - B \qquad \text{(xiii)}$

$$250A + 1000B = -2400$$

Using substitution method of solving simultaneous equations, substitute (xiii) into (xii)

$$250(-4.8 - B) + 1000B = -2400$$

$$-1200 - 250B + 1000B = -2400$$

$$750B = -2400 + 1200$$

$$B = \frac{-1200}{750}$$

$$B = -1.6$$

Substituting the value of $B$ into Eq (xiii)

$$A = -4.8 - (-1.6) = -4.8 + 1.6 = -3.2$$

$\therefore \qquad i_1(t) = -1.6e^{-1000t} - 3.2e^{-250t} + 4.8$

$$i_1(t) = 4.8 - 1.6e^{-1000t} - 3.2e^{-250t} \text{ A} \qquad \text{(xi)}$$

But $i_2$ from Eq (i) is given by

$$R_1L_1 + L_1\frac{di_1}{dt} + R_1i_2 = V$$

$$50i_1 + 0.1\frac{di_1}{dt} + 50i_2 = 240 \qquad \text{(xii)}$$

Substituting the Eq (xi) into Eq (xii)

$$50(4.8 - 1.6e^{-1000t} - 3.2e^{-250t}) + 0.1\frac{d}{dt}(4.8 - 1.6e^{-1000t} - 3.2e^{-250t}) + 50i_2 = 240$$

$$240 - 80e^{-1000t} - 160e^{-1000t} + 80e^{-250t} + 50i_2 = 240$$

$$e^{-1000t}(160-80)+e^{-250t}(80-160)+50i_2=0$$

$$80e^{-1000t}-80e^{-250t}=-50i_2$$

$$-50i_2=80(e^{-1000t}-e^{-250t})$$

$$i_2=1.6\,(-e^{-1000t}+e^{-250t})$$

$$i_2=1.6e^{-250t}-1.6e^{-1000t}\ \text{A} \qquad \text{(xiii)}$$

## 4.2 Exercise

**1.** On the circuit of Fig.1. are 3 passive elements, with a voltage and a current defined for each. Determine eat at t=0$^-$ and at t = 0$^+$

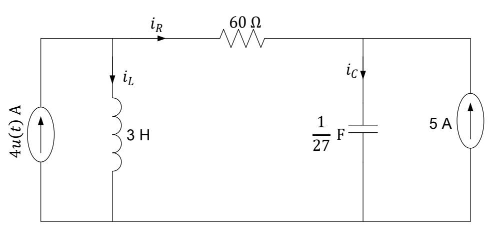



**Figure 1**

2. For Fig.2 find: (i) $\alpha$ (ii) $\omega$ (iii) $i(0^+)$ (iv) $\left.\frac{di}{dt}\right|_{t=0^+}$ (v) $i(0.5\ ms)$

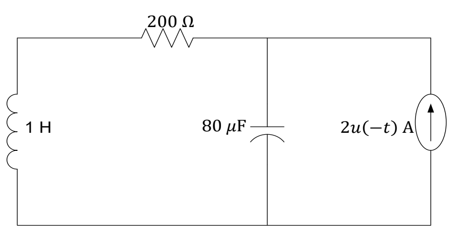



**Figure 2**

**3.** For Fig.3. Find: (i) $i(t)$ (ii) $i(20\ \text{ms})$

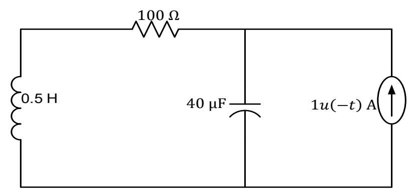



**Figure 3**

**4.** For Fig. 4 Given $i_s=5u(-t)-10u(t)$ A, determine:
$(i) i_L(0^-)$ $(ii) v_c(0^+)$ $(iii) v_R(0^+)$ $(iv) i_L(\infty)$

$(v)\ i_L(0.015\ \text{s})$

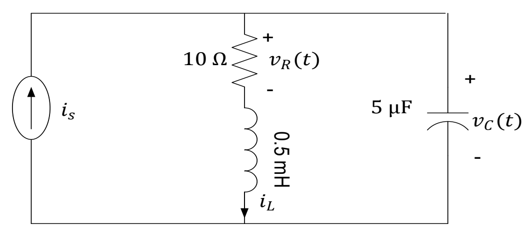



**Figure 4**

**5.** For Fig. 3, determine (i) $\alpha$ (ii) $\omega$ (iii) $i(0^+)$ (iv) $\left.\frac{di}{dt}\right|_{t=0^+}$ (v) $i(t), t>0$ (vi) hence $i(10\ ms)$

**6.** Given $i_s=30u(-t)-60u(t)$A for Fig. 4, determine: (i) $i_L(0^-)$ (ii) $v_c(0^+)$ (iii) $v_R(0^+)$ (iv) $i_L(\infty)$ (v) $i_L(t), t>0$, (vi) hence $i_L(0.2\ \text{ms})$

**7.** Find the complete response $v$ and then $i$ for $t > 0$ in the circuit of Fig.5

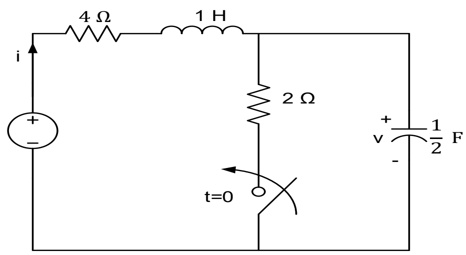



**Figure 5**

**Answer:** $\boldsymbol{i = 2 - 6e^{-2t} + 4e^{-3t}}$ **A**

**8.** In the RLC circuit of Fig. 6, $V$ is a dc source. Write a formal expression for the initial charge on the capacitor is zero.

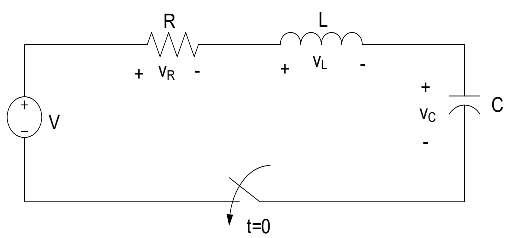



**Figure 6**

**Answer:** $\boldsymbol{P_1 = \frac{-R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \equiv -\alpha + \beta}$

$\boldsymbol{P_2 = \frac{-R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \equiv -\alpha - \beta}$

**9.** State the initial condition to evaluate the constants of integration in **Exercise 8**. Define $\omega_o$, the resonance frequency and express the characteristic root in terms of $\alpha$ and $\omega_o$

**Answer:** $\boldsymbol{P_1, P_2} = -\alpha \pm \sqrt{\alpha^2 - \boldsymbol{\omega_0^2}}$

**10.** Find an expression for $v_c(t)$ for $t > 0$ in the circuit below Fig. 7

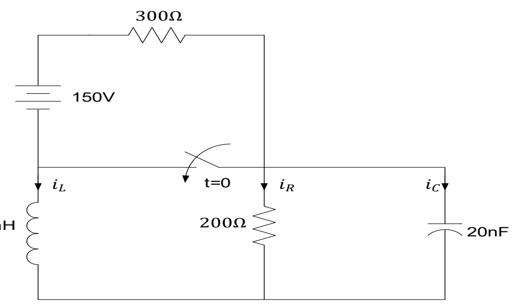



**Figure 7**

**Answer:** $\boldsymbol{v_c(t) = 90e^{-5000t} - 20e^{-200000t}}$ **V**

**11.** After being opened for a long time, the switch in Fig. 8 close at $t = 0$. Find (a) $i_L(0^-)$; (b) $v_L(0^-)$; (c) $i_R(0^+)$; (d) $i_c(0^+)$; (e) $v_c(0.2\text{ s})$

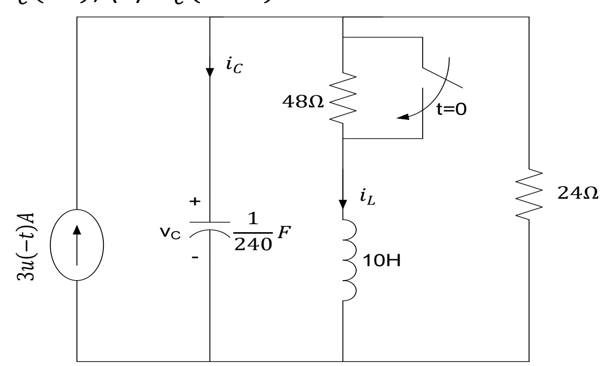



**Figure 8**

**Answer:**

**1 A; 48 V; 2 A; −3 A; −17.54** V

# CHAPTER 5

# TWO-PORT NETWORK

## 5.0 Introduction

A port is a pair of terminals at which a signal may enter or leave a network. The rule is that what goes in is what equally comes out. But for a two-port network, we have two of such a pair of terminals. One pair may be used as input for, say, an energy signal while the other pair is used as the output for the load see Fig. 5.1.

Two-Port networks find applications as important building blocks in electronic systems, automatic control systems, communication systems, and transmission and distribution systems.

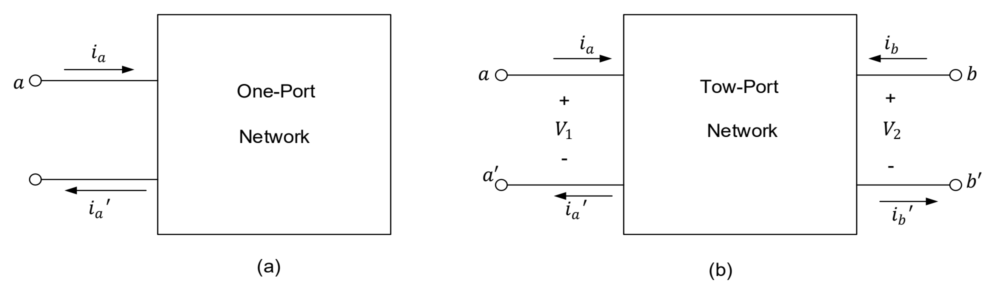



**Figure 5.1**

As mentioned earlier, for each part, what goes in is what comes out, and we're not much concerned with the internal circuitry of the network (i.e., viewed differently, it might just be regarded as the proverbial block box!) Note the conventional direction of the arrow at the top of (output) port b, see Fig. 5.2b for the Two-Port network. Also, it's assumed that for simplicity, the network is a linear one, and no independent source(s) are involved, while there may be dependent source(s).

When a quotient involves terms at the same port, it is known as a driving point term, while if it involves terms at two different parts, it is called a transfer term. The quotient is normally that of transform pairs, and for a two-port network can be impedance, admittance, or simply voltage or current gains, since we're obviously talking about quotients of voltages and/or currents:

$$Gain = \begin{cases} \text{Voltage transfer function} \qquad G_{12}(s) = \dfrac{V_2(s)}{V_1(s)} \\ \text{current transfer function} \propto_{12} (s) = \dfrac{I_2(s)}{I_1(s)} \end{cases}$$

Transfer impedance function $Z_{12}(s) = \frac{V_1(s)}{I_2(s)}$

(Reverse) Transfer impedance function $Z_{21}(s) = \frac{V_2(s)}{I_1(s)}$

Transfer admittance function $Y_{12}(s) = \frac{I_1(s)}{V_2(s)}$

Reverse transfer admittance function $Y_{21}(s) = \frac{I_2(s)}{V_1(s)}$

All of the above involve signals at two different ports, hence the term "transfer". There may also be driving point function whereby the quotient is of signals at the same port for example: $\frac{V_1(s)}{I_1(s)}, \frac{V_2(s)}{I_2(s)}, \frac{I_1(s)}{V_1(s)}, \frac{I_2(s)}{V_2(s)}$. (During point functions of like terms are identically unity).

For two-port networks, there are four different sets of parameters, of which we'll in this textbook deal with z, y and t parameters only while the rest will only come up for an honourable mention!

The parameters are:

1. $z$ (Or impedance) parameters:

$$\left.\begin{matrix} V_1 = f_1(I_1, I_2) = z_{11}I_1 + Z_{12}I_2 \\ V_2 = f_2(I_1, I_2) = z_{21}I_1 + Z_{22}I_2 \end{matrix}\right\} \Rightarrow \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

2. $y$ (Or admittance) parameters:

$$\left.\begin{matrix} I_1 = f_1(V_1, V_2) = y_{11}V_1 + y_{12}V_2 \\ I_2 = f_2(V_1, V_2) = y_{21}V_1 + y_{22}V_2 \end{matrix}\right\} \Rightarrow \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

3. $h$ (Or hybrid or "mixed") parameters:

$$\left.\begin{matrix} V_1 = f_1(I_1, V_2) = h_{11}I_1 + h_{12}V_2 \\ I_2 = f_2(I_1, V_2) = y_{21}I_1 + h_{22}V_2 \end{matrix}\right\} \Rightarrow \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ V_2 \end{pmatrix}$$

There may also be "inverse hybrid" where the letter merely exchanges – $V$ for $I$; $I$ for $V$, and $h$ for $g$:

3a. $$\left.\begin{matrix} I_1 = f_1(V_1, I_2) = g_{11}\, V_1 + g_{12}I_2 \\ V_2 = f_2(V_1, I_2) = g_{21}\, V_1 + g_{22}I_2 \end{matrix}\right\} \Rightarrow \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ I_2 \end{pmatrix}$$

4. $t-$ (or transmission or ABCD) parameters:

$$\left.\begin{aligned} V_1 &= f_1(V_2, I_2) = t_{11}V_2 - t_{12}I_2 \\ I_1 &= f_2(V_2, I_2) = t_{21}\,V_2 - t_{22}I_2 \end{aligned}\right\} \Longrightarrow \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}\begin{pmatrix} V_2 \\ I_2 \end{pmatrix}$$

(The reason for the minus sign will become apparent, later).

As pointed out earlier, the sets of parameters to cover are: z, y and t. As we shall in this textbook concentrate more on the t parameters, that is, transmission parameters, not least because of its relationship to other "sister" courses in electrical as well as electronics engineering.

## 5.1 Impedance Parameters 'Z'

Impedance and admittance parameters are commonly used in the synthesis of filters. They are also useful in the design and analysis of impedance-matching networks and power distribution networks.

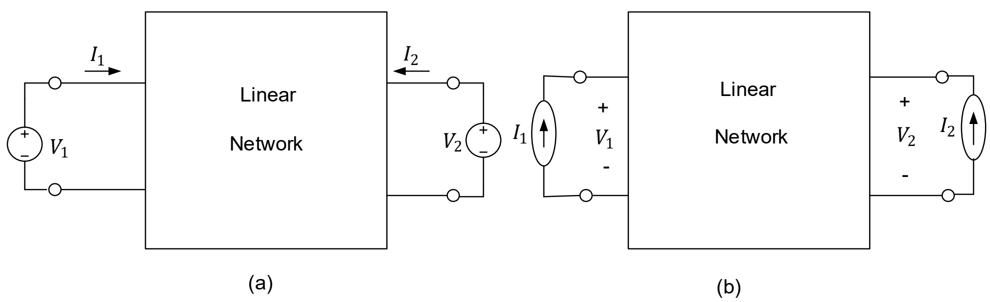



**Figure 5.2 The linear two-port network: (a) driven by voltage sources, (b) driven by current sources**

We discuss impedance parameters in this section and admittance parameters in the next section.

A two-port network may be voltage-driven as in Fig. 5.2(a) or current-driven as in Fig. 5.2(b). From either Figs. 5.2(a) or (b), the terminal voltages can be related to the terminal currents as

$$\boxed{\begin{aligned} V_1 &= z_{11}I_1 + z_{12}I_2 \\ V_2 &= z_{21}I_1 + z_{22}I_2 \end{aligned}} \qquad 5.1$$

or in matrix form as

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = [z]\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \qquad 5.2$$

where the z terms are called the impedance parameters, or simply z parameters, and have units of ohms.

The values of the parameters can be evaluated by setting $I_1 = 0$ (input port open-circuited) or $I_2 = 0$ (output port open-circuited). Thus,

$$\boxed{\begin{array}{ll} z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0}, & z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} \\ z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}, & z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} \end{array}} \qquad 5.3$$

Since the z parameters are obtained by open-circuiting the input or output port, they are also called the open-circuit impedance parameters. Specifically,

$z_{11} =$ Open circuit input impedance

$z_{12} =$ Open circuit transfer impedance from port 1 to port 2 5.4

$z_{21} =$ Open-circuit transfer impedance from port 2 to port 1

$z_{22} =$ Open circuit output impedance

According to Eq. (5.3), we obtain $z_{11}$ and $v_{21}$ by connecting a open-circuited voltage $V_1$ (or a current source $I_1$) to port I with port as in Fig. 5.3(a) and finding $I_1$ and $V_2$ we then get

$$z_{11} = \frac{V_1}{I_1}, \quad z_{21} = \frac{V_2}{I_1} \qquad 5.5$$

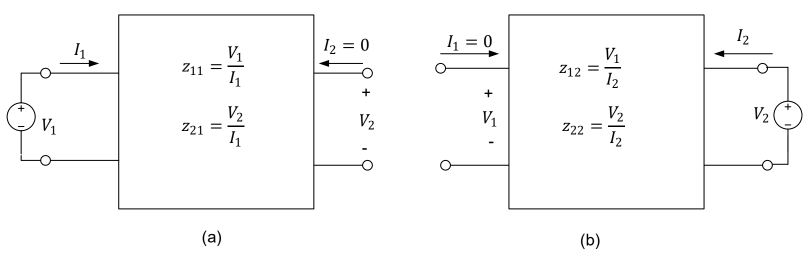



**Figure 5.3 Determination of the z-parameters: (a) finding $z_{11}$& $z_{21}$ (b) finding $z_{12}$ & $z_{22}$**

Similarly, we obtain $z_{12}$ and by $z_{22}$ connecting a voltage $V_2$ (or a current source $I_2$) to port 2 with port I open-circuited as in Fig. 5.3(b) and finding $I_2$ and $V_1$; we then get

$$z_{12} = \frac{V_1}{I_2}, \quad z_{22} = \frac{V_2}{I_2} \qquad 5.6$$

The above procedure provides us with a means of calculating or measuring the z parameters.

Sometimes $z_{11}$ and $z_{22}$ are called driving-point impedances, while $z_{21}$ and $z_{12}$ are called transfer impedances. A driving-point impedance is the input impedance of a two-terminal (one-port) device. Thus, $z_{11}$ is the input driving-point impedance with the output port open-circuited, while $z_{22}$ is the output driving-point impedance with the input port open-circuited.

When $z_{11} = z_{22}$, the two-port network is said to be symmetrical. This implies that the network has mirror like symmetry about some center line; that is, a line can be found that divides the network into two similar halves.

When the two-port network is linear and has no dependent sources, the transfer impedances are equal ($z_{12} = z_{21}$), and the two-port is said to be reciprocal. This means that if the points of excitation and response are interchanged, the transfer impedances remain the same. As illustrated in Fig. 5.4, a two-port is reciprocal if interchanging an ideal voltage source at one port with an ideal ammeter at the other port gives the port same ammeter reading.

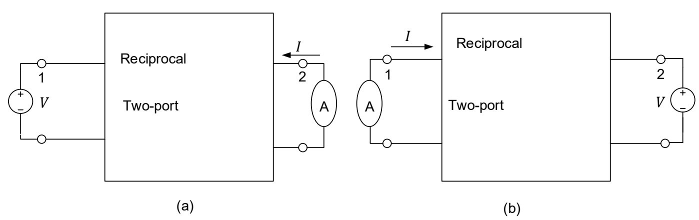



**Figure 5.4 Interchanging a voltage source at one port with an ideal ammeter at the other port produces the same reading in a reciprocal two-port.**

The reciprocal network yields $V = z_{12}I$ according to Eq. (5.1) when connected as in Fig. 5.4(a), but yields $V = z_{21}I$ when connected as in Fig. 5.4(b). This is possible only if $z_{12} = z_{21}$. Any two-port that is made entirely of resistors, capacitors, and inductors must be reciprocal. A reciprocal network can be replaced by the T-equivalent circuit in Fig. 5.5(a). If the network is not reciprocal, a more general equivalent network is shown in Fig. 5.5(b); notice that this figure follows directly from Eq. (5.1).

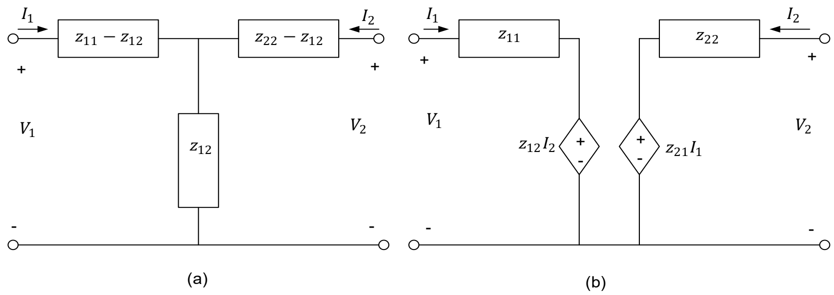



**Figure 5.5 (a) T-equivalent circuit (for reciprocal case only), (b) general equivalent circuit**

It should be mentioned that for some two-port networks. the -parameters do not exist because they cannot be described by Eq. (5.1). As an example, consider the ideal transformer of Fig.5.6. The defining equations for the two-port network are:

$$V_1 = \frac{1}{n}V_2, \qquad and \quad I_1 = -nI_2 \qquad 5.7$$

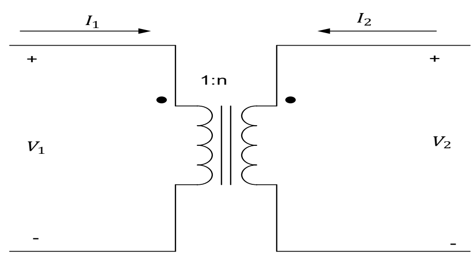



**Figure 5.6 An ideal transformer has no z-parameters**.

Observe that it is impossible to express the voltages in terms of the currents, and vice versa, as Eq. (5.1) requires. Thus, the ideal transformer has no z. parameters. However, it does have hybrid parameters as we shall see in **Section 5.4**.

**Example 5.1**: Determine the z parameters for the circuit in Fig. 5.7.

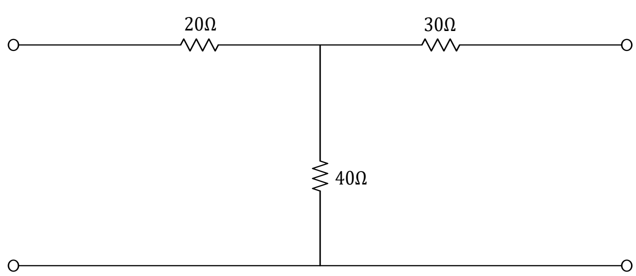



**Figure 5.7**

Solution

METHOD I To determine $z_{11}$ and $z_{21}$, we apply a voltage source $V_1$ to the Input port and leave the output port open as in Fig.5.8(a). Then,

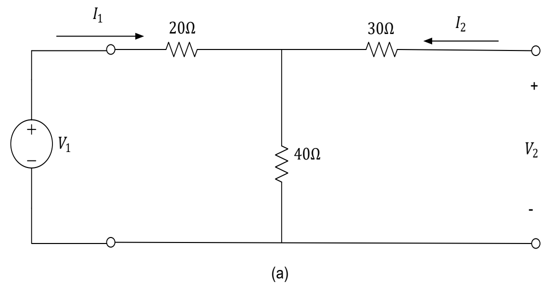



**Figure 5.8 (a) finding $z_{11}$& $z_{21}$**

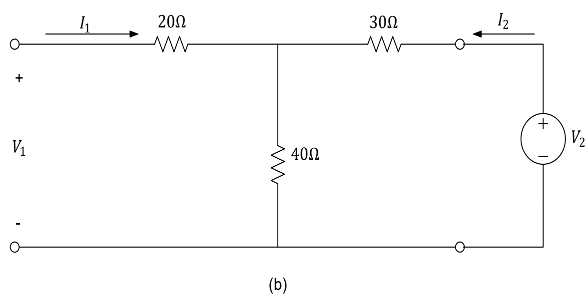



**Figure 5.8 (b) finding $z_{12}$& $z_{22}$**

$$z_{11} = \frac{V_1}{I_1} = \frac{(20+40)I_1}{I_1} = 60\ \Omega$$

that is, $z_{11}$ is the input impedance at port 1.

$$z_{21} = \frac{V_2}{I_1} = \frac{40I_1}{I_1} = 40\ \Omega$$

To find $z_{12}$ and $z_{22}$, we apply a voltage source $V_2$ to the output port and leave the input port open as in Fig. 5.8(b). Then,

$$z_{12} = \frac{V_1}{I_2} = \frac{40I_2}{I_2} = 40\ \Omega,\ \ z_{22} = \frac{V_2}{I_2} = \frac{(30+40)I_2}{I_2} = 70\ \Omega$$

Thus,

$$[z] = \begin{bmatrix} 60\ \Omega & 40\ \Omega \\ 40\ \Omega & 70\ \Omega \end{bmatrix}$$

METHOD 2 Alternatively, since there is no dependent source in the given circuit, $z_{12} = z_{21}$ and we can use Fig. 5.5(a). Comparing Fig. 5.7 with Fig. 5.5(a), we get

$$z_{12} = 40\ \Omega = z_{21}$$

$$z_{11} - z_{12} = 20\ \Omega \quad \Longrightarrow \quad z_{11} = 20 + z_{12} = 60\ \Omega$$

$$z_{22} - z_{12} = 30\ \Omega \quad \Longrightarrow \quad z_{22} = 30 + z_{12} = 70\ \Omega$$

**Example 5.2**: Find $I_1$ and $I_2$ in the circuit in Fig. 5.9.

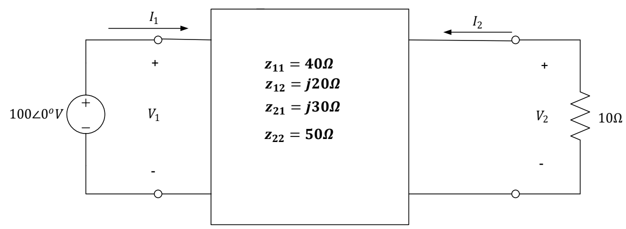



**Figure 5.9**

Solution

This is not a reciprocal network. We may use the equivalent circuit in Fig. 5.5(b) but we can also use Eq. (5.1) directly. Substituting the given z parameters into Eq. (5.1),

$$V_1 = 40I_1 + j20I_2 \qquad 5.8$$

$$V_2 = j30I_1 + 50I_2 \qquad 5.9$$

Since we are looking for $I_1$ and $I_2$, we substitute

$$V_1 = 100\angle 0^o, \quad V_2 = -10I_2$$

into Eqs. (5.8) and (5.9), which become

$$100 = 40I_1 + j20I_2 \qquad 5.10$$

$$-10I_2 = j30I_1 + 50I_2 \quad \Longrightarrow \quad I_1 = j2I_2 \qquad 5.11$$

Substituting Eq. (5.11) into Eq. (5.10) gives

$$100 = j80I_2 + j20I_2 \quad \Longrightarrow \quad I_2 = \frac{100}{j100} = -j$$

From Eq. (5.11), $I_1 = j2(-j) = 2$.

Thus, $I_1 = 2\angle 0^o\ A, \ \ I_2 = 1\angle - 90^o\mathrm{A}$

## 5.2 Admittance Parameters

In the previous section we saw that impedance parameters may not exist for a two-port network. So, there is a need for an alternative means of describing such a network. This need may be met by the second set of parameters, which we obtain by expressing the terminal currents in terms of the terminal voltages. In either Fig. 5.10(a) or (b), the terminal currents can be expressed in terms of the terminal voltages as.

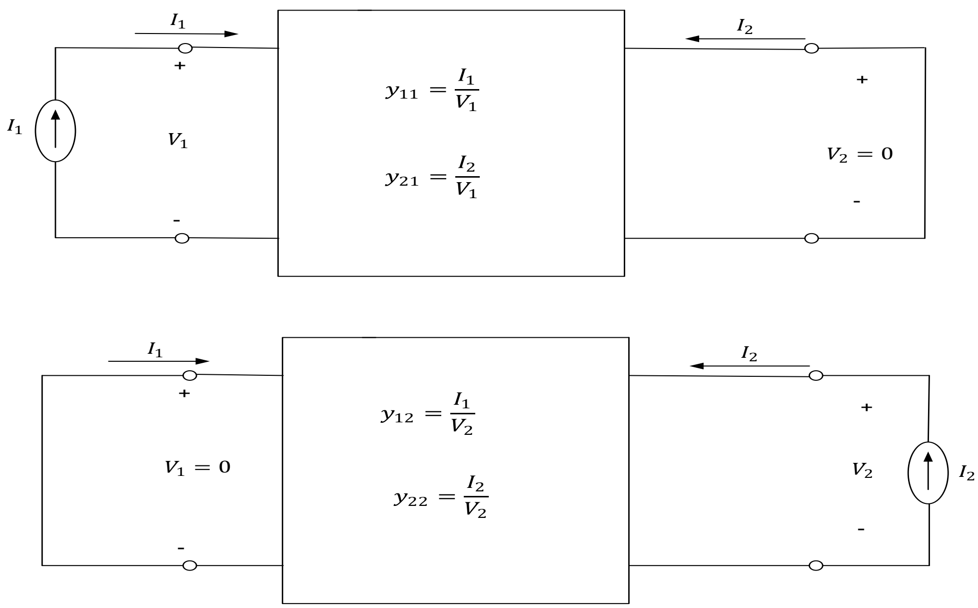



**Figure 5.10 Determination of the y-parameters: (a) finding $y_{11}$& $y_{21}$, (b) finding $y_{12}$ & $y_{22}$**

$$\boxed{\begin{aligned} I_1 &= y_{11}V_1 + y_{12}V_2 \\ I_2 &= y_{21}V_1 + y_{22}V_2 \end{aligned}} \quad 5.12$$

or in matrix form as

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = [y] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad 5.13$$

The $y$ terms are known as the admittance parameters (or, simply, y parameters) and have units of Siemens.

The values of the parameters can be determined by setting $V_1 = 0$ (input port short-circuited) or $V_2 = 0$ (output port short-circuited). Thus,

$$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}, \quad y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$$
$$y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}, \quad y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} \qquad 5.14$$

Since the $y$ parameters are obtained by short-circuiting the input or output port. They are also called the short-circuit admittance parameters. Specifically,

$y_{11}$ =Short-circuit input admittance

$y_{12}$ =Short-circuit transfer admittance from port 2 to port 1 5.15

$y_{21}$ =Short-circuit transfer admittance from port I to port 2

$y_{22}$ =Short-circuit output admittance

Following Eq. (5.14), we obtain $y_{11}$ and $y_{21}$ by connecting a current $I_1$ to port I and short-circuiting port 2 as in Fig. 5.10(a), finding $V_1$ and $I_2$ and then calculating

$$y_{11} = \frac{I_1}{V_1}, \quad y_{21} = \frac{I_2}{V_1} \qquad 5.16$$

Similarly, we obtain $y_{12}$ and $y_{22}$ by connecting a current source $I_2$ to port 2 and short-circuiting port 1 as in Fig. 5.10(b). Finding $I_1$ and $V_2$ and then getting

$$y_{12} = \frac{I_1}{V_2}, \quad y_{22} = \frac{I_2}{V_2} \qquad 5.17$$

This procedure provides us with a means of calculating or measuring the $v$ parameters. The impedance and admittance Parameters are collectively referred to as immittance parameters.

For a two-port network that is linear and has no dependent sources, the transfer admittances are equal $(y_{12} = y_{21})$. This can be proved in the same way as for the z parameters. A reciprocal network $(y_{12} = y_{21})$ can be modeled by the $\pi$-equivalent circuit in Fig.5.11(a). If the network is not reciprocal, a more general equivalent network is shown in Fig.5.11(b).

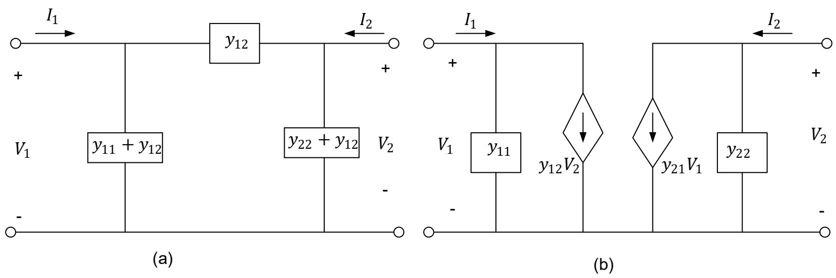



**Figure 5.11(a) π −equivalent circuit (for reciprocal case only), (b) general equivalent circuit**

**Example 5.3**: Obtain the $y$ parameters for the π network shown in Fig. 5.12

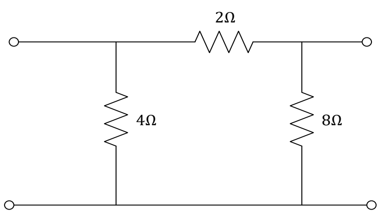



**Figure 5.12**

Solution:

METHOD 1 To find $y_{11}$ and $y_{21}$, short-circuit the output port and connect a current source $I_1$ to the input port as in Fig. 5.13(a). Since the $8\ \Omega$ resistor is short-circuited, the $2\ \Omega$ resistor is in parallel with the $4\ \Omega$ resistor. Hence,

$$V_1 = I_1(4 \parallel 2) = \frac{4}{3}I_1, \quad y_{11} = \frac{I_1}{V_1} = \frac{I_1}{\frac{4}{3}I_1} = 0.75\ \text{S}$$

By current division,

$$-I_2 = \frac{4}{4+2}I_1 = \frac{2}{3}I_1, \;\; y_{21} = \frac{I_2}{V_1} = \frac{-\frac{2}{3}I_1}{\frac{4}{3}I_1} = 0.5\ \text{S}$$

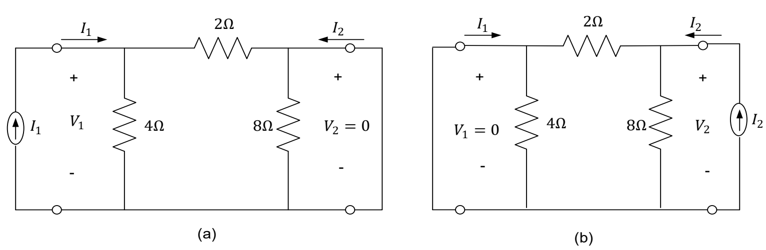



**Figure 5.13**

To get $y_{12}$ and $y_{22}$, short-circuit the input port and connect a current source $I_2$ to the output port as in Fig. 5.13(b). The $4\ \Omega$ resistor is short-circuited so that the $2\ \Omega$ and $8\ \Omega$ resistors are in parallel.

$$V_2 = I_2(8 \parallel 2) = \frac{8}{5}I_2, \quad y_{22} = \frac{I_2}{V_2} = \frac{I_2}{\frac{8}{5}I_2} = \frac{8}{5} = 0.625\ \text{S}$$

By current division,

$$-I_1 = \frac{8}{8+2}I_2 = \frac{4}{5}I_2, \quad y_{12} = \frac{I_1}{V_2} = \frac{-\frac{4}{5}I_2}{\frac{8}{5}I_2} = -0.5\ \text{S}$$

METHOD 2 Alternatively, comparing Fig. 5.12 with Fig 5.11(a).

$$y_{12} = -\frac{1}{2}S = y_{21}$$

$$y_{11} + y_{12} = \frac{1}{4} \quad \Longrightarrow \quad y_{11} = \frac{1}{4} - y_{12} = 0.75\ \text{S}$$

$$y_{22} + y_{12} = \frac{1}{8} \quad \Longrightarrow \quad y_{22} = \frac{1}{8} - y_{12} = 0.625\ \text{S}$$

as obtained previously.

**Example 5.4**: Determine the $y$ parameter for the two-port shown in Fig.5.14

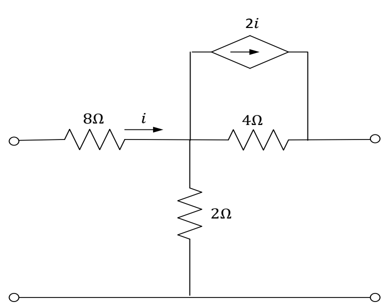



**Figure 5.14**

Solution

We follow the same procedure in the previous example. To get $y_{11}$ and $y_{21}$, we use the circuit in Fig.5.15(a). in which port 2 is short circuited and a current source is applied to port 1. At node 1,

$$\frac{V_1 - V_o}{8} = 2I_1 + \frac{V_o}{2} + \frac{V_o - 0}{4}$$

But $\mathrm{I_1} = \dfrac{\mathrm{V_1 - V_o}}{8}$, therefore

$$0 = \frac{V_1 - V_o}{8} + \frac{3V_o}{4}$$

$$0 = V_1 - V_o + 6V_o \quad \Longrightarrow \quad V_1 = -5V_o$$

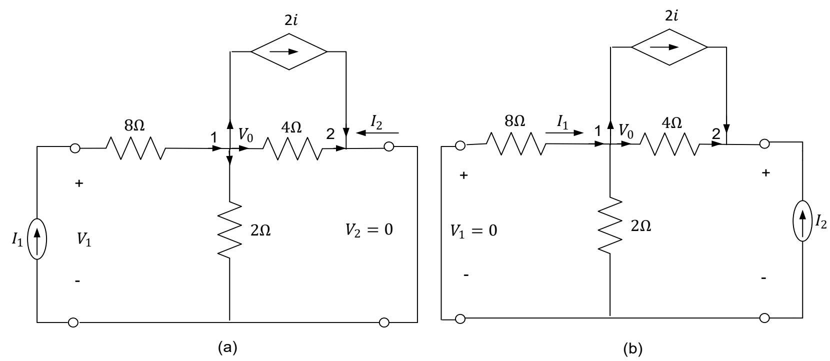



**Figure 5.15(a) finding $y_{11}$ and$y_{21}$, (b) finding $y_{12}$ and $y_{22}$**

Hence,

$$I_1 = \frac{-5V_o - V_o}{8} = -0.75V_o$$

And

$$y_{11} = \frac{I_1}{V_1} = \frac{-0.75V_o}{-5V_o} = 0.15 \text{ S}$$

At node 2,

$$\frac{V_o - 0}{4} + 2I_1 + I_2 = 0$$

Or

$$-I_2 = 0.25V_o - 1.5V_o = -1.25V_o$$

Hence,

$$y_{21} = \frac{I_2}{V_1} = \frac{1.25V_o}{-5V_o} = -0.25 \text{ S}$$

Similarly, we get $y_{12}$ and $y_{22}$ using Fig.5.15(b). at node 1

$$\frac{0 - V_o}{8} = 2I_1 + \frac{V_o}{2} + \frac{V_o - V_2}{4}$$

But, $I_1 = \frac{0 - V_o}{8}$ therefore,

$$0 = -\frac{V_o}{8} + \frac{V_o}{2} + \frac{V_o - V_2}{4}$$

Or $$0 = -V_o + 4V_o + 2V_o - 2V_2 \quad \Longrightarrow \quad V_2 = 2.5V_o$$

Hence,

$$y_{12} = \frac{I_1}{V_2} = \frac{-V_o/8}{2.5V_o} = -0.05 \text{ S}$$

At node 2

$$\frac{V_o - V_2}{4} + 2I_1 + I_2 = 0$$

Or $$-I_2 = 0.25V_o - \frac{1}{4}(2.5V_o) - \frac{2V_o}{8} = -0.625V_o$$

$$y_{22} = \frac{I_2}{V_2} = \frac{0.625V_o}{2.5V_o} = 0.25 \text{ S}$$

Notice that $y_{12} \neq y_{21}$ in this case, since the network is not reciprocal.

## 5.3 Transmission Line Parameters (ABCD-Parameters)

Since there are no restrictions on which terminal voltages and currents should be considered independent and which should be dependent variables, we expect to be able to generate many sets of parameters.

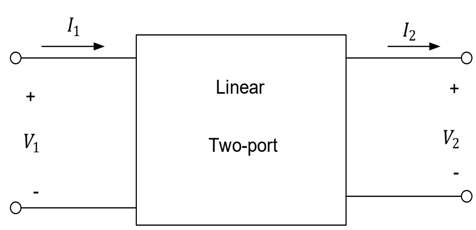



**Figure 5.16 Terminal variable used to define the ABCD parameters**

Another set of parameters relates the variables at the Input port to those at the output port. Thus,

$$\boxed{\begin{aligned} V_1 &= AV_2 - BI_2 \\ I_1 &= CV_2 - DI_2 \end{aligned}} \qquad 5.18$$

Or

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = [T] \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \qquad 5.19$$

Eqs (5.18) and (5.19) relate the input variables ($V_1$ and $I_1$) to the output variables ($V_2$ and $-I_2$). Notice that in computing the transmission parameters. $-I_2$ is used rather than $I_2$ because the current is considered to be leaving the network, as shown in Fig. 5.18, as opposed to entering the network as in Fig. 5.1(b). This is done merely for conventional reasons; when you cascade two-ports (output to input). It is most logical to think of $I_2$ as leaving the two-port. It is also customary in the power industry to consider $I_2$ as leaving the two-port.

The two-port parameters in Eqs. (5.18) and (5.19) provide a measure of how a circuit transmits voltage and current from a source to a load. They are useful in the analysis of transmission lines (such as cable and fiber) because they express sending-end variables ($V_1$ and $I_1$) in terms of the receiving-end variables ($V_2$ and $-I_2$). For this reason, they are called transmission parameters. They are also known as ABCD parameters. They are used in the design of telephone systems, microwave networks, and radars.

The transmission line parameter is best illustrated by taking on a very simple, practical example using a T-type network:

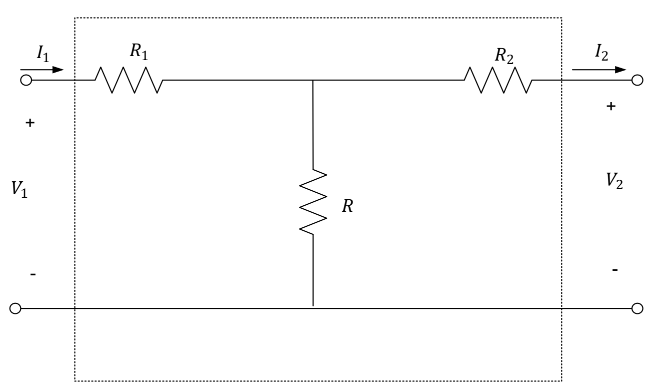



**Figure 5.17 A T-type network**

(For $I_2$, negative reversal, means $I_2$ pointing +ive left is equivalent to $-I_2$ pointing –ve right)

Before taking on the actual example, a quick method for determining $t_{11}, t_{12}, t_{21}, t_{22}$, or equivalently, A, B, C, D respectively, is in order:

From set 4 above:

$$t_{11} = \left.\frac{V_1}{V_2}\right|_{I_2=0} \qquad 5.20$$

A numeric ("unitless") being a quotient of two like quantities.

$$t_{12} = -\left.\frac{V_1}{I_2}\right|_{V_2=0} \qquad 5.21a$$

With the unit of "Ohms"

$$t_{21} = \left.\frac{I_1}{V_2}\right|_{I_2=0} \qquad 5.21b$$

With the unit of mhos (Ʊ) or Siemens(S)

$$t_{22} = -\left.\frac{I_1}{I_2}\right|_{V_2=0} \qquad 5.22$$

Again, with no unit

So, transmission parameters are another version of "hybrid" since it's a mixture of different units (and no units). In keeping faith with the earlier designations, $t_{11}\left(-\frac{V_1}{V_2}\right)$ can be conceptualized as some port of "reverse (negative) voltage gain', while $t_{22}\left(-\frac{I_1}{I_2}\right)$ can be thought of as "reverse because itse(negative) current gain" $(= \propto_{21})$. $t_{12}$ is (negative) transfer impedance function $-z_{12}(s)$, while $t_{21}$ is (positive) transfer admittance function $y_{12}$ S.

In summary

$$A = t_{11} = \left.\frac{V_1}{V_2}\right|_{I_2=0}, \text{open circuit reverse voltage gains.} \qquad 5.23$$

$$C = t_{21} = \frac{I_1}{V_2}\bigg|_{I_2=0}, \text{open circuit transfer admittance} \qquad 5.24$$

$$B = t_{12} = \frac{V_1}{(-I_2)}\bigg|_{V_2=0}, \text{short circuit transfer impedance} \qquad 5.25$$

$$D = t_{22} = \frac{I_1}{(-I_2)}\bigg|_{V_2=0}, \text{short circuit reverse current gain} \qquad 5.26$$

To determine the ABCD parameters by "direct" (recommended) method, let's refer to the $T$ network on Fig. 5.17 by way of illustration.

$$\Rightarrow A = \frac{V_1}{V_2}\bigg|_{I_2=0} = I_1\frac{(R_1+R)}{I_1(R)} = \frac{(R_1+R)}{R(numeric)},$$

After recalling mesh analysis and noting the various impedance (resistance) being "seen" by $V_1, V_2$, respectively and applying Ohm's law.

$R_2$ is "dead" owing to the zeroing of $I_2$

$$C = \frac{I_1}{V_2}\bigg|_{I_2=0} = \frac{I_1}{I_1(R)} = \frac{1}{R}\,\mho\ (mho) \text{ or siemens (S)}$$

Next, we fine

$$D = \frac{I_1}{-I_2}\bigg|_{V_2=0} - I_2 = I_1\frac{R}{(R+R_2)} = \frac{I_1}{(-I_2)} = \frac{(R+R_2)}{R} \quad it\ is\ unitless\Big\}$$

Because it's easier and merely involves current division rule

Lastly, with $V_2$ shorted, $R$ and $R_2$ are in parallel, and together they're in series with $R_1$ Ohm's law at input port

$$\Rightarrow \frac{V_1}{I_1} = R_1 + (R \parallel R_2)$$

$$B = \frac{V_1}{-I_2}\bigg|_{V_2=0} = \frac{I_1\left[R_1 + \frac{RR_2}{(R+R_2)}\right]}{(-I_2)}$$

From above $\frac{I_1}{(-I_2)} = (R + R_2)/R$

$$\Rightarrow \quad B = \left(\frac{R+R_2}{R}\right)\left(\frac{R_1R + R_1R_2 + RR_2}{R+R_2}\right)$$

$$= \frac{(R_1R + R_1R_2 + RR_2)}{R}\Omega$$

Because it possesses the dimension of resistance as unit

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \frac{[R_1 + R]}{R} & \frac{[R_1R + R_1R_2 + RR_2]}{R}\Omega \\ \frac{1}{R}\Omega & \frac{[R + R_2]}{R} \end{pmatrix}$$

**Example 5.5:** Set the values of $R_1 = 2\ \Omega,\ R = 10\ \Omega,\ R_2 = 4\ \Omega$ in the T-type network of Fig. 5.17. Determine the transmission line parameters.

Solution:

$$A = \frac{(2 + 10)}{10} = 1.2$$

$$B = \frac{[(2)(10) + (2)(4) + (10)(4)]}{10} = \frac{68}{10} = 6.8\ \Omega$$

$$C = \frac{1}{R} = \frac{1}{10} = 0.1\ \mho(\text{mho})\text{or siemer (S)}$$

$$D = \frac{(10 + 4)}{10} = 1.4$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1.2 & 6.8\ \Omega \\ 0.1\ \mho & 1.4 \end{pmatrix}$$

The determinant of this square matrix is:

$$AD - BC = (1.2)(1.4) - (6.8)(0.1) = 1.68 - 0.68 = 1$$

And leads us, by chance to the statement:

When the determinant $AD - BC = t_{11}t_{22} - t_{12}t_{21}$ is equal to **unity,** then the network is known as a **reciprocal network**.

Meaning that in terms of the transmission or inverse transmission parameters, a network is reciprocal if

$$\boxed{\boldsymbol{AD - BC = 1,\ \ ad - bc = 1}}$$

A transmission parameter (ABCD) network is said to be ***symmetric*** (possess symmetry) if: $\mathbf{A} = \mathbf{D}$. Is the network in the foregoing example symmetric or not?

**Example 5.6:** Lets double the respective values of the given data in **Example 5.5** for the T-type network of Fig. 5.17 to be $R_1 = 4\ \Omega, R = 20\ \Omega, R_2 = 8\ \Omega$. Determine the ABCD parameters.

Solution:

$$\Rightarrow \quad A = \frac{(4+20)}{20} = 1.2$$

$$B = \frac{[(4)(20) + (4)(8) + (20)(8)]}{20} = \frac{272}{20} = 13.6\ \Omega$$

$$C = \frac{1}{20} = 0.05\ \mho$$

$$D = \frac{(20+8)}{20} = 1.4$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1.2 & 13.6\ \Omega \\ 0.05\ \mho & 1.4 \end{pmatrix}$$

One can see that, upon doubling each of the $R\ is$; $A$ and $D$ are unaffected (the numeric remain unchanged); $B$ is doubled from its previous value; and $C$ is halved from its former value!

The determinant of this square matrix,

$$\Delta = (1.2)(1.4) - (13.6)(0.05) = 1.68 - 0.68 = 1$$

As before so, saying the network elements up or down all by the same factor, does not alter the reciprocity of a given $T$ network.

**Example 5.7**: Determine the ABCD parameters of the $\pi$-type network shown in Fig. 5.18.

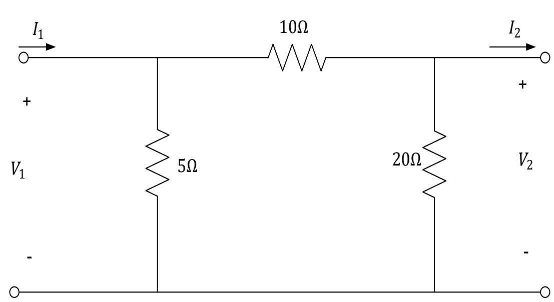



**Figure 5.18 A typical $\pi$-network**

Now, for a look at a pie $-(\pi -)[or\ \Delta - (delta -]$ connected two-port network:

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{I_1[5 \parallel (10+20)]}{I_1\left[\frac{5}{5+10+20}\right] \times 20} = \frac{(5)(30)/(5+30\,)}{(5\times 20)/35}$$

$$= \frac{(5)(30)(35)}{(5\times 20)(35)} = 1.5$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{I_1}{I_1(5\times 20)/35} = \frac{35}{100} = 0.35\ \mho$$

$$D = \left.\frac{I_1}{-I_2}\right|_{V_2=0} = \frac{I_1}{I_1\left[\frac{5}{5+10}\right]} = \frac{15}{5} = 3$$

20 Ω having been shorted out and then using current division

$$B = \left.\frac{V_1}{-I_2}\right|_{V_2=0} = \frac{I_1(5 \parallel 10)}{\left(\frac{I_1}{3}\right)} = \frac{\left[\frac{5\times 10}{(5+10)}\right]}{\left(\frac{1}{3}\right)}$$

Using the result for B above $= \dfrac{50\times 3}{15} = 10\ \Omega$

$$t = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1.5 & 10\ \Omega \\ 0.35\ \mho & 3 \end{pmatrix}$$

It should be pointed out that, in each of the two cases above, i.e., the $T-$ and $\pi-$ connected networks, putting $I_2$ “properly” without a negative sign, would merely result in negating the values for B and D. this is trivial as long as we are clear about the sign convention!

Let’s examine what results when two $T-$ connected two-port networks are connected in cascade.

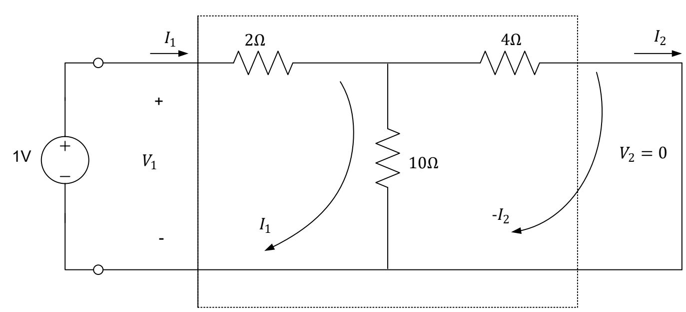



**Figure 5.19 T-type network connected in cascade**

**Example 5.8:** Let's recall the $T-$ connected network in Fig. 5.17 and use an alternative (less recommended) method for finding the transmission (ABCD) parameters by imposing a 1 V source voltage as an input voltage $V_1$ as seen in Fig. 5.19.

Solution:

We "impose" 1 V source (Fig.5.19) representing $V_1$, which is an independent variable along with $I_1$

$$\text{First, we determine } \mathrm{t_{12}} = \mathrm{B} = \mathrm{B} = \left.\frac{\mathrm{V_1}}{-\mathrm{I_2}}\right|_{\mathrm{V_2}=0}$$

$$\mathrm{R_{eq}} = 2 + (10 \parallel 4) = 2 + \frac{40}{14} = \frac{68}{14}$$

$$\text{Current division: } -\mathrm{I_2} = \left[\frac{(1\mathrm{V})}{\left(\frac{68}{14}\right)\Omega}\right] \times \left[\frac{10}{(10+4)}\right] = \left(\frac{14}{68}\right) \times \left(\frac{10}{14}\right) = \frac{10}{68}$$

$$\Longrightarrow \quad \mathrm{B} = \frac{1}{(-\mathrm{I_2})} = \frac{68}{10} = 6.8\ \Omega$$

And so on with A, C, and D

Mesh 1: $\quad V_1 = 12I_1 + 10I_2 \qquad$ (i)

Mesh 2: $\quad V_2 = 10I_1 + 14I_2 \qquad$ (ii)

From (ii) $\quad 10I_1 = V_2 - 14I_2$

Dividing (ii) by 10

$$\text{z}\Longrightarrow \quad I_1 = 0.1V_2 - 1.4I_2 \qquad \text{(iii)}$$

Comparison Eq (iii) with the second equation in Eq.5.18 shows that:

$$C = 0.1\ \mho \quad and\ D = 1.4$$

From mesh 1:
$$V_1 = 12(0.1V_2 - 1.4I_2) + 10I_2$$
$$V_1 = 1.2V_2 - 16.8I_2 + 10I_2$$
$$V_1 = 1.2V_2 - 6.8I_2, \qquad \text{(iv)}$$

Comparison Eq (iv) with the first equation in Eq. 5.18 shows that

$$A = 1.2, B = 6.8$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1.2 & 6.8\ \Omega \\ 0.1\ \mho & 1.4 \end{pmatrix}, \qquad as\ before$$

Even though I do not recommend this later method because of its cumbersomeness, I'd admit that some might find it actually more palatable than the previous method owing to the fact (as it appears) that it might be impervious to error. However, this is for those who feel at home with mesh analysis or node-voltage analysis (at this stage everybody should!). note, however, that in this case, the current $I_2$ is in anticlockwise, not clockwise, direction as in classical mesh analysis.

**Example 5.9**: Find the transmission parameters for the two-port network in Fig. 5.20

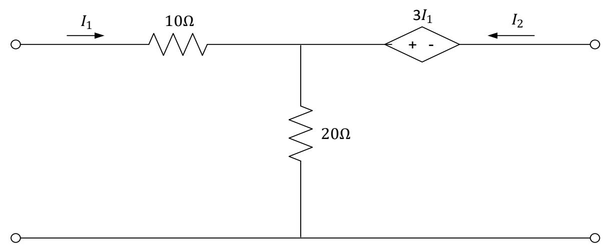



**Figure 5.20**

Solution:

To determine A and C, we leave the output port open as in Fig. 5.21(a) so that $I_2 = 0$ and place a voltage source $V_1$ at the input pon. We have

$$V_1 = (10 + 20)I_1 = 30I_1 \quad and \quad V_2 = 20I_1 - 3I_1 = 17I_1$$

Thus,

$$A = \frac{V_1}{V_2} = \frac{30I_1}{17I_1} = 1.765, \quad C = \frac{I_1}{V_2} = \frac{I_1}{17I_1} = 0.0588\ \mathrm{S}$$

To obtain B and D, we short-circuit the output port so that $V_2 = 0$ as shown in Fig. 5.21(b) and place a voltage source $V_1$ at the input port. At node a in the circuit of Fig. 5.21(b), KCL gives

$$\frac{V_1 - V_o}{10} - \frac{V_o}{20} + I_2 = 0 \qquad 5.27$$

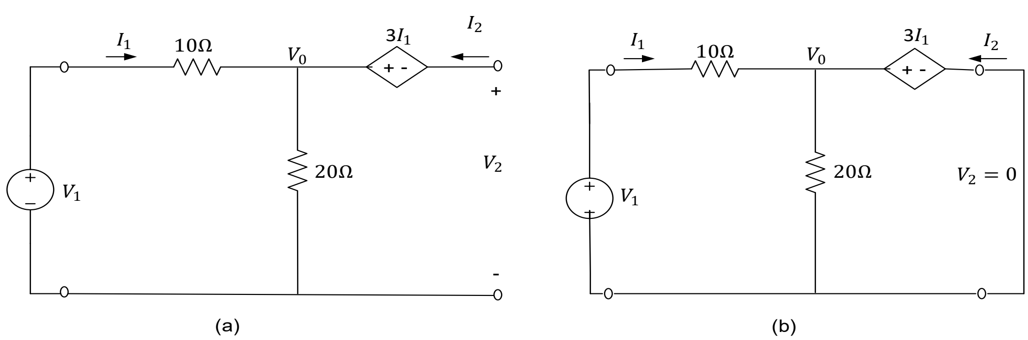



**Figure 5.21 (a) finding A and C (b) finding B and D**

But $V_o = 3I_1$ and $I_1 = (V_1 - V_o)/10$. Combining these gives

$$V_o = 3I_1 V_1 = 13I_1 \quad 5.28$$

Substituting $V_o = 3I_1$ into Eq (5.27) and replacing the first term with $I_1$

$$I_1 - \frac{3I_1}{20} + I_2 = 0 \quad \Longrightarrow \quad \frac{17}{20}I_1 = -I_2$$

Therefore

$$D = -\frac{I_1}{I_2} = \frac{20}{17} = 1.176, \quad B = -\frac{V_1}{I_2} = \frac{-13I_1}{\left(-\frac{17}{20}\right)I_1} = 15.29\ \Omega$$

**Example 5.10**: The ABCD parameters of the two-port network in Fig.5.22 are

$$\begin{bmatrix} 4 & 20\ \Omega \\ 0.1\ \text{S} & 2 \end{bmatrix}$$

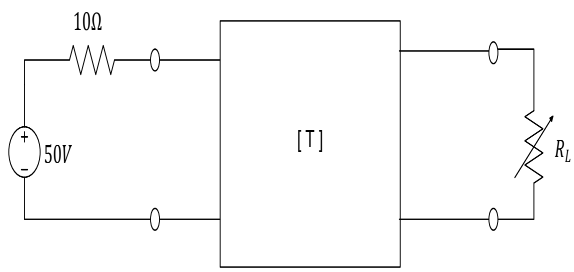



**Figure 5.22**

The output port is connected to a variable load for maximum power transfer. Find $R_L$ and the maximum power transferred.

Solution

What we need is to find the Thevenin equivalent ($Z_{Th}$ and $V_{Th}$,) at the load or output port. We find $Z_{Th}$, using the circuit In Fig. 5.23(a). Our goal is to get $Z_{Th} = \frac{V_2}{I_2}$. Substituting the given ABCD parameters into Eq. (5.28), we obtain

$$V_1 = 4V_2 - 20I_2 \quad 5.29.1$$

$$I_1 = 0.1V_2 - 2I_2 \quad 5.29.2$$

At the input port, $V_1 = -10I_1$. Substituting this into Eq. (5.29.1) gives

$$-10I_1 = 4V_2 - 20I_2 \quad 5.29.3$$

$$I_1 = -0.4V_2 + 2I_2$$

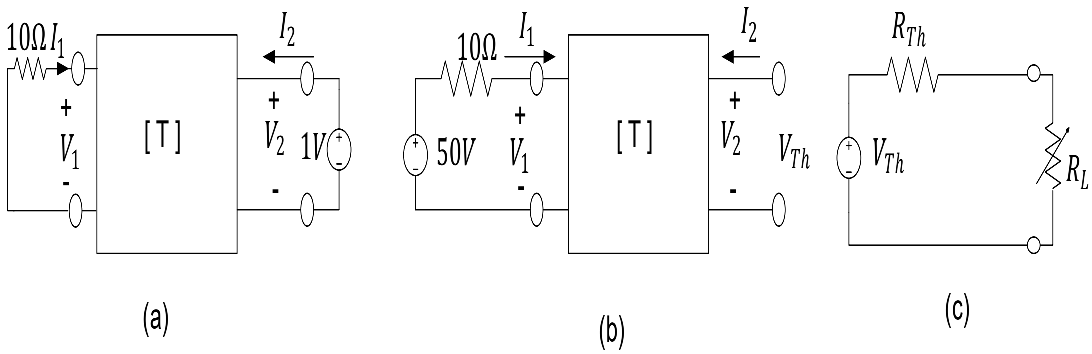



**Figure 5.23 (a) finding $Z_{Th}$ (d) finding $V_{Th}$ (c) finding R$_L$ for maximum power transfer**

Setting the right-hand sides of Eqs. (5.29.2) and (5.29.3) equal.

$$0.1V_2 - 2I_2 = -0.4V_2 + 2I_2 \quad \Longrightarrow \quad 0.5V_2 = 4I_2$$

Hence, $$Z_{Th} = \frac{V_2}{I_2} = \frac{4}{0.5} = 8\ \Omega$$

To find $V_{Th}$, we use the circuit in Fig. 5.23(b). At the output port $I_2 = 0$and at the input port $V_1 = 50 - 10I_1$. Substituting these into Eqs. (5.29.1) and (5.29.2).

$$50 - 10I_1 = 4V_2 \qquad 5.29.4$$

$$I_1 = 0.1V_2 \qquad 5.29.5$$

Substituting Eq. (5.29.5) into Eq. (5.29.4),

$$50 - V_2 = 4V_2 \quad \Longrightarrow \quad V_2 = 10$$

Thus. $$V_{Th} = V_2 = 10\ \mathrm{V}$$

The equivalent circuit is shown in Fig. 5.23(c). For maximum power transfer,

$$R_L = Z_{Th} = 8\Omega$$

The maximum power is

$$P = I^2R_L = \left(\frac{V_{Th}}{2R_L}\right)^2 R_L = \frac{V_{Th}^2}{4R_L} = \frac{100}{4 \times 8} = 3.125\ \mathrm{W}$$

### 5.4 Relationships Between Parameters

Since the six sets of parameters relate the same input and output terminal variables of the same two-port network, they should be interrelated. If two sets of- parameters exist, we can relate one set to the other set. Let us demonstrate the process with two examples.

Given the $z$ parameters, let us obtain the $y$ parameters. From Eq. (5.2),

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = [z] \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \qquad 5.30$$

Or

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = [z]^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \qquad 5.31$$

Also, from Eq. (5.9),

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = [y] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \qquad 5.32$$

Comparing Eqs. (5.31) and (5.32), we have that

$$[y] = [z]^{-1} \qquad 5.33$$

The Adjoint or the $[z]$ matrix is

$$\begin{bmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{bmatrix}$$

and its determinant is

$$\Delta_z = z_{11}z_{22} - z_{12}z_{21}$$

Substituting these into Eq. (5.33), we get

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \frac{\begin{bmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{bmatrix}}{\Delta_z} \qquad 5.34$$

Equating terms yields

$$y_{11} = \frac{z_{22}}{\Delta_z}, \quad y_{12} = -\frac{z_{12}}{\Delta_z}, \quad y_{21} = \frac{z_{21}}{\Delta_z}, \quad y_{22} = \frac{z_{11}}{\Delta_z} \qquad 5.35$$

Table 5.1 provides the conversion formulas for the six sets of two port parameters. Given one set of parameters, Table 5.1 can be used to find other parameters. For example, given the $T$ parameters, we find the corresponding $h$ parameters in the fifth column of the third row.

**Table 5.1 Conversion of two-port parameters**

| | z | | y | | T | |
|---|---|---|---|---|---|---|
| $z$ | $z_{11}$ | $z_{12}$ | $\frac{y_{22}}{\Delta_y}$ | $\frac{y_{12}}{\Delta_y}$ | $\frac{A}{C}$ | $\frac{\Delta_T}{C}$ |
| | $z_{21}$ | $z_{22}$ | $\frac{y_{21}}{\Delta_y}$ | $\frac{y_{11}}{\Delta_y}$ | $\frac{1}{C}$ | $\frac{D}{C}$ |
| $y$ | $\frac{z_{22}}{\Delta_z}$ | $-\frac{z_{12}}{\Delta_z}$ | $y_{11}$ | $y_{12}$ | $\frac{D}{B}$ | $-\frac{\Delta_T}{B}$ |
| | $-\frac{z_{21}}{\Delta_z}$ | $\frac{z_{11}}{\Delta_z}$ | $y_{21}$ | $y_{22}$ | $-\frac{1}{B}$ | $\frac{A}{B}$ |
| $T$ | $\frac{z_{11}}{z_{21}}$ | $\frac{\Delta_z}{z_{21}}$ | $-\frac{y_{22}}{y_{21}}$ | $-\frac{1}{y_{21}}$ | $A$ | $B$ |
| | $\frac{1}{z_{21}}$ | $\frac{z_{22}}{z_{21}}$ | $-\frac{\Delta_y}{y_{21}}$ | $-\frac{y_{11}}{y_{21}}$ | $C$ | $D$ |

Also, given that $z_{21} = z_{12}$ for a reciprocal network, we can use the table to express this condition in terms of other parameters. It can also be shown that

But

$$[t] \neq [T]^{-1} \qquad 5.36$$

**Example 5.11**: Find $[z]$ of a two-port network if

$$[T] = \begin{bmatrix} 10 & 1.5\ \Omega \\ 2\ \text{S} & 4 \end{bmatrix}$$

Solution

If $A = 10,\ B = 1.5,\ C = 2,\ D = 4$, the determinant of the matrix is

$$\Delta_T = AD - BC = 40 - 3 = 37$$

From Table 5.1

$$z_{11} = \frac{A}{C} = \frac{10}{2} = 5, \quad z_{12} = \frac{\Delta_T}{C} = \frac{37}{2} = 18.5$$

$$z_{21} = \frac{1}{C} = \frac{1}{2} = 0.5, \quad z_{22} = \frac{D}{C} = \frac{4}{2} = 2$$

Thus,

$$[z] = \begin{bmatrix} 5 & 18.5 \\ 0.5 & 2 \end{bmatrix} \Omega,$$

**Example 5.12**: Obtain the $y$ parameters of the op amp circuit in Fig. 5.24. Show that the circuit has no parameters.

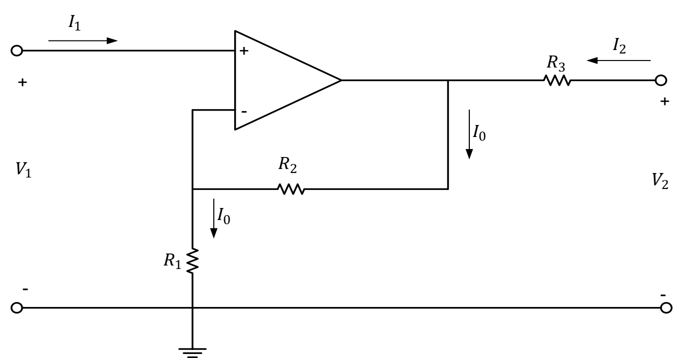



**Figure 5.24**

Solution:

Since no current can enter the input terminals of the op amp. $I_1 = 0$, which can be expressed in terms of $V_1$ and $V_2$ as

$$I_1 = 0V_1 + 0V_2 \qquad 5.37$$

Comparing this with Eq. (5.12) gives

$$y_{11} = 0 = y_{12}$$

Also,

$$V_2 = R_3 I_2 + I_o(R_1 + R_2)$$

where $I_o$, is the current through $R_1$ and $R_2$. But $I_o = V_1/R_1$. Hence,

$$V_2 = R_3 I_2 + \frac{V_1(R_1 + R_2)}{R_1}$$

which can be written as

$$I_2 = -\frac{(R_1 + R_2)}{R_1 R_3} V_1 + \frac{V_2}{R_3}$$

Comparing this with Eq. (5.12) shows that

$$y_{21} = -\frac{(R_1 + R_2)}{R_1 R_3}, \quad y_{22} = \frac{1}{R_3}$$

The determinant of the $[y]$ matrix is

$$\Delta_y = y_{11}y_{22} - y_{12}y_{21} = 0$$

Since $\Delta_y = 0$. the $[y]$ matrix has no inverse; therefore, the $[z]$ matrix does not exist according to Eq. (5.33). Note that the circuit is not reciprocal because of the active element.

## 5.5 For Two-Cascaded Two-Port Network

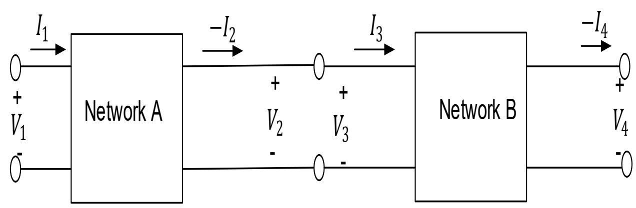



**Figure 5.25 Cascaded Two port network**

$$A: V_1 = t_{A_{11}} V_2 - t_{A_{12}} I_2 = t_{A_{11}} V_3 + t_{A_{12}} I_3 \qquad 5.38$$

$$I_1 = t_{A_{21}} V_2 - t_{A_{22}} I_2 = t_{A_{21}} V_3 + t_{A_{22}} I_3$$

$$B: \ V_3 = t_{B_{11}} V_4 - t_{B_{12}} I_4 \qquad 5.39$$

$$I_3 = t_{B_{21}} V_4 - t_{B_{22}} I_4$$

So, $V_1 = t_{A_{11}}\left(t_{B_{11}} V_4 - t_{B_{12}} I_4\right) + t_{A_{12}}\left(t_{B_{21}} V_4 - t_{B_{22}} I_4\right)$

$$= \left(t_{A_{11}} t_{B_{11}} + t_{A_{12}} t_{B_{12}}\right) V_4 - \left(t_{A_{11}} t_{B_{12}} + t_{A_{12}} t_{B_{22}}\right) I_4$$

$$I_1 = t_{A_{21}}\left(t_{B_{11}} V_4 - t_{B_{12}} I_4\right) + t_{A_{22}}\left(t_{B_{21}} V_4 - t_{B_{22}} I_4\right)$$

$$= -\left(t_{A_{21}} t_{B_{21}} + t_{A_{22}} t_{B_{22}}\right) I_4$$

$$\text{C:}\left(t_{A_{12}} t_{B_{11}} + t_{A_{22}} t_{B_{21}}\right) V_4 \qquad 5.40$$

$$\text{D:}\left(t_{A_{11}} t_{B_{12}} + t_{A_{12}} t_{B_{22}}\right) I_4 \qquad 5.41$$

So, to transit from port 1 to port 4:

Combining Eqs.(5.38 to 5.41) we have Eq.5.42

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} t_{A_{11}} t_{B_{11}} + t_{A_{12}} t_{B_{21}} & t_{A_{11}} t_{B_{12}} + t_{A_{12}} t_{B_{22}} \\ t_{A_{21}} t_{B_{11}} + t_{A_{22}} t_{B_{21}} & t_{A_{21}} t_{B_{12}} + t_{A_{22}} t_{B_{22}} \end{pmatrix} \quad 5.42$$

But: $$\begin{pmatrix} t_{A_{11}} & t_{A_{12}} \\ t_{A_{21}} & t_{A_{22}} \end{pmatrix} \begin{pmatrix} t_{B_{11}} & t_{B_{12}} \\ t_{B_{21}} & t_{B_{22}} \end{pmatrix} = \begin{pmatrix} t_{A_{11}} t_{B_{11}} + t_{A_{12}} t_{B_{21}} & t_{A_{11}} t_{B_{12}} + t_{A_{12}} t_{B_{22}} \\ t_{A_{22}} t_{B_{11}} + t_{A_{22}} t_{B_{21}} & t_{A_{21}} t_{B_{21}} + t_{A_{12}} t_{B_{22}} \end{pmatrix} \quad 5.43$$

Eq.5.43 shows that for two-port networks in a cascade, the net ABCD parameters are just the (successive) products of the individual ABCD parameters. Applications wise, this unique property comes in quite handy!

### 5.6 Further Examples on Two-Port Network

**1.** Determine the z parameters for the circuit shown in Fig 5.26. the output port includes a controlled voltage source.

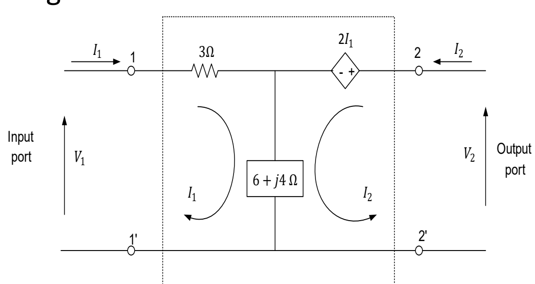



**Figure 5.26**

Solution: since the actual parameters of the circuit are known, and the circuit is relatively simple, the z parameters may be determined by writing the two loop equations.

$$V_1 = [3 + (6 + j4)]I_1 + [6 + j4]I_2 V_2 - 2I_1 = [6 + j4]I_1 + [6 + j4]I_2$$

Simplifying $V_1 = [9 + j4]I_1 + [6 + j4]I_2 V_2 = [8 + j4]I_1 + [6 + j4]I_2$

Thus, the z parameters are

$$z_{11} = (9 + j4)\Omega \;\; z_{12} = (6 + j4)\Omega \;\; z_{21} = (8 + j4)\Omega \;\; z_{22} = (6 + j4)\Omega$$

**2.** Draw the z parameters model for the circuit of Fig.5.26

Solution: see Fig.5.27

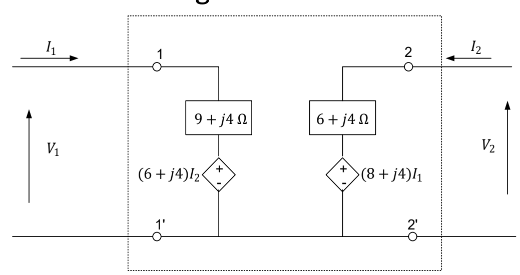



**Figure 5.27**

**3.** The following open-circuit currents and voltage were determined experimentally for an unknown two-port:

$$\left.\begin{aligned} V_1 &= 100\angle 0^o\ V \\ V_2 &= 75\angle 0^o\ V \\ I_2 &= 12.5\angle 0^o\ A \end{aligned}\right|_{I_2=0} \qquad \left.\begin{aligned} V_1 &= 30\angle 0^o\ V \\ V_2 &= 50\angle 0^o\ V \\ I_2 &= 5\angle 0^o\ A \end{aligned}\right|_{I_1=0}$$

Determine the z parameters.

$$z_{11} = \left.\frac{V_1}{I_1}\right| = \frac{100}{12.5} = 8\Omega \; z_{12} = \left.\frac{V_1}{I_2}\right| = \frac{30}{5} = 6\Omega$$

$$z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{75}{12.5} = 6\Omega \; z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = \frac{50}{5} = 10\Omega$$

**4.** Draw a z parameter model for the circuit of **Prob 3**

Solution: see Fig.5.28

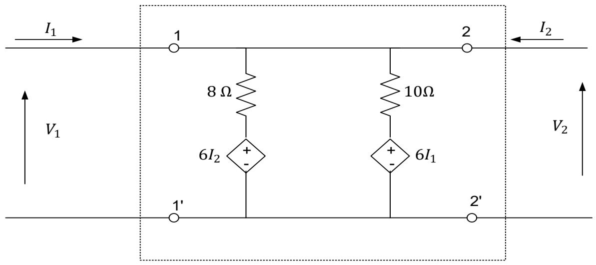



**Figure 5.28**

**5.** Determine the z parameters for the network of Fig 5.29

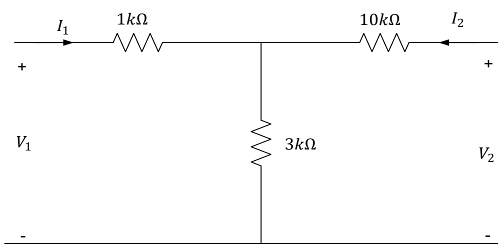



**Figure 5.29**

Solution: the loop equations become

$$V_1 = 4000I_2$$

$$V_2 = 3000I_1 + 13{,}000I_2$$

Thus, $z_{11} = 4\ k\Omega$,

$$z_{12} = z_{21} = 3\ k\Omega z_{22} = 13\ k\Omega$$

**6.** In a T network $Z_1 = 3\ \angle 0^o\Omega$, $Z_2 = 4\angle 90^o\Omega$, $Z_3 = 3\angle -90^o\Omega$. Find the z parameters

Solution:

$$z_{11} = Z_1 + Z_3 = 3\ \angle 0^o + 3\angle{-90^o}$$
$$= 4.242\ \angle{-45^o}\Omega$$
$$z_{12} = z_{21} = Z_3 = 3\ \angle{-90^o}\Omega$$
$$z_{22} = Z_2 + Z_3$$
$$= 4\ \angle 90^o + 3\ \angle{-90^o} = 1\angle 90^o\Omega$$

**7.** Determine the z parameters of a T network having $Z_1 = (3 + j4)\ \Omega$, $Z_2 = 1\angle{-90^o}$, and $Z_3 = (3 + j2)\ \Omega$

Solution:

$$z_{11} = Z_1 + Z_3 = (3 + j4) + (3 + j2)$$
$$= 6+= 8.484\ \angle 45^o\Omega$$
$$z_{12} = z_{21} = Z_3 = 3 + j4 = 3 + j4\ \Omega$$
$$z_{22} = Z_2 + Z_3 = -j1 + (3 + j4)$$
$$= 3 + j3$$
$$= 4.242\ \ \angle 45^o\Omega$$

**8.** Find the z parameters of the network of Fig.5.30

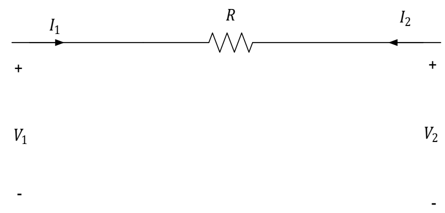



**Figure 5.30**

Solution:

$I_1$ and $I_2$ are not independent, the z parameters cannot be found

**9.** Find the y parameters of the network of Fig.5.30

Solution:

$I_1 = -I_2$, with $V_2 = 0$, we obtain

$$y_{11} = \frac{1}{R} \text{and} \quad y_{21} = -\frac{1}{R}$$

With $V_1 = 0$, we have

$$y_{21} = -\frac{1}{R} \text{and} \quad y_{22} = \frac{1}{R}$$

**10.** Determine the z parameters of the network of Fig.5.31

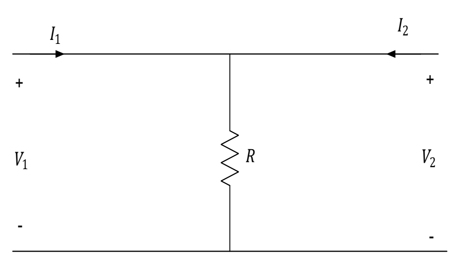



**Figure 5.31**

Solution:

since $V_1 = V_2$

With $I_2 = 0$ $z_{11} = R$

and $z_{21} = R$

With $I_1 = 0$ $z_{12} = R$

and $z_{22} = R$

**11.** Obtain the y parameters of the network of Fig.5.31

Solution: since $V_1$ and $V_2$ are not independent, the y parameters cannot be found

**12.** Find the z parameters of the network of Fig.5.31

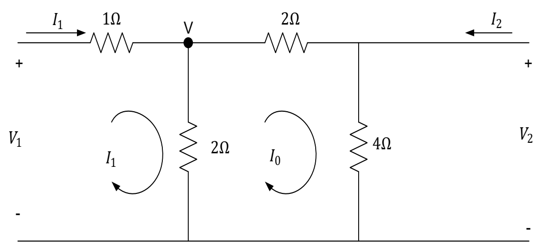



**Figure 5.32**

Solution: $I_2 = 0$ $V_2 = 4I$

and $2I_1 = (2 + 2 + 4)I = 8I = 2V_2$

Thus, $z_{21} = \frac{V_2}{I_1} = \frac{2}{2} = 1\Omega = z_{21}$

$$V_1 = (1)I_1 + 2(I_1 - I)$$

$$= (1)I_1 + 2\left(I_1 - \frac{1}{4}I_1\right) = \frac{5}{2}I_1$$

Thus,

$z_{11} = \frac{V_1}{I_1} = \frac{5}{2}\ \Omega$

$I_1 = 0$ $V_2 = 2I_2$

or $z_{22} = \frac{V_2}{I_2} = 2\ \Omega$

**13.** Find the y parameters of the network of Fig. 5.32

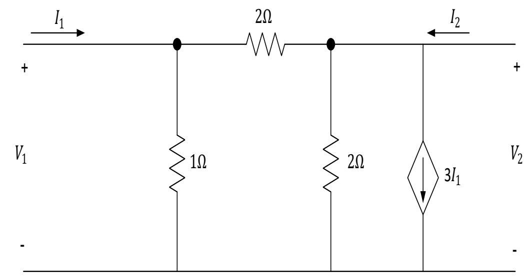



**Figure 5.33**

Solution:

Redrawing the diagram of Fig.5.32 we have Fig. 5.33

$$V_2 = 0 I_1 = \frac{V_1}{1+1} = \frac{V_1}{2}\ and\ \frac{I_1}{V_1} = y_{11}$$

$$= \frac{1}{2}\ S$$

$$I_2 = -\frac{I_1}{2} = -\frac{1}{2}\left(\frac{V_1}{2}\right)$$

$$= -\frac{1}{4}V_1\ \ and\ \frac{I_2}{V_1} = y_{12} = -\frac{1}{4} = y_{21}$$

$$V_2 = 0; I_2 = \frac{\frac{V_2}{8}}{5}$$

Hence, $\frac{I_2}{V_2} = y_{22} = \frac{1}{\frac{8}{5}} = \frac{5}{8}$ S

**14.** Find the z parameters of the network of Fig.5.35b

Solution:

$$z_{11} = R_y + R_y + 2R_y,\quad z_{12} = R_y$$

$$= z_{21},\quad z_{22} = R_y + R_y$$

$$= 2R_y$$

**15.** As a 2-port network, determine the transmission parameters of the transformer as shown in Fig 5.36

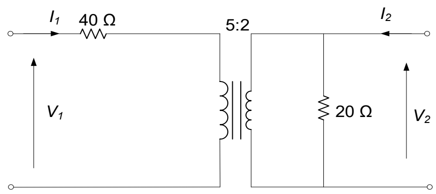



**Figure 5.36**

Solution:

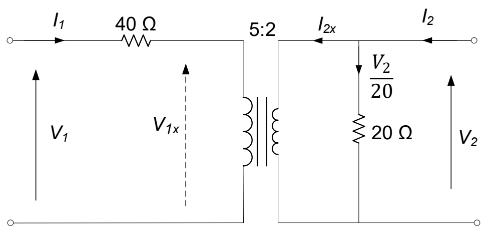



**Figure 5.37**

From Fig. 5.37, the transformer's turns ratio is:

$$\frac{V_{1_x}}{V_2} = \frac{5}{2} = \frac{I_{2_x}}{I_1}$$

$$I_2 = \frac{V_2}{20} + I_{2_x} = \frac{V_2}{20} + 2.5I_1$$

$$\Rightarrow I_1 = -\frac{V_2}{20 \times 2.5} - \frac{I_2}{2.5}$$

$$= -0.02V_2 + 0.4I_2 \qquad *$$

$$V_1 = 40I_1 + V_{1_x} = 40I_1 + 2.5V_2$$

$$= 40(-0.02V_2 + 0.4I_2) + 2.5V_2$$

$$V_1 = -0.8V_2 + 16I_2 + 2.5V_2$$

$$V_1 = 1.7V_2 + 16I_2 \qquad **$$

From * and **, the transmission parameters (ABCD) are then:

$$\begin{bmatrix} 1.7 & 16\ \Omega \\ -0.02\ \mho & 0.4 \end{bmatrix}$$

## 5.7 Images Impedance

This is a concept used in electronic network design and analysis and mostly in filter design: By image impedance is implied the impedance "seen" looking into a port of a network particularly two-port networks (although it's also applicable to networks with more than two parts).

For a two-port network, $Z_i$ stands for the impedance seen looking into port 1 when port 2 is terminated with image impedance $Z_{i2}$. $Z_{i1}$ and $Z_{i2}$ will generally not be equal to except for symmetrical or anti-symmetrical networks.

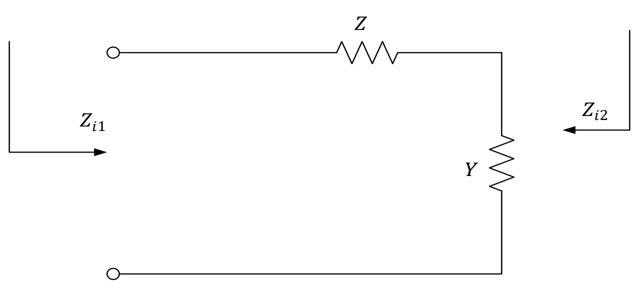



**Figure 5.38 'L' network**

Fig. 5.38 is a simple "L" network with series impedance $Z$ and shunt admittance $Y$, with the respective port image impedances.

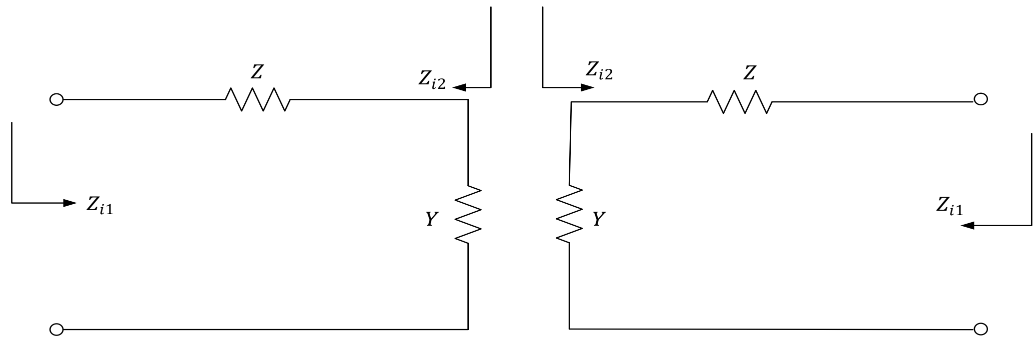



**Figure 5.39 T-section of L-network**

To determine the image impedance, a "T" section is formed from back-to-back L-sections, as in Fig. 5.39.

$$Z_{i1} = Z + \frac{1}{2Y + \frac{1}{Z+Z_{i1}}} \qquad 5.44$$

To determine $Z_{i1}$, we "eliminate" $Z_{i2}$ by forming back-to-back L-section in T-section (The two $Z_{i2}$'s oppose each other) $Z$ and $Z_{i1}$ are in series and their series admittance is now in parallel with the two Y's. As per the rule, their net admittance is determined by simple addition. The reciprocal of this sum, gives rise to an impedance that is in series with the Z on the left and the impedance "seen" by the $Z_{i1}$ on the left is this series combination.

From the foregoing without going through the whole rigmarole,

$$Z_{i1}{}^{2} = Z^{2} + \frac{Z}{Y} \qquad 5.45$$

To determine $Z_{i2}$, form a $\pi$ −section from back-to-back L-section as in Fig. 5.40:

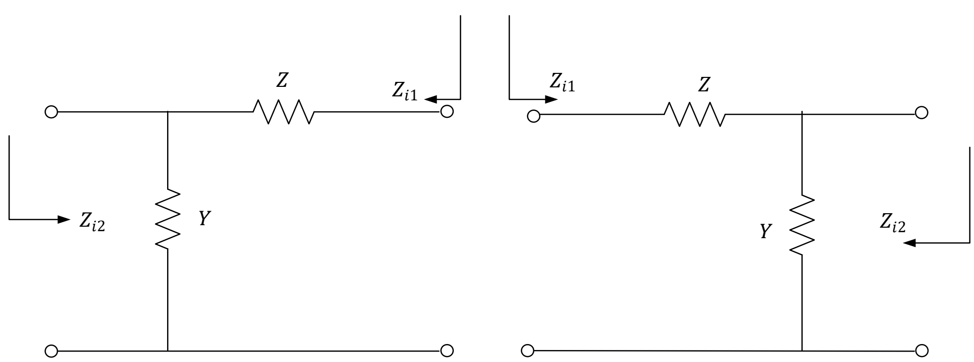



**Figure 5.40 $\pi$-section of L-network**

Working with admittances (which can be easily derived from the "dual" of that for $Z_{i1}$ just treated in Eq 5.44:

$$Y_{i2} = Y + \cfrac{1}{2Z \quad \frac{1}{Y+Y_{i2}}}$$

$$\Longrightarrow \quad \boxed{Y_{i2} = Y^2 + \frac{Y}{Z}} \qquad 5.46$$

Combining Eqs 5.44 and 5.46 we have Eq 5.47

$$\frac{Z}{Y} = {Z_{i1}}^3 - Z^2 = {Y_{i2}}^2 - Y^2$$

$$\frac{Z_{i1}}{Y_{i2}} = \sqrt{\frac{Z^2 + \frac{Z}{Y}}{Y^2 + \frac{Y}{Z}}} = \sqrt{\frac{Z\left(Z + \frac{1}{Y}\right)}{Y\left(Y + \frac{1}{Z}\right)}}$$

$$\approx \sqrt{\frac{Z}{Y}\frac{Z}{Y}} = Z/Y \qquad 5.47$$

For practical determination of image impedance, we measure short-circuit impedance $Z_{sc}$, which is the impedance of port 1 where port 2 is short-circuited, and $Z_{oc}$, that is, impedance of port 1 when port 2 is short-circuited.

$$\Longrightarrow \quad Z_{i1} = \sqrt{(Z_{sc})(Z_{oc})} \qquad 5.48$$

The "black box" configuration of the network is in Fig. 5.1. For filter design, L network is referred to as a half section. Two half-sections in cascade are equivalent to a T section or $\pi$ section, depending on their respective orientations.

$$\left({Z_i}^2 \to \frac{Z}{Y}\right) \qquad 5.49$$

**Definition of image impedance:** The input impedance of an infinitely long chain of cascaded identical networks (with the parts arranged so that like impedance faces like). For reciprocal network $(AD - BC = 1)$

$$Z_{i1} = \sqrt{\frac{AB}{CD}} \qquad 5.50$$

$$Z_{i2} = \sqrt{\frac{CB}{CA}} \qquad 5.51$$

Image propagation term $\psi = \cos^{-1}\sqrt{AD}$ 5.52

$\psi$ for a transmission line segment is equivalent to the (propagation constant of transmission line) x (the length)

When used for filter design, $Z_{i1}$ the image impedance for port 1, is termed $Z_{iT}$, and $Z_{i2}$ is termed $Z_{i\pi}$

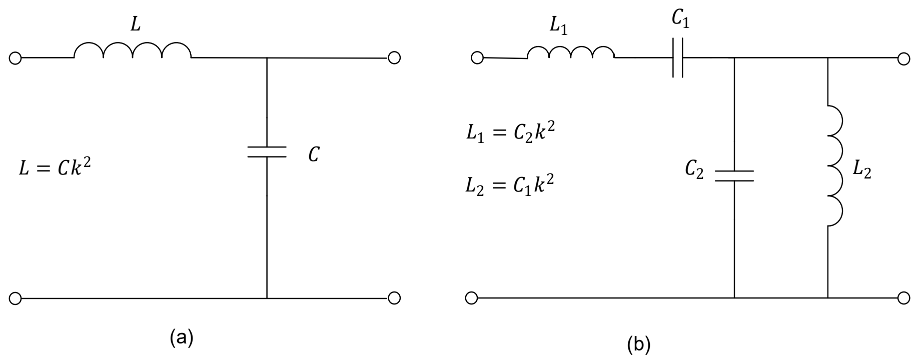



**Figure 5.41 (a) constant 'k' low-pass filter half section (b) Constant 'k' band-pass filter half section**

$$k^2 = Z/Y \qquad 5.53$$

$k(\Omega)$ is the limiting value of $Z_i$ as the size of the section (in terms of values of its components, inductances, capacitances etc.) approached zero, while keeping at its initial value. It also represents the image impedance of the section at resonance, in the case of band-pass filters, or at $\omega = 0$ in the case of a low-pass filter.

For low pass half-section,

$$k = \sqrt{\frac{j\omega L}{j\omega C}} = \sqrt{\frac{L}{C}}$$

For image impedances,

$$\lim_{Z,Y\to 0} Z_i = k \qquad 5.54$$

Image impedance:

$$Z_{iT}{}^2 = Z^2 + k^2$$
$$\frac{1}{Z_{i\pi}{}^2} = Y_{i\pi}{}^2 = Y^2 + \frac{1}{k^2} \qquad 5.55$$

Given that the filters do not contain any resistive elements, image impedance in the pass band of the filter is real, and in the stop band it is purely imaginary.

For low-pass half section:

$$Z_{iT}{}^2 = -(\omega L)^2 + \frac{L}{C} \qquad 5.56$$

Transition occurs at cut-off frequency given by $\omega_c = 1/\sqrt{LC}$

At less than $\omega_c$, $Z_{iT} = L\sqrt{\omega_c{}^2 - \omega^2}$ is real. At $> \omega_c, Z_{iT} = jL\sqrt{\omega^2 - \omega_c{}^2}$ is imaginary. When the electrical properties of a 4-terminal network (2port) are unaffected even after interchanging input and output terminals, the network is called a **symmetrical network**. Otherwise, it is called **asymmetrical or unsymmetrical**.

Asymmetrical networks have the following electrical properties.

1. Iterative impedance
2. Image impedance
3. Image transfer constant

### 5.7.1 Iterative Impedance

It is the impedance measured at one pair of terminals of a network in the chain of infinite networks.

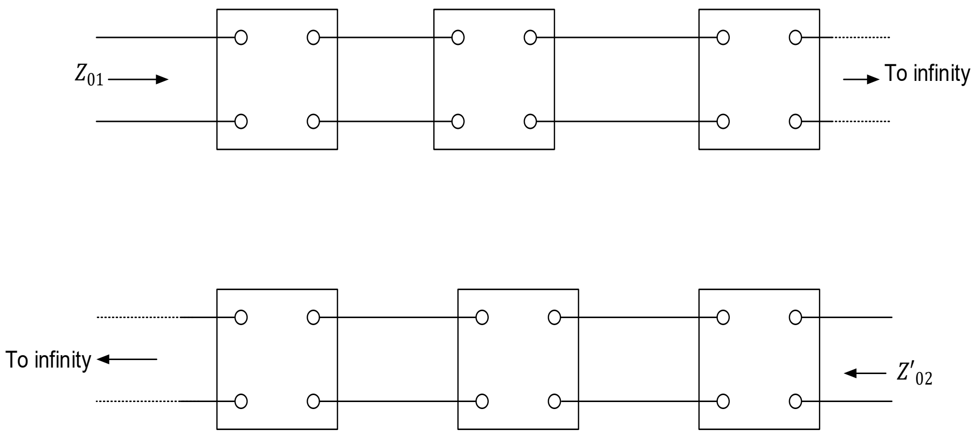



**Figure 5.42 One pair of a network**

It can also be viewed, as the impedance measured at any pair of terminals of the network when the other pair of terminals is terminated in the impedance of the same values as shown in Fig. 5.43.

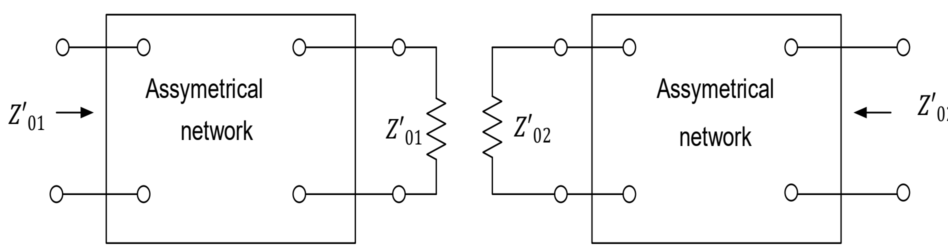



**Figure 5.43**

Iterative impedance for any asymmetrical networks are of different values when measured at different parts of the network, and these are represented by $Z_{01}$ and $Z_{02}$ respectively at port 1 and port 2.

## 5.8 Insertion Loss

When a network is inserted between a generator and a load, load current decreases and hence power delivered to the load also decreases. The loss in power delivered to the load by insertion of the network is known as insertion loss. This is generally expressed in decibels and nepers.

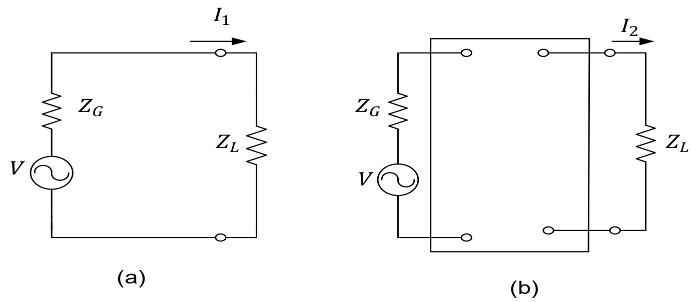



**Figure 5.44**

In Fig.5.44(a), there's no network between the generator and the load. But in Fig.5.44(b), a 4-terminal network is inserted between generator and load.

Current through $Z_L$ with the network inserted is $I_2$.

insertion loss $= \alpha = \ln\left|\frac{I_1}{I_2}\right|$ nepers, or

$\alpha = 20\log\left|\frac{I_1}{I_2}\right|$ decibels

In terms of power ration,

$\alpha = \frac{1}{2}\ln\left|\frac{P_1}{P_2}\right|$ nepers, or

$\alpha = 10\log\left|\frac{P_1}{P_2}\right|$ decibels

So, insertion loss is equivalent to the number of nepers or decibels by which current in load, or power delivered to the load, is charged due to insertion of a network.

When the current delivered to the load is greater than that from the source $(I_2 > I_1)$, then there's negative loss, i.e. insertion gain.

**Example 5.13**: For the circuit Fig. 5.45, determine insertion loss when network N is inserted between load and source.

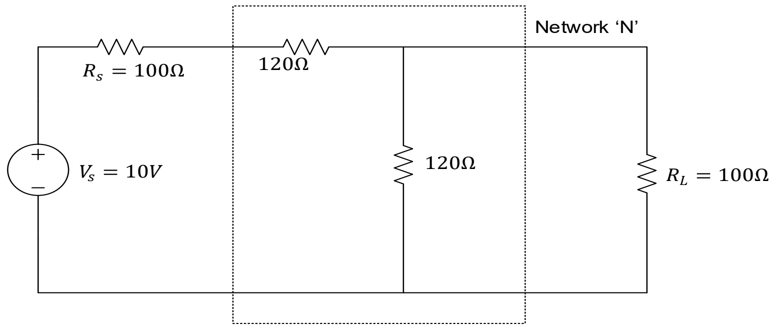



**Figure 5.45**

Solution:

To solve for $I_1$ without N network, consider the circuit of Fig. 5.46.

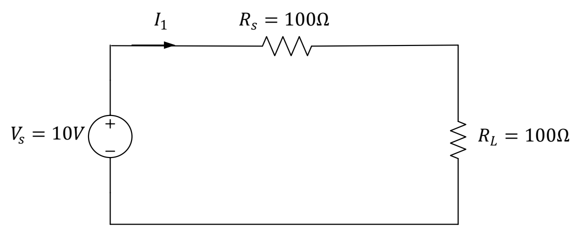



**Figure 5.46 Solving for $I_1$ without network N**

$$I_1 = \frac{V_s}{(R_s + R_L)} = \frac{10}{(100 + 100)} = 0.5 \text{ A}$$

To solve for the total current in the circuit of Fig. 5.45 with N inserted,

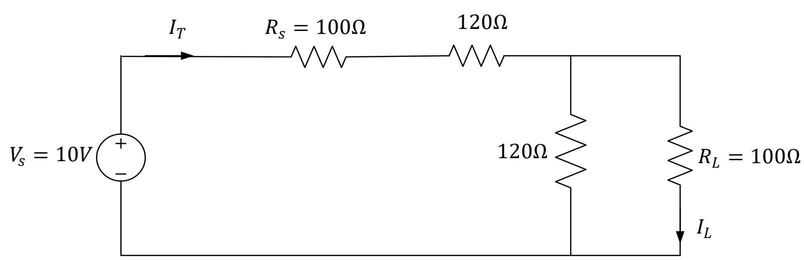



**Figure 5.47 Solving for $I_T$ with network N included**

$$I_T(\text{'I' total}) = \frac{10}{(100 + 120 + 120 \parallel 100)}$$

$$= \frac{10}{\left(220 + \frac{12000}{220}\right)} = \frac{(10)(11)}{(2420 + 6000)}$$

$$= \frac{11}{302} = 0.0364 \text{ A}$$

$I_L$ (by current divider rule) $= 0.364 \left[\frac{120}{(100+120)}\right] = 0.0199$

Insertion loss $\alpha = 20\log\left(\frac{I_1}{I_2}\right) = 20\log\left(\frac{0.05}{0.0199}\right) = 8.00\text{ dB}$, or $\alpha = \ln\left(\frac{0.05}{0.0199}\right) = 0.921$ neper.

**Example 5.14**: For the network Fig. 5.48, determine the insertion loss.

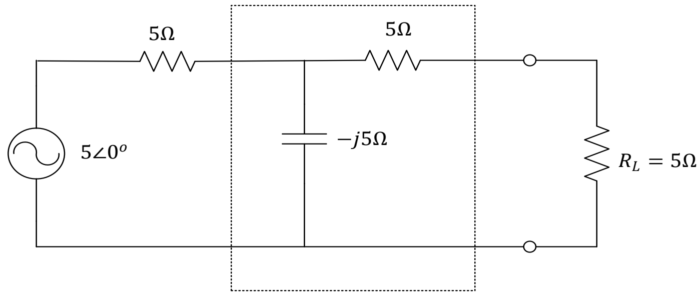



**Figure 5.48**

Solution:

Before insertions we consider Fig. 5.49 to solve for $I_1$ as in the proceeding example.

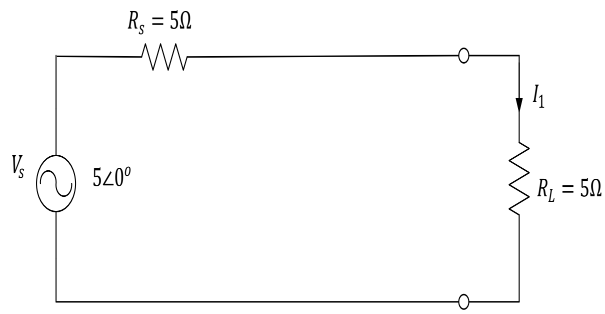



**Figure 5.49**

$$I_1 = \frac{V_s}{(R_s + R_L)} = 5\frac{\angle 0^o}{(5+5)} = 0.5\angle 0^o\text{A}$$

After insertion we consider Fig.5.50 to calculate the total current $I_T$ using voltage divider rule.

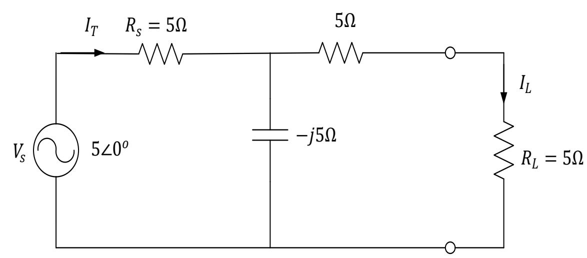



**Figure 5.50**

$$I_T = \frac{5\angle 0^o}{[5 + 10 \parallel (-j5)]} = \frac{5\angle 0^o}{\left[5 + \frac{(10)(-j5)}{(10-j5)}\right]}$$

$$= \frac{5\angle 0^o}{(7-j4)} = \frac{5\angle 0^o}{8.06\ \angle{-29.74^o}} = 0.62\ \angle 29.74^o$$

$$I_L = \frac{0.62\ \angle 29.74^o(-j5)}{10-j5} = \frac{(0.62\angle 29.74^o)(5\angle{-90^o})}{11.18\ \angle 26.57^o}$$

$$0.277\angle{-86.83^o}$$

$$\alpha = 20\log\left|\frac{0.5\angle 0^o}{0.277\angle{-8.83^o}}\right| = 20\log\left|\frac{0.5}{0.277}\right| = 5.13dB$$

## 5.9 Symmetrical Network

Shown in Fig. 5.51 is a typical (T-type) symmetrical network:

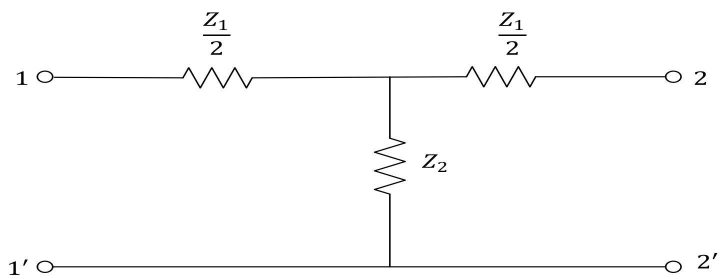



**Figure 5.51**

Characteristics impedance $Z_0 = \sqrt{(Z_{0c})(Z_{sc})}$, where $Z_{0c},\ Z_{sc}$ are open and short circuit impedance, respectively see Figs. 5.53 and 5.54.

Total series arm impedance and shunt arm impedance must be $Z_1$ and $Z_2$ respectively.

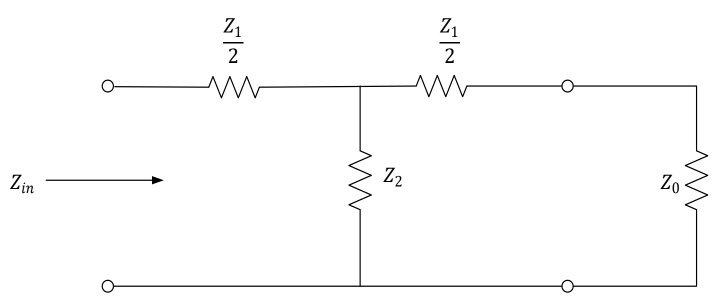



**Figure 5.52 Input impedance looking when looking into the input of Fig 5.51 circuit**

For the circuit of Fig. 5.52,

$$Z_{in} = Z_0 = \frac{Z_1}{2} + \left[Z_2 \parallel \left(\frac{Z_1}{2} + Z_0\right)\right]$$

$$\Rightarrow Z_0 = \frac{Z_1}{2} + \frac{Z_2\left(\frac{Z_1}{2} + Z_0\right)}{Z_2 + \frac{Z_1}{2} + Z_0}$$

$$\Rightarrow \quad Z_0\left(Z_1+\frac{Z_1}{2}+Z_0\right)=\left(\frac{Z_1}{2}\right)\left(Z_2+\frac{Z_1}{2}+Z_0\right)+\frac{Z_1Z_0}{2}+Z_2Z_0$$

$$\Rightarrow Z_2Z_0+\frac{Z_1Z_0}{2}+{Z_0}^2=\frac{Z_1Z_2}{2}+\frac{{Z_1}^2}{4}+\frac{Z_1Z_0}{2}+\frac{Z_1Z_2}{2}+Z_2Z_0$$

$$\Rightarrow \quad {Z_0}^2=\frac{{Z_1}^2}{4}+Z_1Z_2$$

$$\Rightarrow \quad Z_0=\sqrt{\frac{{Z_1}^2}{4}+Z_1Z_2}$$

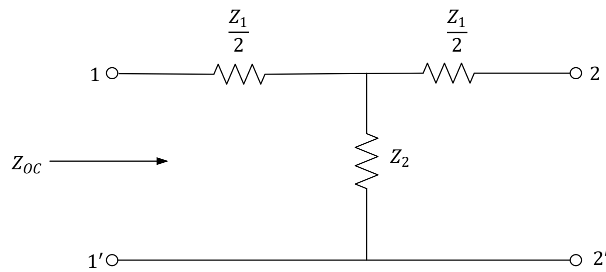



**Figure 5.53 A short circuit Equivalent circuit of Fig 5.51**

For the circuit of Fig. 5.53,

$$Z_{1OC}=Z_{2OC}=Z_{OC}=\frac{Z_1}{2}+Z_2$$

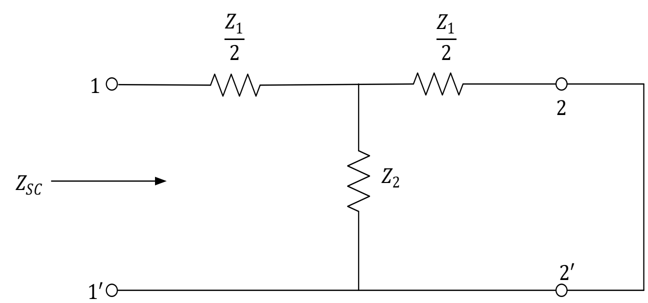



**Figure 5.54 A short circuit Equivalent circuit of Fig 5.51**

For the circuit Fig. 5.53,

$$Z_{1sc}=Z_{2sc}=Z_{sc}=\frac{Z_1}{2}+\left(\frac{Z_1}{2}\right)\Big\|\,Z_2$$

$$=\left(\frac{Z_1}{2}\right)+\frac{\left(\frac{Z_1Z_2}{2}\right)}{\left(\frac{Z_1}{2}+Z_2\right)}$$

$$= \frac{{Z_1}^2}{4} + \frac{Z_1Z_2 + \frac{Z_1Z_2}{2}}{\frac{Z_1}{2} + Z_2}$$

$$(Z_{0c})(Z_{sc}) = \left(\frac{Z_1}{2} + Z_2\right)\left(\frac{Z^2}{4} + Z_1Z_2\right)$$

$$= \frac{{Z_1}^2}{4} + Z_1Z_2 = {Z_0}^2 \Longrightarrow Z_0 = \sqrt{(Z_{0c})(Z_{sc})}$$

= (Geometric mean of open and short circuit impedances measured at any pair of terminals)

**Example 5.15**: For the network shown in the Fig. 5.55, determine the characteristics impedance, open and short circuit impedances

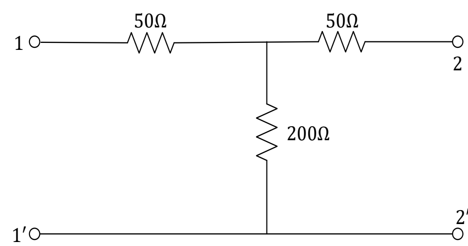



**Figure 5.55**

Solution:

$$Z_0 = \sqrt{\frac{{Z_1}^2}{4} + Z_1Z_2} = \sqrt{\frac{100^2}{4} + (100)(200)} = \sqrt{2{,}500 + 20{,}00} = 150\ \Omega$$

$$Z_{0c} = \frac{Z_1}{2} + Z_2 = 50 + 200 = 250\ \Omega$$

$$Z_{sc} = 50 + 50 \parallel 200 = 50 + \frac{10{,}000}{250} = 90\ \Omega$$

$$\text{Or}, Z_{sc} = \frac{Z_0}{Z_{0c}} = \frac{150}{250} = 90\ \Omega$$

**Example 5.16**: A symmetrical T-network comprising pure resistances has open-and short circuit impedances of $400\ \angle 0^o\ \Omega, 300\angle 0^o\ \Omega$, respectively. Design a symmetrical T-network

Refer to Fig. 5.53.

Solution:

$$Z_{oc} = \frac{Z_1}{2} + Z_2 = 400$$

$$Z_{sc} = \frac{Z_1}{2} + \left(Z_2 \parallel \frac{Z_1}{2}\right) = 300$$

$$\Longrightarrow \quad {Z_{oc}}^2 = \frac{{Z_1}^2}{4} + {Z_2}^2 + Z_1 Z_2 = 160{,}000$$

$$\Longrightarrow \quad {Z_2}^2 = 160{,}000 - \left(\frac{{Z_1}^2}{4} + Z_1 Z_2\right)$$

$${Z_2}^2 = 160{,}000 - {Z_o}^2 = 160{,}000 - (Z_{oc})(Z_{sc})$$
$$= 160{,}000 - (400)(300) = 160{,}000 - 120{,}000 = 40{,}000$$
$$\Longrightarrow \quad Z_2 = 200\ \Omega$$
$$\frac{Z_1}{2} + Z_2 = 400 = \frac{Z_1}{2} + 200 \Longrightarrow \frac{Z_1}{2} = 200\ \Omega$$

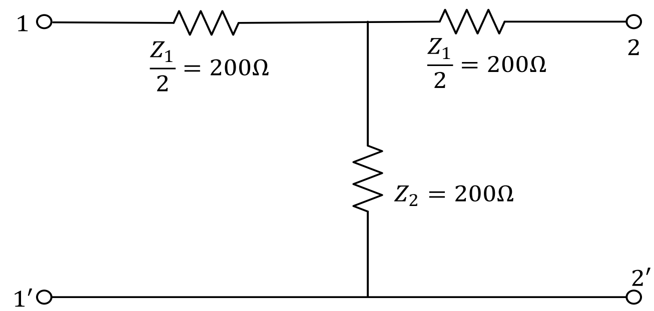



**Figure 5.56**

## 5.10 Exercise

1. A two-port network has transmission parameters with matrix $\begin{pmatrix} 4A & 2B \\ 2C & 4D \end{pmatrix}$

(a) Determine its input impedance at port 1 with $I_2 = 0$

(b) Determine its input impedance at port 2, given that $I_1 = 0$

2. For Fig. 1, given that the transmission parameters of the 2port network are$A = C = 1, B = 2, D = 3,$

(a) Determine the value of $Z_{in}$

(b) Determine same if $I_2$ is reversed

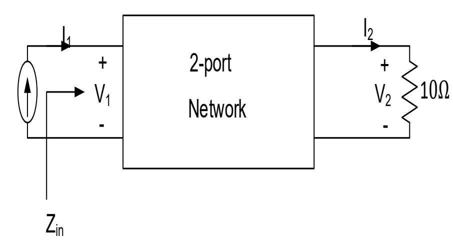



**Figure 1**

3. A two-port network has transmission parameters with matrix $\begin{pmatrix} 2A & B \\ C & 2D \end{pmatrix}$

(a) Determine its input impedance at port 1 with $I_2 = 0$

(b) Determine its input impedance at port 2, given that $I_1 = 0$

4. For Fig. 2, given that the transmission parameters of the 2port network are

$$A = C = 2, B = 4, D = 6,$$

(a) Determine the value of $Z_{in}$

(b) Determine same if $I_2$ is reversed

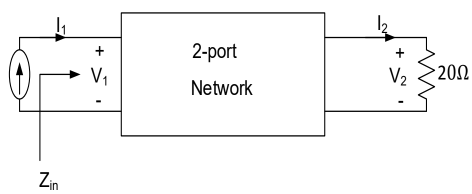



**Figure 2**

5. The 2-port network in Fig. 3 has transmission parameters with matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$

(i) Determine its input impedance at port 1, given that $I_2 = 0$

(ii) Determine its input impedance at port 2, given that $I_1 = 0$

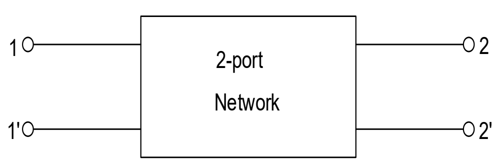



**Figure 3**

6. An ideal transformer, depicted in Fig. 4 has a turns-ratio of 2:1. Considering high voltage side as port 1, and low voltage side as port 2, what are the transmission line parameters of the transformer?

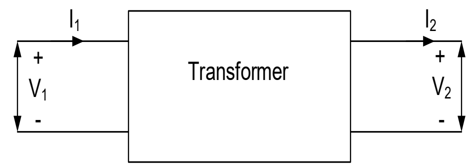



**Figure 4**

7. Shown is a two-port network with transmission matrix T, with parameters $T_{ij}$. A 1 Ω resistor is connected in series at terminals a (port 1) as in Fig. 5. What are the ABCD parameters of the modified 2-port network (the dashed box)?

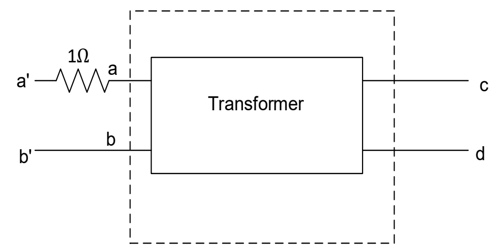



**Figure 5**

8. As a 2-port network, determine the transmission parameters of the transformer as shown in Fig. 6

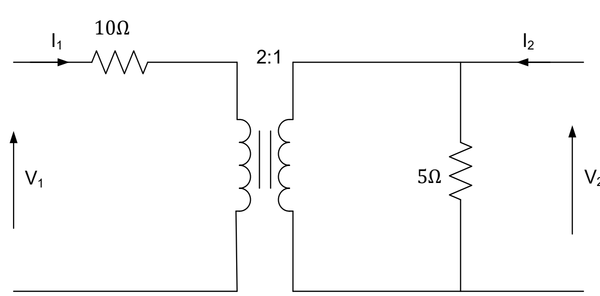



**Figure 6**

9. A two-port network has the parameters $A = 1 + j1, B = 2\ \Omega, C = 1 + j1.55$ and $D = 3$. What are the

input current and voltage, when the output is a current of $100\ \mathrm{mA}$ through a resistive load of $10\ \Omega$?

10. (a) Define Insertion loss for a given network

(b) For the network in Fig. 7 determine the insertion in decibels and nepers, after inserting network N (dashed box) between the source and the load

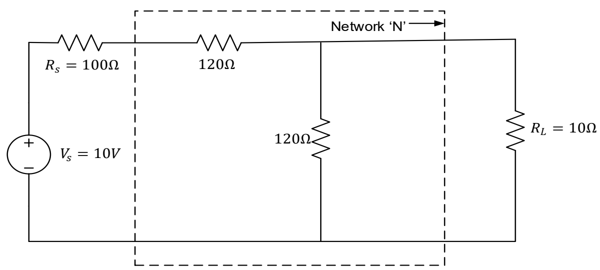



**Figure 7**

11. Find the z parameters of the two-port network in Fig. 8.

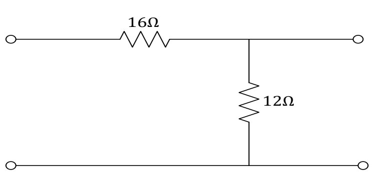



**Figure 8**

**Answer:**$z_{11} = 28\Omega, z_{12} = z_{21} = z_{22} = 12\Omega$

12. Calculate $I_1$ and $I_2$ in the two-port of Fig. 9.

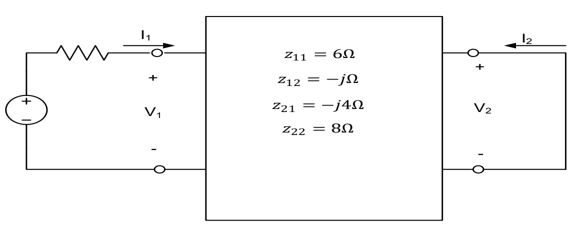



**Figure 9**

**Answer**: $200\angle 30^o\ mA, 100\angle 120^o\ mA$

13. Obtain the $y$ parameters for the circuit in Fig. 10.

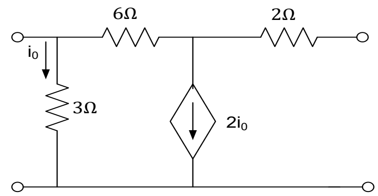



**Figure 10**

**Answer**: $y_{11} = 0.625\ \mathrm{S}, y_{12} = -0.125\ \mathrm{S}, y_{21} = 0.375\ \mathrm{S}, y_{22} = 0.125\ \mathrm{S}$

14. Find the transmission parameters for the circuit in Fig. 11

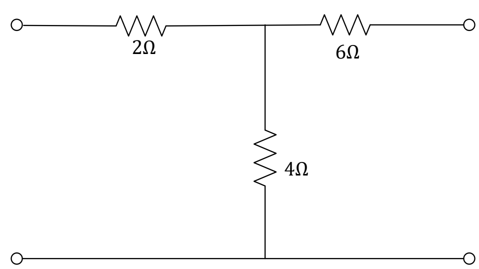



**Figure 11**

**Answer:**$A = 1.5,\ B = 22\ \Omega,\ C = 125\ \mathrm{mS},\ \ \mathrm{D} = 2.5$

14. Obtain the $y$ parameter for the $T$ network shown in Fig. 12

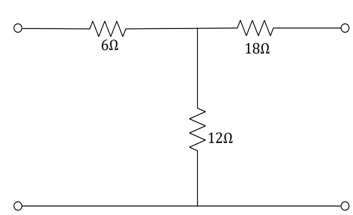



**Figure 12**

# CHAPTER 6

# THE COMPLEX FREQUENCY PLANE

## 6.0 Introduction

In our sinusoidal circuit analysis, we have learned how to find voltages and currents in a circuit with a constant frequency source. If we let the amplitude of the sinusoidal source remain constant and vary the frequency, we obtain the circuit's frequency response. The frequency response may be regarded as a complete description of the sinusoidal steady-state behavior of a circuit as a function of frequency.

*The frequency response of a circuit is the variation in its behavior with change in signal frequency*

The sinusoidal steady-state frequency responses of circuits are of significance in many applications, especially in communications and control systems. A specific application is in electric filters that block out or eliminate signals with unwanted frequencies and pass signals of the desired frequencies. Filters are used in radio, TV, and telephone systems to separate one broadcast frequency from another.

We begin this chapter by considering the frequency response of simple circuits using their transfer functions. We then consider Bode plots, which are the industry-standard way of presenting frequency response. We also consider series and parallel resonant circuits and encounter important concepts such as resonance, quality factor, cutoff frequency, and bandwidth. We discuss different kinds of filters and network scaling. In the last section, we consider one practical application of resonant circuits and two applications of filters.

## 6.1 Transfer Function

The transfer function $H(\omega)$ (also called the network function) is a useful analytical tool for finding the frequency response of a circuit. In fact, the frequency response of a circuit is the plot of the circuit's transfer function $H(\omega)$ versus $\omega$, with $\omega$ varying from $\omega = 0$ to $\omega = \infty$

A transfer function is the frequency-dependent ratio of a forced function to a forcing function (or of an output to an input). The idea of a transfer function was implicit when we used the concepts of impedance and admittance to relate voltage and current. In general, a linear network can be represented by the block diagram shown in Fig. 6.1.

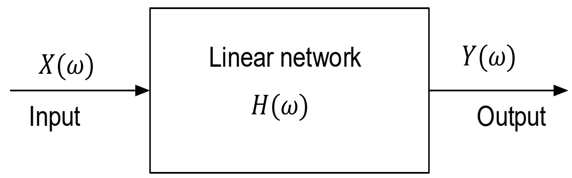



**Figure 6.1**

*The transfer function $H(\omega)$ of a circuit is the frequency-dependent ratio of a phasor output $Y(\omega)$ (an element voltage or current) to a phasor input $X(\omega)$ (source voltage or current).*

Thus,

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)}} \qquad 6.1$$

assuming zero initial conditions. Since the input and output can be either voltage or current at any place in the circuit, there are four possible transfer functions:

$$H(\omega) = \text{Voltage gain} = \frac{V_o(\omega)}{V_i(\omega)} \qquad 6.2a$$

$$H(\omega) = \text{Current gain} = \frac{I_o(\omega)}{I_i(\omega)} \qquad 6.2b$$

$$H(\omega) = \text{Transfer Impedance} = \frac{V_o(\omega)}{I_i(\omega)} \qquad 6.2c$$

$$H(\omega) = \text{Transfer Admittance} = \frac{I_o(\omega)}{V_i(\omega)} \qquad 6.2d$$

where subscripts $\mathrm{i}$ and $o$ denote input and output values. Being a complex quantity, $H(\omega)$ has a magnitude $H(\omega)$ and a phase $\phi$; that is, $H(\omega) = H(\omega)\angle\phi$

To obtain the transfer function using Eq. (6.2), we first obtain the frequency-domain equivalent of the circuit by replacing resistors, inductors, and capacitors with their impedances $R, j\omega L$ and $1/j\omega C$. We then use any circuit technique(s) to obtain the appropriate quantity in Eq. (6.2). We can obtain the frequency response of the circuit by plotting the magnitude and phase of the transfer function as the frequency varies. A computer is a real time-saver for plotting the transfer function.

The transfer function $H(\omega)$ can be expressed in terms of its numerator polynomial $N(\omega)$ and denominator polynomial $D(\omega)$ as

$$\boxed{H(\omega) = \frac{N(\omega)}{D(\omega)}} \qquad 6.3$$

where $N(\omega)$ and $D(\omega)$ are not necessarily the same expressions for the input and output functions, respectively. The representation of $H(\omega)$ in Eq. (6.3) assumes that common numerator and denominator factors in $H(\omega)$ have canceled, reducing the ratio to lowest terms. The roots of $N(\omega) = 0$ are called the zeros of $H(\omega)$ and are usually represented as $j\omega = z_1, z_2$ …. Similarly, the roots of $D(\omega) = 0$ are the poles of $H(\omega)$ and are represented as $j\omega = p_1, p_2$ ….,.

*A zero, as a root of the numerator polynomial, is a value that results in a zero value of the function. A pole, as a root of the denominator polynomial, is a value for which the function is infinite.*

To avoid complex algebra, it is expedient to replace jo temporarily with s when working with $H(\omega)$ and replace $s$ with $j\omega$ at the end.

**Example 6.1**: For the RC circuit in Fig. 6.2(a), obtain the transfer function $V_o/V_s$ and its frequency response. Let $v_s = V_m \cos \omega t$

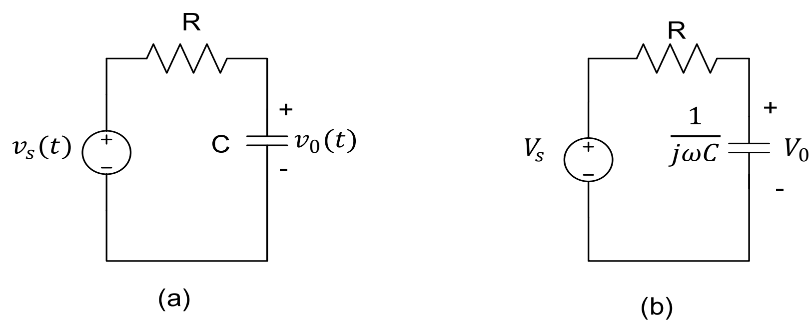



**Figure 6.2(a) time-domain RC circuit (b) frequency-domain RC circuit**

Solution

The frequency-domain equivalent of the circuit is in Fig. 6.2(b). By voltage division, the transfer function is given by

$$H(\omega) = \frac{V_o}{V_s} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC}$$

We obtain the magnitude and phase of $H(\omega)$ as

$$H = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}, \quad \phi = -\tan^{-1}\frac{\omega}{\omega_0}$$

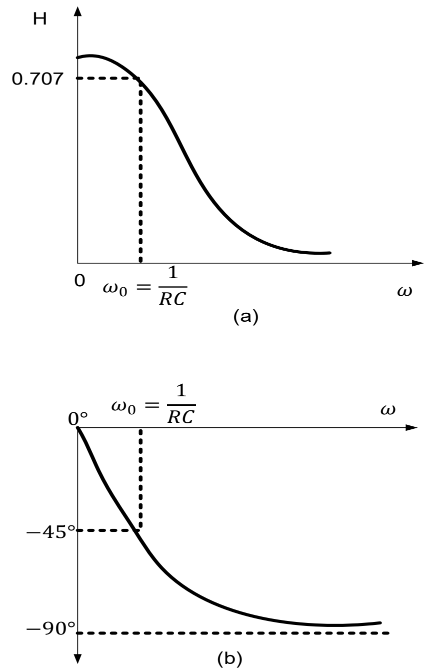



**Figure 6.3**

Where $\omega_0 = 1/RC$. To plot $H$ and $\phi$ for $0 < \omega < \infty$, we obtain their values at some critical points and then sketch.

At $\omega = 0, H = 1$ and $\phi = 0$. At $\omega = \infty$, $H = 0$ and $\phi = -90^o$. Also, at $\omega = \omega_0, H = 1/\sqrt{2}$ and $\phi = -45^o$. With these and a few more points as shown in Table 6.1, we find that the frequency response is as shown in Fig. 6.3. Additional features of the frequency response in Fig. 6.3 will be explained in **Section 7** on lowpass filters.

Table 6.1 (for example, 6.1)

| $\frac{\omega}{\omega_0}$ | $H$ | $\phi$ | $\frac{\omega}{\omega_0}$ | $H$ | $\phi$ |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 10 | 0.1 | $-80^o$ |
| 1 | 0.71 | $-45^o$ | 20 | 0.05 | $-87^o$ |
| 2 | 0.45 | $-63^o$ | 100 | 0.01 | $-89^o$ |
| 3 | 0.32 | $-72^o$ | $\infty$ | 0 | $-90^o$ |

**Example 6.2**: For the circuit in Fig.6.4, calculate the gain $I_o(\omega)/I_i(\omega)$ and its poles and zeros

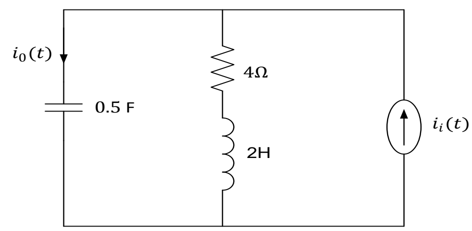



**Figure 6.4**

Solution:

By current division

$$I_o(\omega) = \frac{4 + j2\omega}{4 + j2\omega + \frac{1}{j0.5\omega}} I_1(\omega)$$

Or

$$\frac{I_o(\omega)}{I_i(\omega)} = \frac{j0.5\omega(4 + j2\omega)}{1 + j2\omega + (j\omega)^2} = \frac{s(s+2)}{s^2 + 2s + 1}, \qquad s = j\omega$$

The zeros are at

$$s(s+2) = 0 \quad \Longrightarrow \quad z_1 = 0, z_2 = -2$$

The poles are at

$$s^2 + 2s + 1 = (s+1)^2 = 0$$

Thus, there is a repeated pole (or double pole) at $p = -1$

## 6.2 Poles and Zeros Consulate

The complex frequency $'s'$ has both real and imaginary parts in general $s = \sigma + j\omega$, where $\sigma$ (lower case letter sigma of Greek alphabets) in the neper frequency and $\omega$ (lower-case letter omega) is the radian frequency. Any forced response can be portrayed graphically as a function of the complex frequency.

For the driving point (taken at the same terminal) input impedance: $\vec{Z}(s) = 4 + 5s\ \Omega$, to study how the impedance varies graphically with $\sigma$, we set

$$s = \sigma + j0 \Longrightarrow \vec{Z}(\sigma) = 4 + 5s = 4 + 5\sigma\ \Omega$$

The root (zero) of this:

$$4 + 5\sigma = 0 \Longrightarrow \sigma = -\frac{4}{5}$$

This pole (the value of $\sigma$ that makes $\vec{Z}(s)$ infinite) is $\sigma = \infty$.

To plot $\left|\vec{Z}(\sigma)\right|$ [absolute value of $\vec{Z}(\sigma)$ versus $\sigma$, we determine the article point since the graph would be a straight line [not crossing into the negative portion of $\vec{Z}(s)$] see Fig. 6.5

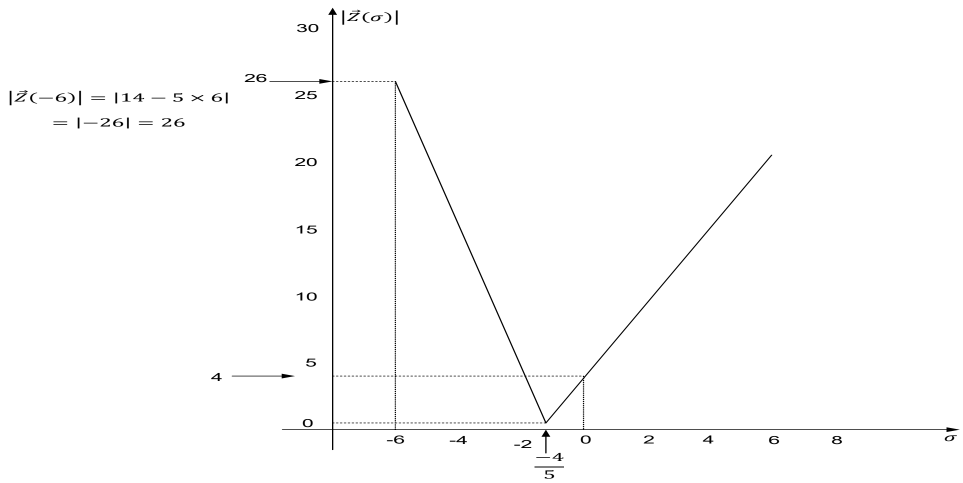



**Figure 6.5**

For the response as a function of $\omega$, we "suppress" $\sigma$, letting $s = j\omega$

$$\Longrightarrow \quad \vec{Z}(j\omega) = 4 + j5\omega$$

$$\Longrightarrow \quad \left|\vec{Z}(j\omega)\right| = \sqrt{16 + 25\omega^2}$$

$$ang\ \vec{Z}(j\omega) = \tan^{-1}\frac{5\omega}{4}$$

For the magnitude, a single pole is at infinity, a minimum at $\omega = 0 \Longrightarrow \left|\vec{Z}(j\omega)\right| = 4$, zero at

$$16 + 25\omega^2 = 0 \ \Longrightarrow \omega = \pm\sqrt{-\frac{16}{25}} = \pm\frac{j4}{5}$$

$$\omega = 0 \Longrightarrow ang\ Z(j\omega = 0)$$

$$\omega = \pm\infty \ \Longrightarrow ang\ \vec{Z}(j\omega) = \pm\frac{\pi}{2}$$

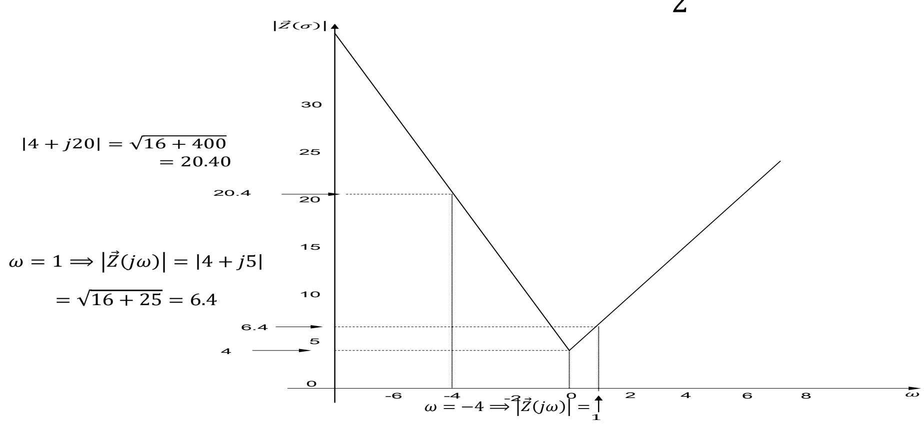



**Figure 6.6**

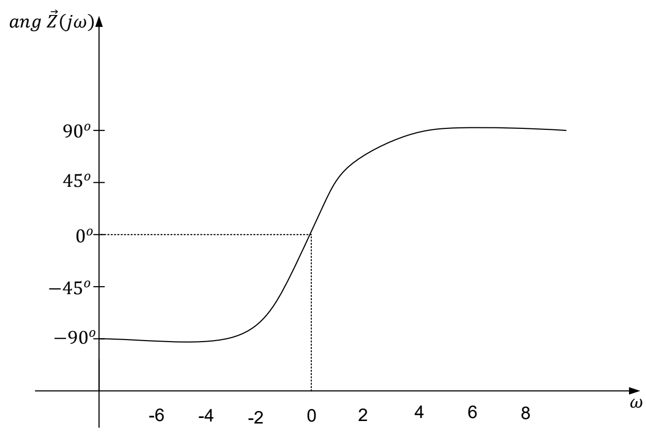



**Figure 6.7**

For the frequency domain function given by

$\vec{F}(s) = \dfrac{(s+3)}{(s+4)}$ , $the\ zero\ is\ at\ s + 3 = 0 \Longrightarrow 5 = -3.$

The pole is at $s + 4 = 0 \Longrightarrow s = -4$ see Fig. 6.8

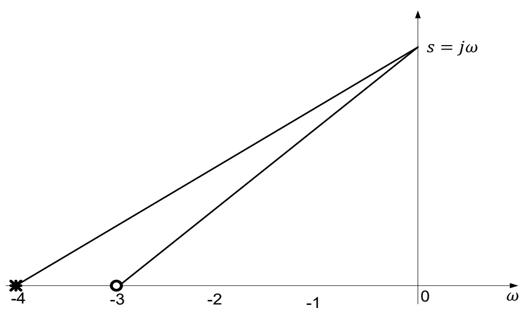



**Figure 6.8**

For plots of $\left|\vec{V}(j\omega)\right|$ and $ang\ \vec{V}(j\omega)\ vs\ \omega$

$$\left|\vec{V}(j\omega)\right| = \left|\frac{(j\omega+3)}{(j\omega+4)}\right| = \frac{|j\omega+3|}{|j\omega+4|} = \sqrt{\frac{(\omega^2+9)}{(\omega^2+16)}}$$

As $\omega$ increases (or decreases) without bound, $\left|\vec{V}(j\omega)\right|$ approaches $\frac{\sqrt{\omega^2}}{\sqrt{\omega^2}} = 1$ as maximum.

At $\omega = 0, \left|\vec{V}(j\omega)\right| = \sqrt{\frac{9}{16}} = \frac{3}{4}$

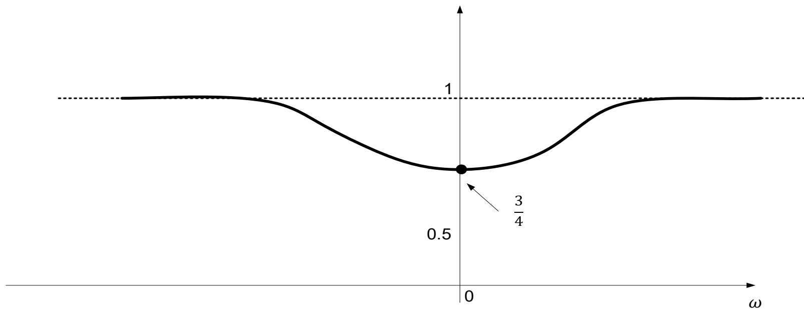



**Figure 6.9**

$$\vec{V}(j\omega) = \frac{(3+j\omega)(4-j\omega)}{16+\omega^2} = \frac{12+j\omega+\omega^2}{16+\omega^2}$$

$$\left|\vec{V}(j\omega)\right| = \frac{(12+\omega^2)^2+\omega^2}{16+\omega^2}$$

$$ang\,\vec{V}(j\omega) = \tan^{-1}\left(\frac{\omega}{12+\omega^2}\right)$$

$$\omega = 0 \Longrightarrow ang\,\vec{V}(j\omega) = \tan^{-1} 0 = 0$$

$$\omega \to \infty\ (or - \infty) \Longrightarrow ang\,\vec{V}(j\omega) \to 0$$

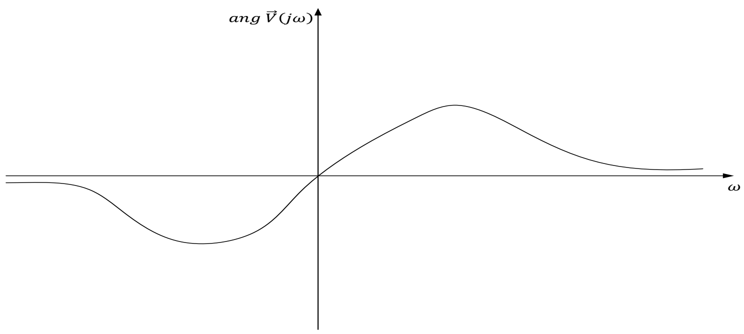



**Figure 6.10**

**Example 6.1**: For the pole-zero constellation shown in Fig. 6.11, obtain an expression for the gain that is a ratio of polynomials in $s$.

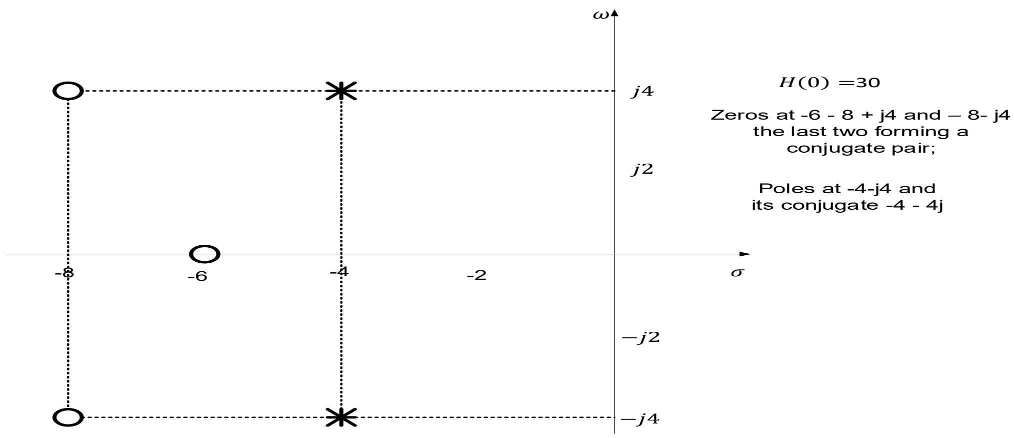



**Figure 6.11**

Solution:

$$H(s) = \frac{k(s+6)(s+8-j\ \ 4)(s+8+j4)}{(s+4-j4)(s+4+j4}$$

$$H(0) = 30 = \frac{k(6)(8-j4)(8+j4)}{(4-j4)(4+j4)} = k\ \frac{6(64+16)}{16+16} = k\frac{6\times 80}{32}$$

$$\Rightarrow k = \frac{(30)(32)}{480} = 2$$

$$\Rightarrow H(s) = \frac{2\ (s+6)(s+8-j4)(s+8+j4)}{(s+4-j4)(s+4+j4)}$$

$$= \frac{2(s+6)[(s+8)^2+4^2]}{(s+4)^2+4^2}$$

$$= \frac{2(s+6)(s^2+16s+64+16)}{s^2+8s+16+16}$$

$$= \frac{2\ (s^3+22s^2+176s+480)}{s^2+8s+32}$$

Note: whenever zeros and/or poles contain imaginary part(s), then they must occur in conjugate part(s). This is in conformity with the physical reality that, although there might occur imaginary quantities theoretically the addition of two complex conjugates always produces a real number!

**Example 6.3**: For each of the pole constellations in the Fig. 6.12 which applies to a voltage gain G, obtain an expression for the gain that is a ratio of polynomials in $s$.

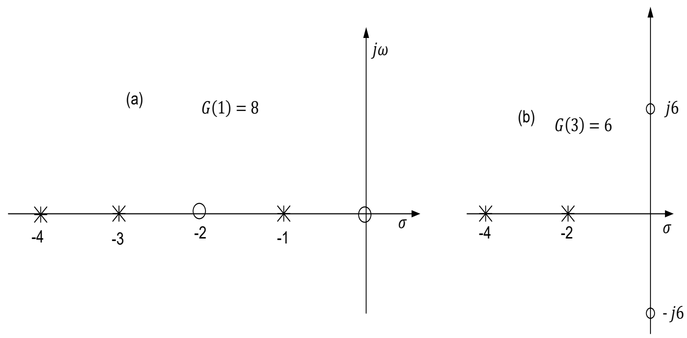



**Figure 6.12**

Solution:

(a) $\dfrac{k \times s(s+2)}{(s+1)(s+3)(s+4)}$

$$\Longrightarrow G(1) = 8 = \frac{k(3)}{(2)(4)(5)}$$

$$\Longrightarrow k = \frac{320}{3}$$

$$G(s) = \left(\frac{320}{3}\right)\frac{(s^2+25)}{(s+1)(s^2+7s+12)}$$

$$= \frac{\left(\frac{320}{3}\right)(s^2+25)}{(s^3+8s^2+19s+12)}$$

$$= \frac{(320s^2+64s)}{(3s^3+24s^2+57s+36)}$$

(b) $\dfrac{k \times (s+j6)(s-j6)}{(s+2)(s-2)(s+4)} = \dfrac{k \times (s^2+36)}{(s^2-4)(s+4)}$

$$G(3) = 6 = \frac{k(45)}{(5)(7)}$$

$$\Longrightarrow k = \frac{(6)(35)}{45} = \frac{14}{3}$$

$$G(s) = \frac{(14s^2+504)}{(3s^3+12s^2-12s-48)}$$

**Example 6.4**: For each of the pole constellations in the Fig. 6.13, which applies to a voltage gain G, obtain an expression for the gain that is a ratio of polynomials in $s$.

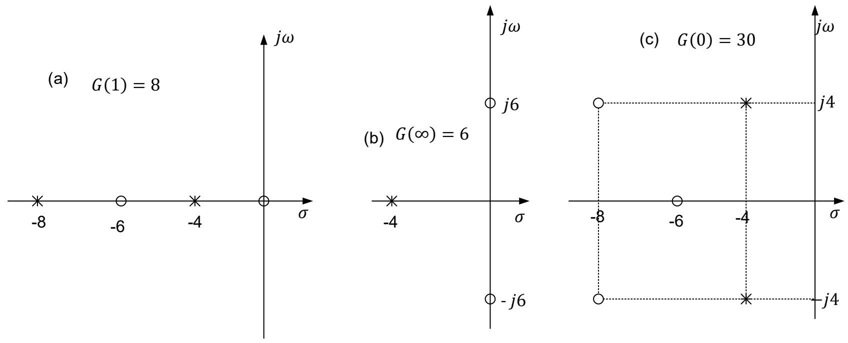



**Figure 6.13**

Solution:

(a) $$G(s) = \frac{k \times s(s+6)}{(s+8)(s+4)}$$

$$\Longrightarrow G(1) = 8 = \frac{k \times (7)(1)}{(9)(5)}$$

$$k = 51.43$$

$$\therefore G(s) = \frac{51.43(s^2 + 6s)}{(s^2 + 12s + 48)}$$

(b) $$G(s) = \frac{k \times (s + j6)(s - j6)}{s(s+4)}$$

$$\Longrightarrow G(\infty) = 6 = \lim_{s \to \infty} ks^2 = k = 6$$

$$\Longrightarrow G(s) = \frac{6(s^2 + 36)}{(s^2 + 4s)}$$

$(c)$ $$G(s) = \frac{k \times (s+6)(s+8+j4)(s+8-j4)}{(s+4+j4)(s+4-j4)}$$

$$G(0) = 30 = \frac{k \times (6)(8+j4)(8-j4)}{(4+j4)(4-j4)}$$

$$30 = 6k \times \frac{(64+16)}{(16+16)}$$

$$\Rightarrow k = \frac{30(32)}{(6)(80)} = 2$$

$$= G(s) = \frac{2(s+6)(s^2+16s+64+16)}{(s^2+8s+16+16)}$$

$$= \frac{2(s^3+225^2+176s+480)}{(s^2+8s+32)}$$

**Example 6.5**: For each of the pole constellations in the Fig. 6.14, which applies to a voltage gain G, obtain an expression for the gain that is a ratio of polynomials in $s$.

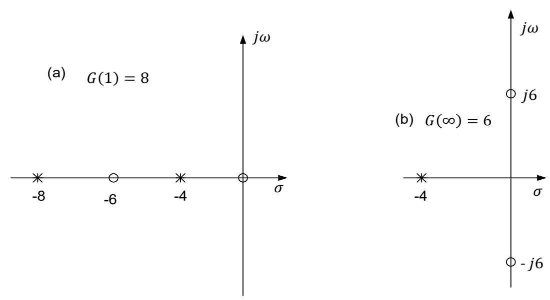



**Figure 6.14**

Solution:

$$(a)\ \frac{K \times s(s+6)}{(s+4)(s+8)} \Rightarrow G(1) = \frac{K \times (7)}{(s)(9)} = 8 \Rightarrow k = 8 \times \frac{45}{7} = \frac{360}{7}$$

$$\Rightarrow G(s) = \frac{360}{7}\frac{s(s+6)}{(s+4)(s+8)} = \frac{360}{7}\frac{s^2+6s}{s^2+125+32}$$

$$(b)\quad G(s) = \frac{K \times (s-j6)(s+j6)}{s(s+4)} = \frac{k(s^2+36)}{s^2+45} \quad \Rightarrow \quad G(0) \neq 6$$

$$\Rightarrow k = \text{indeterminate}$$

$$\Rightarrow \quad G(s) = \frac{k(s^2+36)}{s^2+45} \qquad 0 < k < \infty$$

## 6.3 The Decibel Scale

It is not always easy to get a quick plot of the magnitude and phase of the transfer function as we did above. A more systematic way of obtaining the frequency response is to use Bode plots. Before we begin to construct Bode plots, we should take care of two important issues: the use of logarithms and decibels in expressing gain.

Since Bode plots are based on logarithms, it is important that we keep the following properties of logarithms in mind:

1. $\log P_1 P_2 = \log P_1 + \log P_2$
2. $\log \frac{P_1}{P_2} = \log P_1 - \log P_2$
3. $\log P^n = n \log P$
4. $\log 1 = 0$

In communications systems, gain is measured in bels. Historically, the bel is used to measure the ratio of two levels of power or power gain G; that is,

$$G = \text{Number of bels} = \log_{10} \frac{P_2}{P_2} \qquad 6.4$$

The decibel (dB) provides us with a unit of less magnitude. It is 1/10th of a bel and is given by

$$\boxed{G_{dB} = 10 \ \log_{10} \frac{\mathrm{P}_2}{\mathrm{P}_1}} \qquad 6.5$$

When $P_1 = P_2$, there is no change in power and the gain is 0 dB. If $P_2 = 2P_1$, the gain is

$$G_{dB} = 10 \ \log_{10} 2 \simeq 3\text{dB} \qquad 6.6$$

And when $P_2 = 0.5P_1$ the gain is

$$G_{dB} = 10 \ \log_{10} 0.5 \simeq -3\text{dB} \qquad 6.7$$

Eqs (6.6) and (6.7) show another reason why logarithms are greatly used: The logarithm of the reciprocal of a quantity is simply negative the logarithm of that quantity.

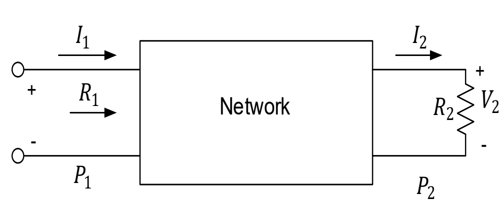



**Figure 6.15**

Alternatively, the gain G can be expressed in terms of voltage or current ratio. To do so, consider the network shown in Fig. 6.15. If $P_1$ is the input power, $P_2$ is the output (load) power, $R_1$ is the input resistance, and $R_2$ is the load resistance, then $P_1 = 0.5\frac{V_1^2}{R_1}$ and $P_2 = 0.5V_2^2/R_2$, and Eq. (6.5) becomes

$$G_{dB} = 10\ log_{10}\frac{P_2}{P_1} = 10\ log_{10}\left[\frac{\frac{V_2^2}{R_2}}{\frac{V_1^2}{R_1}}\right] \qquad 6.8$$

$$= 10\ \log_{10}\left(\frac{V_2}{V_1}\right) + 10\ \log_{10}\frac{R_1}{R_2}$$

$$G_{dB} = 20\ \log_{10}\frac{V_2}{V_1} - 10\ \log_{10}\frac{R_2}{R_1} \qquad 6.9$$

For the case when $R_2 = R_1$, a condition that is often assumed when comparing voltage levels, Eq. (6.9) becomes

$$\boxed{G_{dB} = 20\ \log_{10}\frac{V_2}{V_1}} \qquad 6.10$$

Instead, if $P_1 = I_1^2R_1$ and $P_2 = I_2^2R_2$ for $R_1 = R_2$ we obtain

$$G_{dB} = 20\ \log_{10}\frac{I_2}{I_1} \qquad 6.11$$

Three things are important to note from Eqs. (6.5), (6.10), and (6. 11):

1. That 10 $\log_{10}$ is used for power, while 20 $\log_{10}$ is used for voltage or current, because of the square relationship between them $\left(P = \frac{V^2}{R} = I^2R\right)$
2. That the dB value is a logarithmic measurement of the ratio of one variable to another of the same type. Therefore, it applies in expressing the transfer function H in Eqs. (6.2a) and (6.2b), which are dimensionless quantities, but not in expressing H in Eqs. (6.2c) and (6.2d).
**3.** It is important to note that we only use voltage and current magnitudes in Eqs. (6.10) and (6. I l). Negative signs and angles will be handled independently as we will see in **Section 7**

With this in mind, we now apply the concepts of logarithms and decibels to construct Bode plots.

### 6.4 Exercise

1. For each of the pole constellations in Fig.1 which applies to a voltage gain G, obtain an expression for the gain that is a ratio of polynomials in $s$

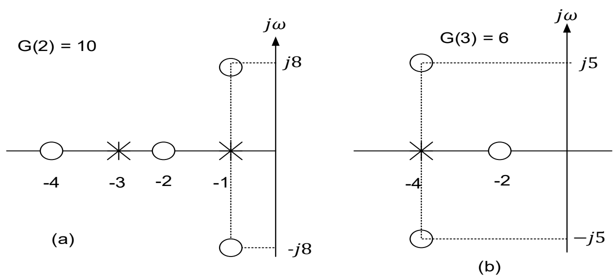



**Figure 1**

2. Calculate $H_{dB}$ for $H(s)$ equal to:

(i) $\dfrac{100}{(s+200)}$ (ii) $100\,(s+200)$

3. Calculate $|H(j\omega)|$ for $H_{dB}$ equal to (i) 30dB (ii) -30dB

4. For each of the pole zero constellations in Fig. 2, which applies to a voltage gain G, obtain an expression for the gain that is a ratio of polynomials in $s$.

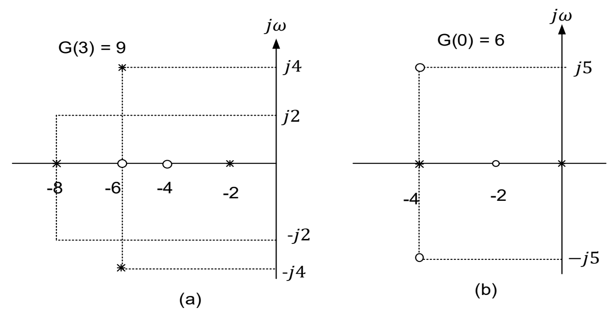



**Figure 2**

5. Calculate $H_{dB}$ for $H(s)$ equal to:

(i) $\dfrac{10}{(s+100)}$ (ii) $10\,(s+100)$

6. Calculate $|H(j\omega)|$ for $H_{dB}$ equal to (i) 40dB (ii) -40dB

7. For each of the pole-zero constellations in Fig. 3 which applies to a voltage gain G, obtain an expression for the gain that it is a ratio of polynomials in $s$

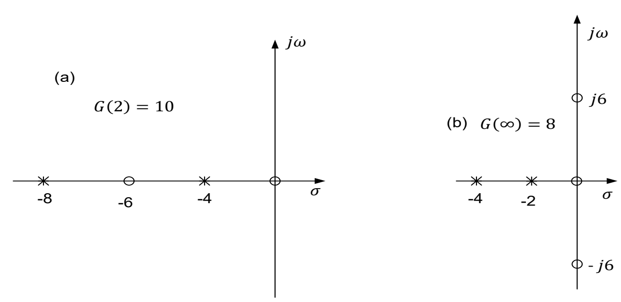



**Figure 3**

8. Calculate $H_{dB}$ for $H(s)$ equal to :

(i) $\dfrac{10}{(s+100)}$ (ii) $10\,(s+100)$

9. Calculate $|H(j\omega)|$ for $H_{dB}$ equal to (i) 40dB (ii) -40dB

10. Calculate $|H(j\omega)|$ for $H_{dB}$ equal to (i) 80dB (ii) -80dB (III) 0

11. For each of the pole-zero constellations in Fig. 4 which applies to a voltage gain G, obtain an expression for the gain that is ratio of polynomials in s ($j\omega$)

# CHAPTER 7

# BODE PLOT

## 7.0 Introduction to Bode plot

Basic of any frequency response is to plot magnitude M and angle Ø against input frequency '$\omega$'. When '$\omega$' is varied from 0 to ∞ there is wide range of variations in M and Ø and hence it becomes difficult to accommodate all such variations with linear scale. Hence H.W. Bode suggested the method in which logarithm values of magnitude is to be plotted against logarithm values of frequencies such plots are called logarithm plots which allows us to show a wide range of variations in magnitude on a single paper.

So, in general Bode plot consists of two plots

1. Magnitude expressed in Logarithm values against logarithm values of frequency called magnitude plot.
2. Phase angle in degrees against Logarithm values of frequency called as phase angle plot.

### 7.0.1 Magnitude plot

The magnitude can be expressed in its Logarithmic values by finding out the value $20\log_{10}|G(j\omega)|$ , which has a unit as decibel denoted by dB.

For Bode plot $|G(j\omega)| = 20\log_{10}|G(j\omega)|\, dB$

Such decibels values are to be plotted against $\log_{10}\omega$ magnitude plot can be shown as in Fig. 7.1

### 7.0.2 The phase angle plot

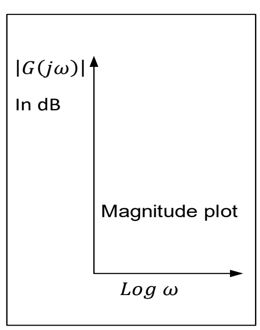



**Figure 7.1**

The Phase Angle Plot: In this angle of $G(j\omega)$ is to be expressed in degrees which is to be plotted against $\log\omega$

The phase angle plot can be shown as in Fig. 7.2.

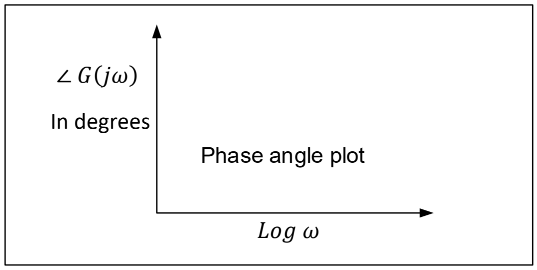



**Figure 7.2**

As for both plots X-axis is Log $\log\omega$ both may be drawn on the same paper with common X-axis.

Note: To predict the closed loop stability from the frequency response of open loop system, the magnitude and phase angle of open loop transfer function $G(j\omega)\ H(j\omega)$is to be plotted against Log $\omega$ and not only $G(j\omega)$.

So, for Bode plot, magnitude in dB and phase angle in degrees are the magnitudes and phase angles of $G(j\omega)\ H(j\omega)$, plotted against $\log\omega$

**7.1 Logarithmic Scales (Semi-log Papers):**

To sketch the magnitude in dB and phase angle in degrees against $\log\omega$ the logarithmic scale is used. This is available on semilog graph paper. In such paper the X-axis is divided into a logarithmic scale which is nonlinear one. While Y-axis is divided into linear scale and hence it is called as semilog paper.

The interesting part about X-axis is the distance between 1 and 2 is greater than distance between 2 and 3 and so on. Similarly, on such a scale, the distance between 1 and 10 is equal to the distance between 10 and 100 or between 100 and 1000 and so on. This distance is called 1 decade.

This is because $\log 1 = 0$ and $\log 10 = 1$. The distance is 1 decade which is divided into 10 parts according to logarithmic scale i.e. $Log\ 2, Log\ 3, \cdots\cdots$

Now $\text{Log}\ 10\ =\ 1$ and $Log\ 100 = 2$. The distance is again $(2 — 1)$ i.e. 1 decade same as between $Log\ 1$ and $Log\ 10$, further divided into parts as $\text{Log}\ 20$, $Log\ 30$,.....

So, X-axis is available, which is divided into two, three, or four such cycles i.e. decades.

So, it is not necessary to find logarithmic value of $\omega$ but the logarithmic scale available takes care of logarithmic value of $\omega$. The advantage of the scale is the wide range of frequencies can be accommodated on a single paper.

As $\log 0 = -\infty$ itis obvious that x-axis cannot be calibrated from 0 but as per requirement the smallest frequency may be selected as starting frequency like 0.01, 0.1 etc. This hardly affects the result of the frequency response.

The Y axis is divided into linear scales similar to standard graph paper.

To clear the idea of semilog paper and decade the graph paper is shown in Fig. 7.3.

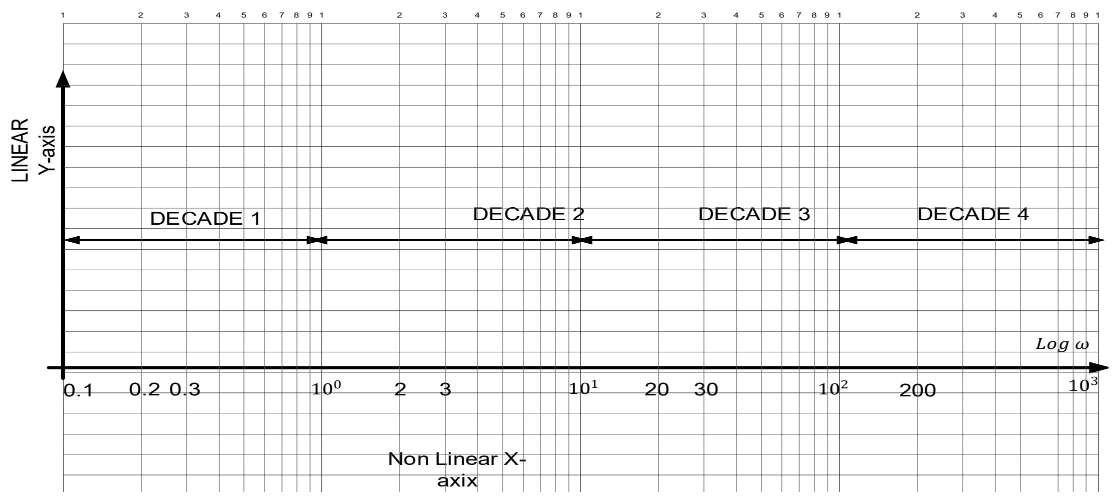



**Figure 7.3 Semi-log paper**

The main advantage using the logarithmic representation is that the multiplication and division of magnitudes get replaced by the addition and subtraction respectively. The experimental determination of the transfer function is easier if frequency response data is presented in the form of the logarithmic plot. Such a plot shows both low frequency and high frequency characteristics in the same diagram.

## 7.2 Standard Form of Open Loop T.F. $G(j\omega)\ H(j\omega)$:

$$\text{Consider } G(s)H(s) = \frac{K' s^Z (s+Z_1)(s+Z_2)\ldots}{s^P (s+P_1)(s+P_2)\ldots}$$

Note that either $s^Z$ or $s^P$ will be present at a time and not both. But this form is not useful to sketch the Bode Plot.

Hence it is necessary to rewrite the $G(s)\ H(s)$ in the time constant form.

$$G(s)\ H(s) = \frac{s^Z K(1+T_1 s)(1+T_2 s)\ \ldots..}{s^P(1+T_a s)(1+T_b s)\ \ldots..}$$

$$Where\ K = \frac{Z_1 \times Z_2 \times \ldots.}{P_1 \times P_2 \times \ldots..} \times K$$

Again either $s^Z$ or $s^P$ is present and not both.

The standard time constant form can be denoted as

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ \ldots..}{s^P(1+T_a s)(1+T_b s)\ \ldots..}$$

K= Resultant system gain $\quad P$ =Type of the system

$T_1, T_2, T_a, T_b, \ldots \ldots.$ =Time constants of different poles and zeros.

Each of the factor involved in $G(s)H(s)$ above will contribute to magnitude and phase angle variations of $G(j\omega)\ H(j\omega)$ in frequency domain. Frequency domain transfer function can be obtained by substituting. $s = j\omega$ in above expression

$$G(j\omega)\ H(j\omega) = \frac{K(1+T_1\, j\omega)(1+T_2\, j\omega)\ \ldots..}{(j\omega)^P(1+T_a\, j\omega)(1+T_b\, j\omega)\ \ldots..}$$

Now basic factors which very frequently occur in the above form can be identified and studied separately.

List of such basic factors is,

1. Resultant system gain K, constant factor. (When $G(j\omega)\ H(j\omega)$ is expressed in time constant form).
2. Poles or zeros at the origin. (Integral and Derivative factors) i.e., $(j\omega)^{\pm P}$ Either poles or zeros at origin will be present.
3. Simple poles and zeros also called as first order factors of the form $(1+j\omega T)^{\pm 1}$
4. Quadratic factors which cannot be factorized 2 into real factors, of the form

$$\left(1 + \frac{2\xi}{\omega_n}s + \frac{s^2}{\omega_n^2}\right) \approx 1 + 2\xi j\left(\frac{\omega}{\omega_n}\right) + \left(\frac{j\omega}{\omega_n}\right)^2$$

Once the behaviour of such factors is clear in frequency domain then by adding logarithmic plots of such factors, the resultant logarithmic plot for any $G(j\omega)\ H(j\omega)$ can be obtained. The process of obtaining logarithmic plots for such factors can be simplified by using asymptotic approximations for each factor. But by adding corrections to such a plot, if necessary, an accurate plot may be obtained.

## 7.3 Bode Plots of Standard Factors of $G(j\omega)\ H(j\omega)$

For each factor procedure to obtain its Bode Plot can be divided into following steps.

Step 1: Replace 's' by '$j\omega$' to convert it to frequency domain.

Step 2: Find its magnitude as a function of $\omega$

Step 3: Express the magnitude in 'dB' by $20\ \log_{10}|G(j\omega)H(j\omega)|$

Step 4: Find phase angle by using $\tan^{-1}\left[\frac{\text{imaginary part}}{real\ part}\right] = \emptyset$ in degrees

Step 5: With required approximations, plot magnitude in dB and phase angle in degrees against $\log\omega$ by varying $\omega$ from 0 to ∞

Let us start with the basic factors one by one.

### 7.3.1 Factor 1: System gain 'K'

$$G(s)H(s) = K$$

i.e.

$$G(j\omega)H(j\omega) = K + j0$$

$$|G(j\omega)H(j\omega)| = \sqrt{K^2 + 0} = K$$

Its 'dB' value $= 20\ \log_{10} K\ \text{dB}$

As gain 'K' is constant, $20\ \log_{10} K$ is always constant though '$\omega$' is varied from 0 to ∞

So, its magnitude plot will be straight line parallel to X-axis.

So, magnitude plot for $K > 1$ is a line parallel to X-axis at a distance of $20\log K$ above $0dB$ reference line. While for $K < 1$ it is at a distance of $20\log K$ below $0\ dB$ reference line.

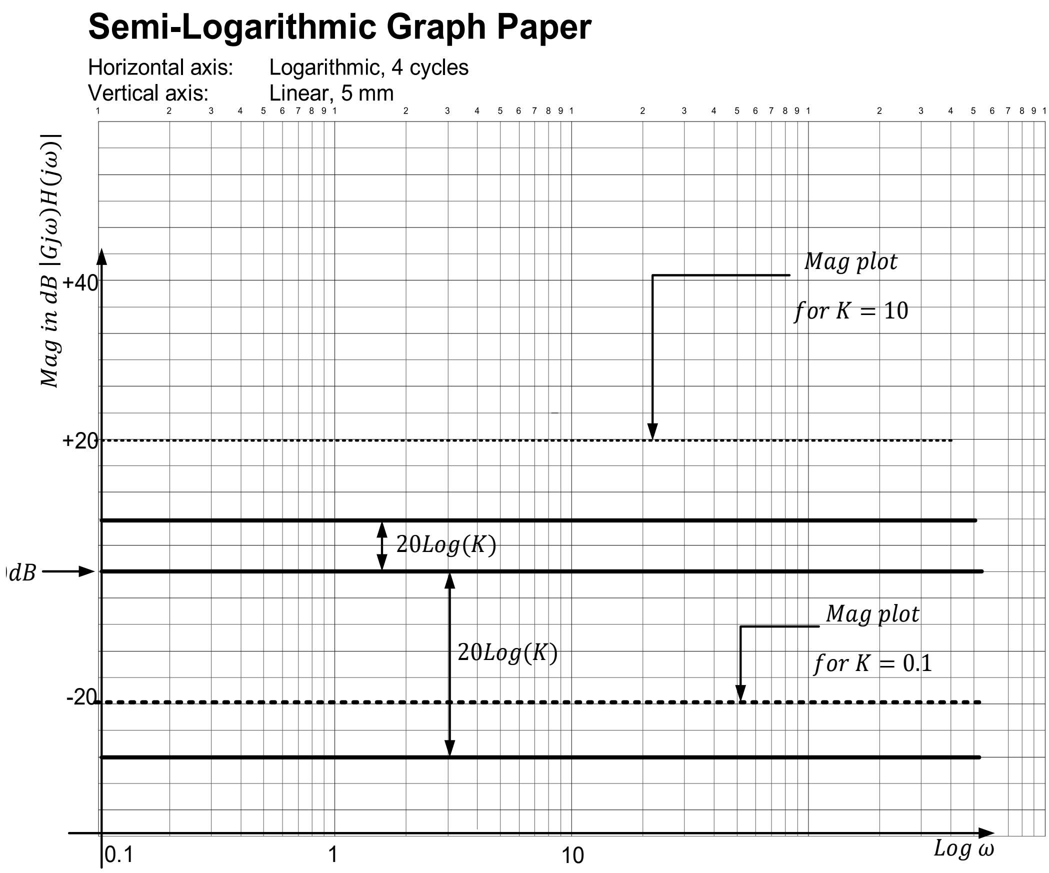



**Fig 7.4 Contribution by K**

This means that in the variation of $|G(j\omega)H(j\omega)|$ effect of 'K' is constant equal to $20\ log\ \ K\ dB$ for all frequencies. This means 'K' shifts the magnitude plot of $|G(j\omega)H(j\omega)|$ by a distance of $20 \log K\ dB$ upwards if $K > 1$ and downwards if

This fact is useful to design 'K' for the required specification. In such case $|G(j\omega)H(j\omega)|$ plot can be plotted with 'K' as unknown and then it just can be shifted upwards or downwards so as to meet the required specification. This shift is nothing but $20 \log K\ dB$, from which required 'K' can be determined.

**Phase angle plot:**

As $\quad G(j\omega)H(j\omega) = K + j0$

$$Corresponding\ \emptyset = tan^{-1}\left(\frac{imaj\ part}{real\ part}\right) = tan^{-1}\frac{0}{K} = 0^o$$

So, it does not affect the phase angle plot as its contribution to phase angle plot is $0^o$

This means that phase plot specifications remain as it 1b for any positive value of '$K$'

But if $'K'$ is negative, it always contributes independent of frequency. $-180'$ to the phage angle plot

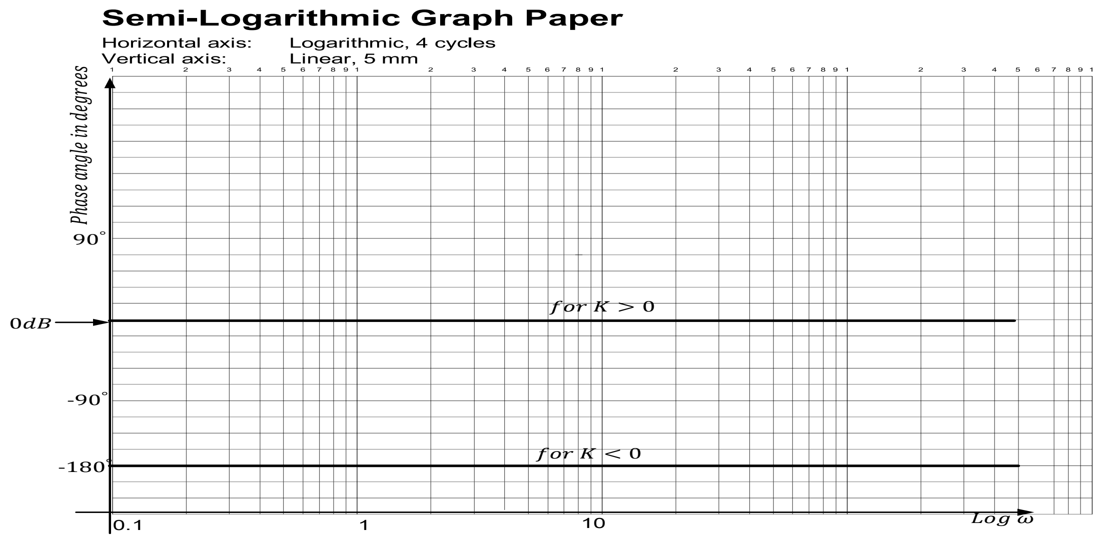



**Figure 7.5 Factor 2: Poles or zeros at the origin $(j\omega)^{\pm\rho}$**

Let us consider for simplicity single pole at the origin

$$G(s)H(s) = \frac{1}{s}$$

$$\therefore \quad G(j\omega) = \frac{1}{j\omega} = \frac{1}{0 + j\omega}$$

$$\therefore \quad \text{For magnitude Plot} \rightarrow |G(j\omega)H(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}} = \frac{1}{\omega}$$

$$\text{Magnitude in dB } = 20 \log \frac{1}{\omega} \text{ dB}$$

$$= 20 \log(\omega)^{-1} \text{ dB}$$

$$= -20 \log \omega$$

Therefore, this equation is similar to $y = mx$ i.e., 1 pole at the origin contributes to the magnitude plot according to the equation $-20 \log \omega$ i.e., according to the straight line of slope $' - 20'$

Let us see the unit of the slope.

Equation is $$|G(j\omega)H(j\omega)| = -20\log\omega$$

If $$\omega = 1 \rightarrow |G(j\omega)H(j\omega)| = 0\text{ dB}$$

$$\omega = 10 \rightarrow |G(j\omega)H(j\omega)| = -20\text{ Db},\ \omega = 100 \rightarrow |G(j\omega)H(j\omega)| = -40\text{ dB}$$

$$\omega = 0.1 \rightarrow |G(j\omega)H(j\omega)| = +20\text{ dB}$$

Now 10 times changes in frequency range is called as 1 decade described earlier i.e. 1 pole at origin reduces the $|G(j\omega)H(j\omega)|$ at the rate of — $20$ dB per decade. Hence the slope of magnitude plot for 1 pole at the origin is called— 20 dB/decade.

So, magnitude plot for 1 pole at origin is a straight line of slope — $20$ dB/decade.

Now at $\omega = 1$ $|G(j\omega)H(j\omega)| = 0$dB i.e., this line intersects the reference 0 dB line at $\omega = 1$.

At $\omega = 0.1$ it has magnitude $+20$ dB while at $\omega = 10$ it has magnitude of $-20$ dB.

As $\omega = 0$ cannot be indicated, starting frequency may be selected as per the requirement. This contribution is valid for range of $\omega$ from 0 to ∞.

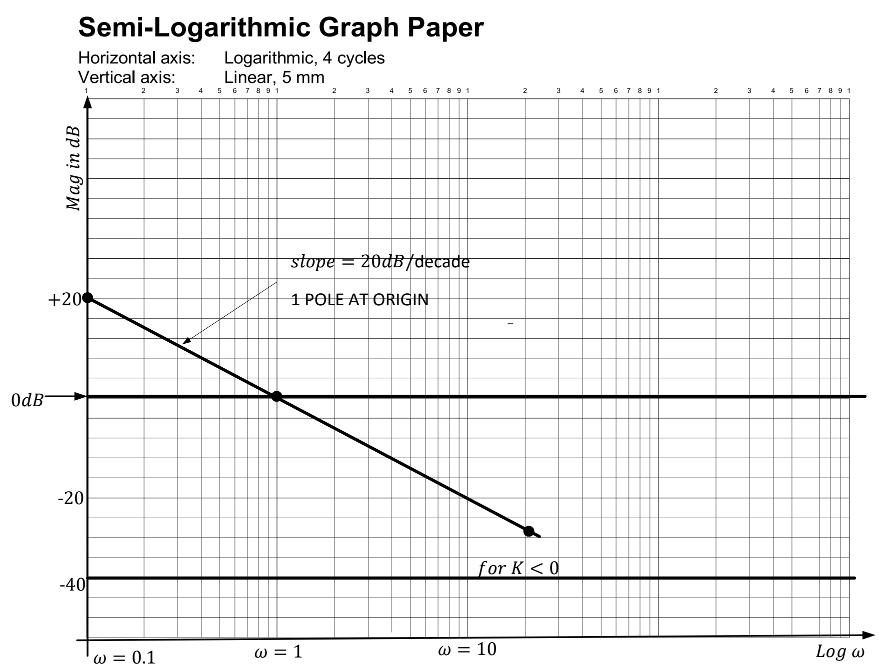



**Figure 7.6 Contribution by 1 pole at origin**

To sketch such a line of slope — $20\ \mathrm{dB/decade}$, first mark the intersection point of $\omega = 1$ with $0\ dB$ line and then go up by $20\ \mathrm{dB}$ for each $1/10^{th}$ reduction in frequency from $\omega = 1$ i.e. $+20\ \mathrm{dB}$ for $\omega = 0.1$, $+40\ \mathrm{dB}$ for $0.01$ or go down by $20\ \mathrm{dB}$ for each 10 times increase in frequency from $\omega = 1$ i.e. — $20\ \mathrm{dB}$ for $\omega = 10$, — $40\ \mathrm{dB}$ for $\omega = 100$ and so on. Then draw a straight line, as shown in the Fig. 7.6

Consider two poles at origin $\quad G(s)H(s) = \frac{1}{s^2}$

$$G(j\omega)H(j\omega) = \frac{1}{j\omega} \cdot \frac{1}{j\omega}$$

$$\therefore \quad |G(j\omega)H(j\omega)| = \frac{1}{\omega} \cdot \frac{1}{\omega} = \frac{1}{\omega^2}$$

$$\therefore \quad |G(j\omega)H(j\omega)| \text{in dB} = 20 \log \frac{1}{\omega^2} = 20 \log(\omega)^{-2} = -40 \log \omega$$

So, it is straight line of slope — $40\ \mathrm{dB/decade}$.

In logarithmic plot, multiplication gets replaced by addition. One point is important to note that at $\omega = 1$, $|G(j\omega)H(j\omega)| = 0\mathrm{dB}$ i.e., this line though has slope — $40\ \mathrm{dB/decade}$it intersects $0\ \mathrm{dB}$ line at $\omega = 1$

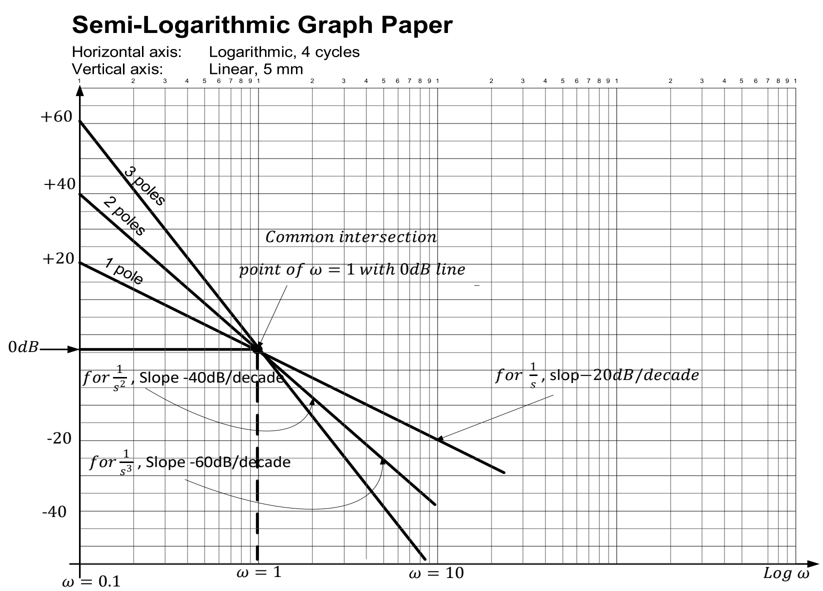



**Figure 7.7 Contribution by poles at origin**

Similarly, for 'P' number of poles at the origin

$$G(s)H(s) = \frac{1}{s^P}$$

$$G(j\omega)H(j\omega) = \frac{1}{\omega} \cdot \frac{1}{\omega} \cdots P \text{ times}$$

$$|G(j\omega)H(j\omega)| = \frac{1}{\omega} \cdot \frac{1}{\omega} \cdots P \text{ times} = \frac{1}{(\omega)^P}$$

$$|G(j\omega)H(j\omega)| \text{in dB} = 20 \log \frac{1}{(\omega)^P} = 20 \log(\omega)^P = -20 \times P \log \omega$$

So, this is a straight line of slope — $20 \times \mathrm{P}$ dB/decade but again intersecting with 0 $dB$ line at $\omega = 1$

Therefore, magnitude plot for 'P' poles at the origin gives a family of lines passing through intersection of $\omega = 1$ and 0 $dB$ line having slope $-20 \times \mathrm{P}$ dB/decade as shown in the Fig. 7.7.

Now if there is zero at the origin i.e.

$$G(s)H(s) = s$$

$$G(j\omega)H(j\omega) = 0 + j\omega$$

$$\therefore \text{ magnitude in dB} = 20 \log \omega \; dB$$

This is equation of a straight line whose slope is $+20$ dB/ decade. The only change is the sign of the slope, for pole it is — 20 dB/decade while for zero it is $+20$ dB/decade but for both, intersection of line with 0 dB occurs at $\omega = 1$ only.

In general, for P number of zeros at the origin

$$G(s)H(s) = s^P$$

$$\therefore \quad G(j\omega)H(j\omega) = j\omega \cdot j\omega \cdot j\omega \cdots P \text{ times}$$

$$\therefore \quad |G(j\omega)H(j\omega)| = \omega^P$$

$$\therefore \quad \text{Magnitude in dB} = 20 \times \mathrm{P} \text{ Log } \omega$$

i.e. $$\text{slope} = +20 \times \mathrm{P} \text{ dB/decade}$$

So, it gives family of lines with slopes as $+20, +40 \ldots\ldots +20 \times P$ dB/decade passing through intersection point of $\omega = 1$ with 0 dB line as shown in the Fig. 7.8.

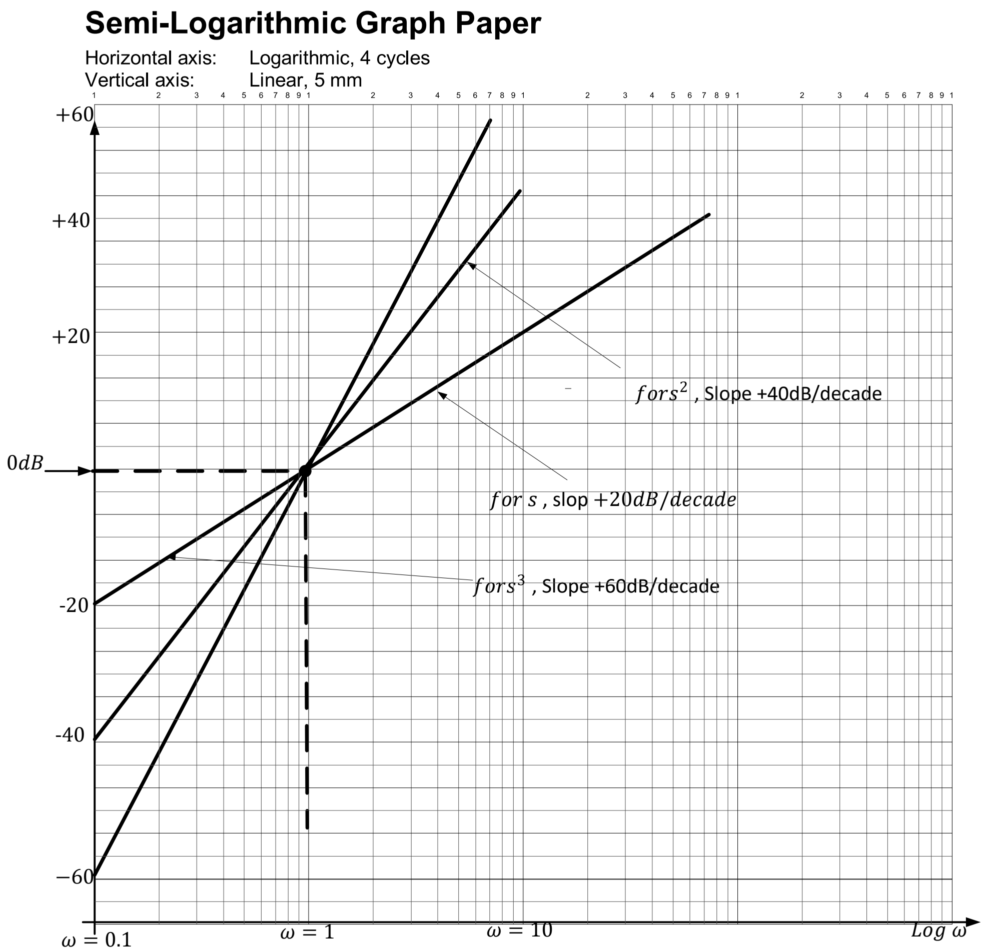



**Figure 7.8 Contribution by zeros at origin**

A zero at the origin increases the magnitude at a rate of $+20$ dB/decade.

Phase Angle Plot: Consider 1 pole at the origin

$$G(s)H(s) = \frac{1}{s} G(j\omega)H(j\omega) = \frac{1}{j\omega}$$

$$\therefore \quad \angle G(j\omega)\, H(j\omega) = \angle\frac{1}{j\omega}\ \angle\frac{1}{j\omega} = \frac{0^o}{90^o \cdot 90^o} = -180^o$$

This is independent of '$\omega$'. So, phase angle plot of 'pole at origin is line parallel to x-axis contributing $-90^o$ to phase angle.

For 2 poles at origin, $\quad G(s)H(s) = \frac{1}{s^2}$

$$\therefore \quad G(j\omega)\ H(j\omega) = \frac{1}{j\omega} \cdot \frac{1}{j\omega}$$

$$\therefore \quad \angle\, G(j\omega)\ H(j\omega) = \angle\frac{1}{j\omega}\angle\frac{1}{j\omega} = \frac{0^o}{90^o \cdot 90^o} = -180^o$$

Angle gets added to each other.

In general P number of poles at the origin contribute $-90^o \times P$ angle to overall phase angle plot. This contribution is irrespective of $\omega$

Similarly, for a zero at the origin,

$$G(s)H(s) = s$$

$$\therefore \quad G(j\omega)\ H(j\omega) = j\omega$$

$$\therefore \quad \angle G(j\omega)H(j\omega) = \angle 0 + j\omega = +\tan^{-1}\frac{\omega}{0} = +90^o$$

1 zero at the origin contributes $+90^o$. The contribution is the same as that of the pole; the only change is its sign. In general, 'P' number of zeros at the origin, the total angle contribution is $+90^o$xP, irrespective of value of $\omega$. This can be shown as in Fig. 7.9.

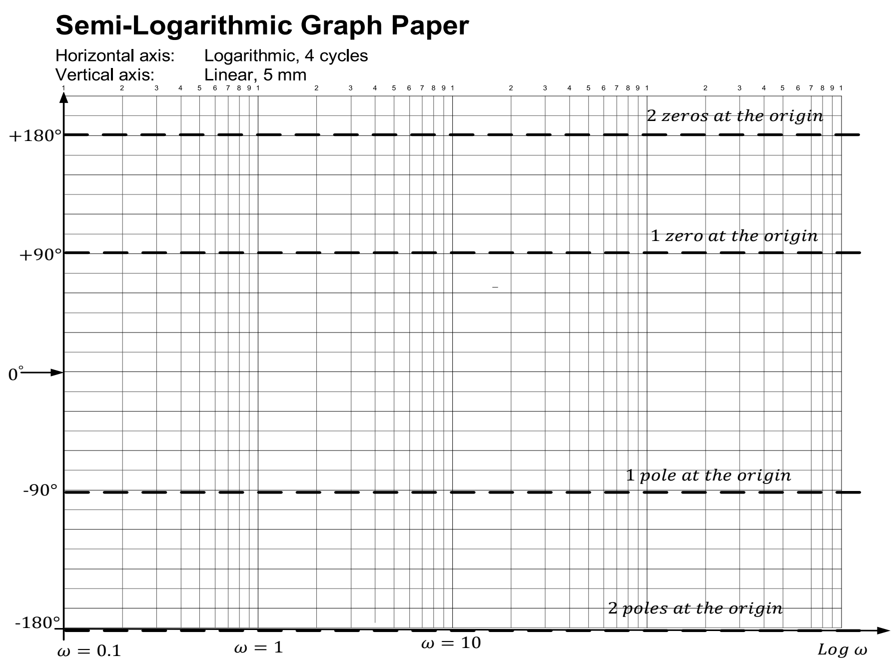



**Figure 7.9 Angle contribution**

Before going to the next factor, let us see the addition of the first two factors on semi-log paper.

The magnitude plots for poles or zeros at the origin are straight lines having slope $-20 \times P$ dB/decade $or$ $+20 \times P$ $d$B/decade respectively passing through intersection point of $\omega = 1$ and $0dB$ line. Now adding magnitude plot of 'K' to the above means to shift the straight line drawn upwards or downwards by $20\ Log\ KdB$ depending . on whether K is greater or less than 1. This shift is experienced, will be same by all points on the straight-line representing poles or zeros at the origin. Hence net addition of 'K' and pole or zeros at origin will be a line parallel to line representing poles or zeros at the origin at a distance of $20\ Log\ K\ dB$ above or below the 0 dB line.

Consider $G(s)\ H(s) = \frac{10}{s}$ so $G(j\omega)\ H(j\omega) = \frac{10}{j\omega}$

### 7.3.2 Factors are:

i. Constant $K = 10$, its contribution to magnitude plot is $20 \log K = 20 \log 10 = +20dB$.
ii. 1 pole at the origin whose magnitude plot is straight line of slope $-20$ dB/decadepassing through intersection point of $\omega = 1$ and $0\ dB$ line.

Now at $\omega = 1$, total magnitude will be addition of magnitudes of K and $1/s$.

i.e. $= 20$ dB due to 'K' $+0\ dB$ due to 1 pole at origin

at $\omega = 1$ $= 20$ dB

i.e. after addition of two lines, intersection point of $\omega = 1$ and 0 dB will shift upwards by 20 dB. So, to draw resultant of the two, we can generalise the procedure as,

i. Draw magnitude plot for K.
ii. Draw straight line representing pole at origin i.e. of slope $-20 \frac{\text{Db}}{\text{decade}}$.
iii. Shift intersection point of $\omega = 1$ and $0\ dB$ on the line representing $20 Log\ K$ line.

iv. +Draw a parallel line to the line representing the pole at the origin from the point obtained in step (iii).

The slope of this line will be the same as the slope of the line representing poles or zeros at the origin. In this example slope of resultant line will be $-20\ \mathrm{dB/decade}$. This is because slope of $20\ Log\ K$ line is $0\ \mathrm{dB/decade}$.

Note: When two lines are added together, the resultant line always has a slope which is algebraic addition of the individual slopes of the two lines which are to be added.

So, magnitude plot for above $G(s)\ H(s)$ is shown on the next page.

**Phase Angle Plot:**

Prepare the table of individual angle contributions and add them to get resultant phase angles.

| $\omega$ | Contribution by K | By 1 pole at origin | Resultant $\emptyset_R$ |
|---|---|---|---|
| 0 | $0^o$ | $-90^o$ | $-90^o$ |
| 10 | $0^o$ | $-90^o$ | $-90^o$ |
| 40 | $0^o$ | $-90^o$ | $-90^o$ |
| 1000 | $0^o$ | $-90^o$ | $-90^o$ |
| ∞ | $0^o$ | $-90^o$ | $-90^o$ |

So, phase angle plot is straight line parallel to x-axis as shown with phase angle $-90^o$

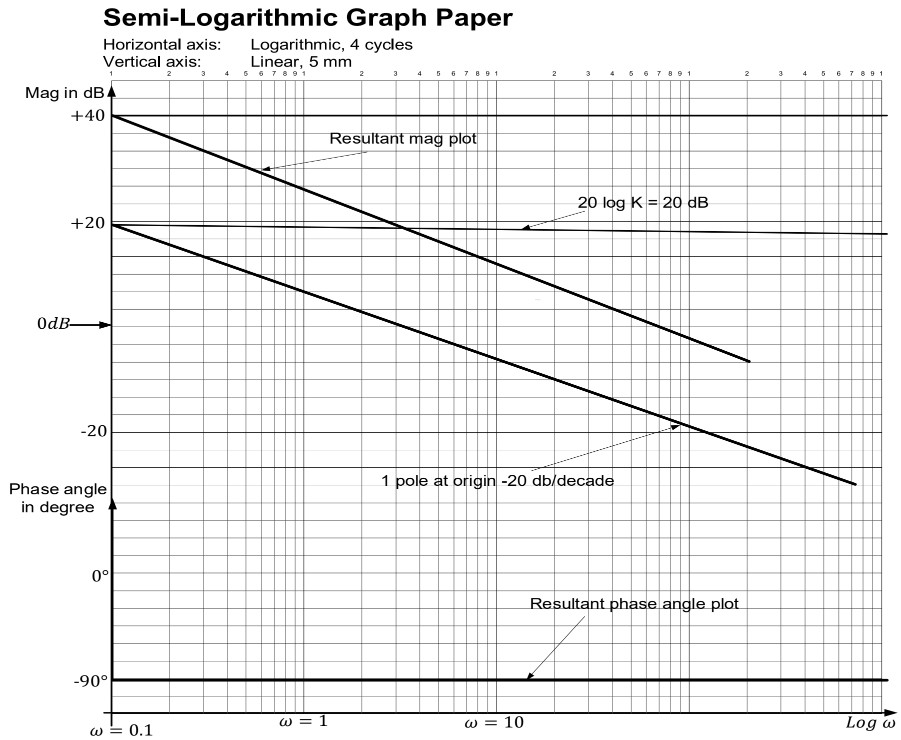



**Figure 7.10**

So, whatever may be the open loop T.F. $G(s)H(s)$ factors, the first step in sketching Bode plot should be the line adding poles at the origin or zeros at the origin and $20\ Log\ K$ line by the procedure discussed above.

From above discussion we can conclude one important fact that "The starting slope of the Bode Plot for the function $G(s)\ H(s)$ gets decided by number of poles or zeros at origin present in $G(s)\ H(s)$."

E.g., Starting slope of Bode Plot for $G(s)H(s) = \frac{10}{s}$ is $-20\ \text{dB/decade}$ as there is one pole at the origin.

Starting slope of Bode Plot for $G(s)H(s) = \frac{20(s+1)}{s^2(s+2)(s+4)}$ will be — $40\ \text{dB/decade}$ as there are 2 poles s at the origin in $G(s)H(s)$. Let us go to the next factor.

### 7.3.3 Factor 3: Simple poles or zeros (First order factors)

$$(1 + Ts)^{\pm 1} \text{ i.e } (1 + j\omega T)^{\pm 1}$$

Let us start with a simple pole

$$G(s)H(s) = (1 + Ts)^{-1} = \frac{1}{(1 + Ts)} \Longrightarrow G(j\omega)\ H(j\omega) = \frac{1}{1 + Tj\omega}$$

$$\therefore \qquad |G(j\omega)\ H(j\omega)| = \frac{1}{\sqrt{1 + (\omega T)^2}} = \left[\sqrt{1 + (\omega T)^2}\right]^{-1}$$

$$\therefore \qquad \text{in dB magnitude} = 20\log[1 + (\omega T)^2]^{-1} = -20\log\sqrt{1 + \omega^2 T^2}\ \text{dB}$$

Now instead of sketching a magnitude plot exactly according to this equation we can approximate this into two regions and can draw a straight-line approximated magnitude plot and then by applying corrections we can modify it to an accurate one if required.

The approximation is,

i. For low frequency range $\omega \ll \frac{1}{T}$ i.e $\omega^2 T^2 \ll 1$ hence can be neglected.

$$\therefore \qquad \text{Magnitude in dB} = -20\log 1 = 0\ \text{dB}$$

So, for low frequencies it is straight line of $0\ dB$ only. Thus, the contribution by such a factor can be completely neglected for low frequency range, as it is very small.

ii. For high frequency range $\omega \gg \frac{1}{T}$ $\therefore 1 \ll \omega^2 T^2$

Magnitude in dB $= -20\log\omega\, T\ \text{dB}$

i.e. it is straight line of slope $-20\ \text{dB/decade}$. As again for every decade (10 times) change in $'\omega'$magnitude still decreases by 20 i.e. slope is $-20\ \text{dB/decade}$. But the intersection of this line with $0\ \text{dB}$ line will give us a range of high frequency and low frequency. i.e. two lines, O dB line for low '$\omega$' and line with slope $-20\ \text{dB/decade}$for high are going to intersect when, $-20\log\omega T = 0\ \text{dB}$

i.e $\qquad \omega T = 1$

i.e $\qquad \omega = \frac{1}{T}$

This frequency at which change of slope from $0\ dB$ to $-20\ \text{dB/decade}$ occurs is called as **Corner Frequency**, denoted by $\omega_c$.

$$\omega_C = \frac{1}{T}$$

Hence asymptotic i.e. approximate magnitude plot for such factor is 0 dB line up to $\omega_C = \frac{1}{T}$ and line of slope $-20\ \mathrm{dB/decade}$, when $\omega > \omega_C$ i.e. above $\omega_C = \frac{1}{T}$. The magnitude plot shown above is called the **Asymptotic Magnitude Plot**.

**Error Application**:

Now let us see how to apply error correction for such asymptotic plots, if required.

The actual equation of magnitude plot is

Magnitude in dB $= -20 \log \sqrt{1 + \omega^2 T^2}$

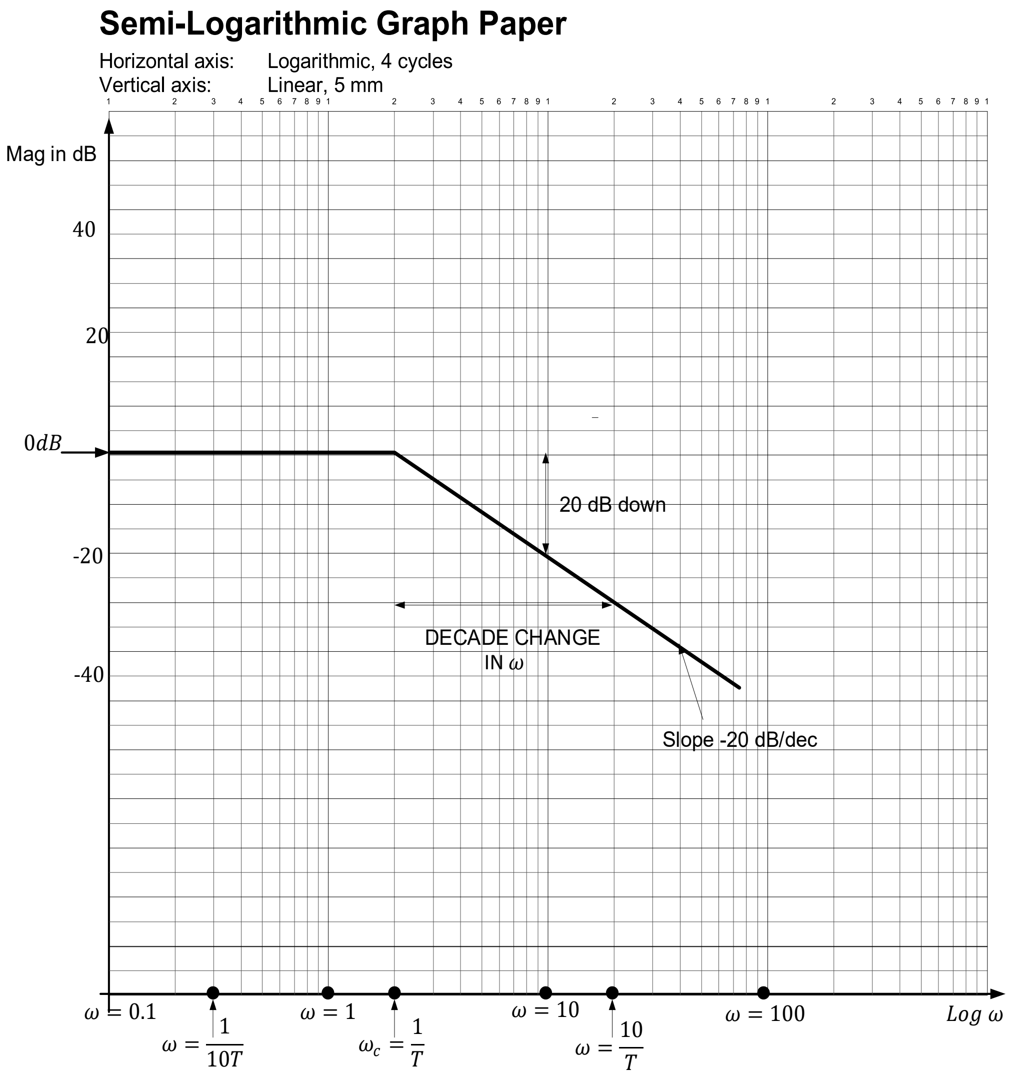



**Figure 7.11**

Now by approximation at $\omega = \omega_C = \frac{1}{T}$, magnitude in $dB = 0$ dB. But actually, it can be calculated as,

$$\text{Actual magnitude in } dB = -20\log\sqrt{1+\omega_c^2 T^2}$$

$$= -20\log\sqrt{1+\frac{1}{T^2}\cdot T^2}\,, \quad \text{substituting } \omega_C = \frac{1}{T} = -20\log\sqrt{2} = -3\text{ dB}$$

Similarly, at $\quad \omega = \dfrac{2}{T}$

Actual magnitude in dB $= -20\log\sqrt{1+\frac{4}{T^2}\cdot T^2} = -20\log\sqrt{5} = -7\text{ dB}$

But approximate magnitude in dB at $\omega = \frac{2}{T}$ is

$$= -20\log\omega T = -20\log\frac{2}{T}\times T$$

$$= -6\text{ dB}$$

∴ Correction at $\omega = \frac{2}{T}$ is $-1$dB i.e. 1 dB down

While at $\omega = \dfrac{1}{2T}$

Actual magnitude in dB $= -20\,Log\sqrt{1+\frac{1}{4T^2}T^2} = -20\log\frac{\sqrt{5}}{4} = -1\text{ dB}$

Approximately $=$ 0 dB as $\omega = \frac{1}{2T}$ is less than $\omega_c$ i.e. in low frequency range.

Correction at $\omega = \frac{1}{2T} = 1$ dB i.e 1 dB down

General error values are

**Table 7.1**

| Frequency | $\omega \to \omega_c$ | $2\omega_c$ | $\frac{\omega_c}{2}$ |
|---|---|---|---|
| Error | 3 dB down | 1dB down | 1dB down |

By applying these errors, the actual magnitude plot may be obtained, if required, as shown in Fig.7.12

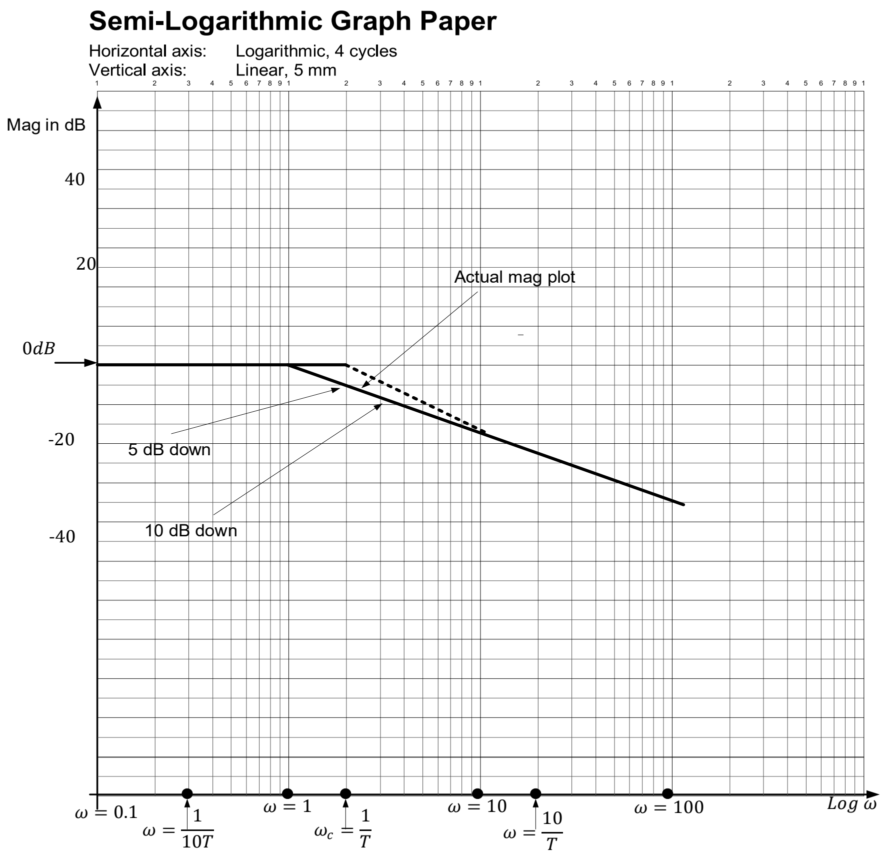



**Figure 7.12**

For simple zero, i.e. first order zero,

$$G(s)\ H(s) = (1 + Ts)$$

$$G(j\omega)H(j\omega) = (1 + j\omega T)$$

$$|G(j\omega)\ H(j\omega)| = \sqrt{1 + (\omega T)^2}$$

$\therefore$ Magnitude in dB $= 20 \log \sqrt{1 + \omega^2 T^2} \cdot \text{dB}$

Hence all the analysis is applicable for a simple zero with change in sign of slope.

The magnitude plot for simple zero is a straight line of 0 dB up to $\omega_c = 1/T$ and then straight line of slope $+20\text{dB/decade}$ for all frequencies more than corner frequency. The errors are $+3dB$ for $\omega_c$ and $+1dB$ for $\omega_c = 2\omega_c$ or $\frac{\omega_c}{2}$. Hence analysis of a simple zero is very much simple when analysis of a simple pole is clear.

It is as shown in Fig. 7.13

Note that varying the value of 'T' i.e. time constant, it shifts the corner frequency '$\omega_c$' to the right or 'left, but the shape remains the same as above.

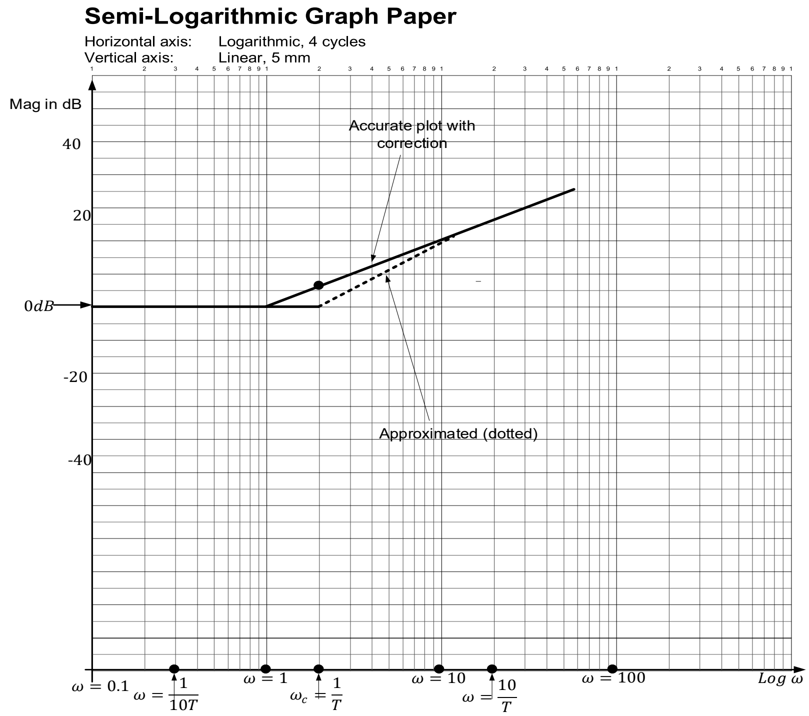



**Figure 7.13**

**Phase Angle Plot**: Consider a simple pole

$$G(s)H(s) = \frac{1}{1+Ts}$$

$$G(j\omega)\,H(j\omega) = \frac{1}{1+j\omega T} \quad \therefore \ \angle G(j\omega)\,H(j\omega) = \frac{0^o}{\tan^{-1}\frac{\omega T}{1}} = \tan^{-1}\omega T$$

While for a simple zero,

$$G(s)\,H(s) = 1 + Ts$$

$$G(j\omega)\,H(j\omega) = 1 + j\omega T \quad \therefore \ \angle G(j\omega)\,H(j\omega) = \tan^{-1}\frac{\omega T}{1} = +\tan^{-1}\omega T$$

So, the shape remains the same. Only the sign of the angle's changes from negative to positive when the factor changes from pole to zero. Such a plot is to be constructed by actually calculating angles for different frequencies. So, make a table as shown in table 7.2.

Table 7.2

| $\omega$ | $\pm\ \tan^{-1}\omega T$ (+for zero, −for pole) |
|---|---|
| $\frac{\omega_c}{10} = \frac{1}{10T}$ | $\pm 5.71^o$ |
| $\frac{\omega_c}{2} = \frac{1}{2T}$ | $\pm 26.6^o$ |
| $\omega_c = \frac{1}{T}$ | $\pm 45^o$ |
| $2\omega_c = \frac{2}{T}$ | $\pm 63.4^o$ |
| $10\omega_c = \frac{10}{T}$ | $\pm 84.3^o$ |

This can be shown as in Fig. 7.14

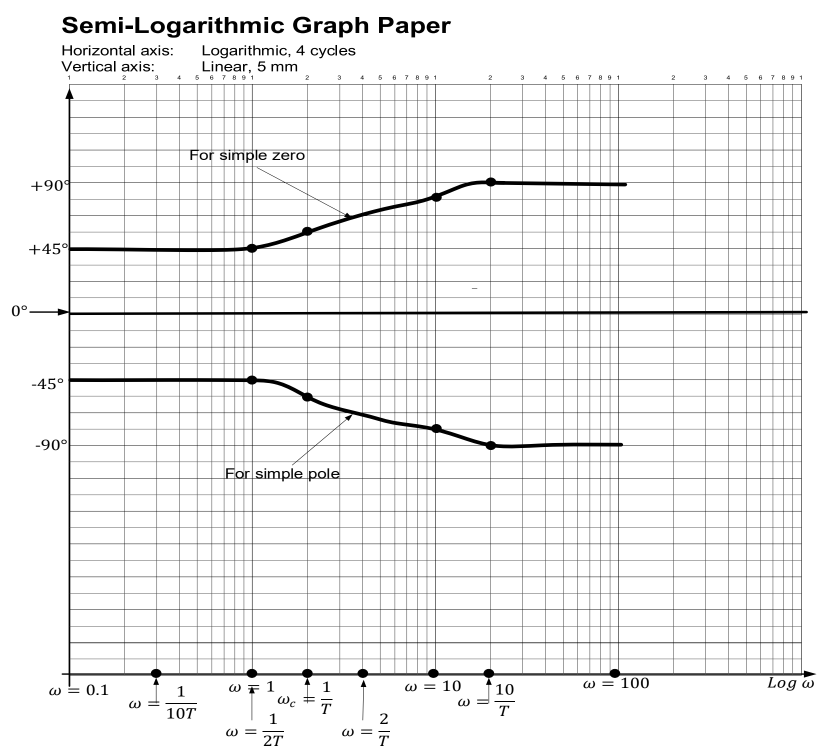



**Figure 7.14**

The shapes will remain the same, for various values of 'T' time constants.

It is important to note that phase angle is $\pm 45^o$ for a zero or pole at $\omega = \omega_c = \frac{1}{T}$

**Example 7.1**: Sketch the Bode Plot for the system having

$$G(s)\ H(s) = \frac{20}{s(1+0.1s)}$$

Sol.: First see that given $G(s)\ H(s)$ is in the proper time constant form or not. If not arrange it in the time constant form. Now identify the factors.

i. $K = 20$

Its magnitude $= 20\ Log\ 20 = +26\ \text{dB}$

ii. 1 pole at origin. Its magnitude plot is straight line passing through intersection point of $\omega = 1$ and $0\ \text{dB}$ with slope — $20\ \text{dB/decade}$.

iii. Simple pole → $\frac{1}{1+0.1s}$ comparing with $\frac{1}{1+Ts}$

$$T = 0.1$$

$$\omega_c = \frac{1}{T} = \frac{1}{0.1} = 10$$

i.e. **Asymptotic magnitude plot** is $0\ \text{dB}$ up to $\omega = \omega_c = 10$ and then straight line of slope — $20\ \text{dB/decade}$. Procedure to Plot resultant

i. Draw $20\ Log\ K$ line.
ii. Draw line for 1 pole at origin
iii. Shift intersection point of $\omega = 1$ and $0\ \text{dB}$ on $20\ log\ K$ line and from this point draw parallel to a line representing 1 pole at origin. This line will have slope — $20\text{dB/decade}$.
iv. This addition of K and pole at origin will continue, as it is till next factor becomes dominant i.e. at $\omega = \omega_c = 10$.

Hence resultant slope from $\omega = 10$ onwards will be (— $20\ \text{dB/decadeas starting}$ slope) + (— $20\ \text{dB/decade}$) due to simple pole i.e. resultant — $40\ \text{dB/decade}$. This will continue up to $\omega \to \infty$ as there is no other factor present in $G(s)H(s)$.

Procedure to draw — $40\ \text{dB/decade}$ line: Mark the point of intersection of $\omega = 10$ and line representing addition of K and $1/s$. From this we want to draw slope of

— $40$ dB/decade. So, whatever is the magnitude of $G(j\omega)\ H(j\omega)$ corresponding to this intersection point, will get reduced by $40$ dB for a decade change in $\omega$ i.e. at $\omega = 100$.

So, mark that point and draw the line. OR on semi-log paper itself, draw the lines of different slopes as $— 20, -40, - 60, — 80, +20\text{dB/decade}$ etc. very light as shown and then draw parallel to these lines of the required slope in magnitude plot. Such lines are shown in Fig.7.15. Draw such lines very light and then draw parallel to these lines from the required point of required slope.

For the phase angle plot prepare the table of angles as below

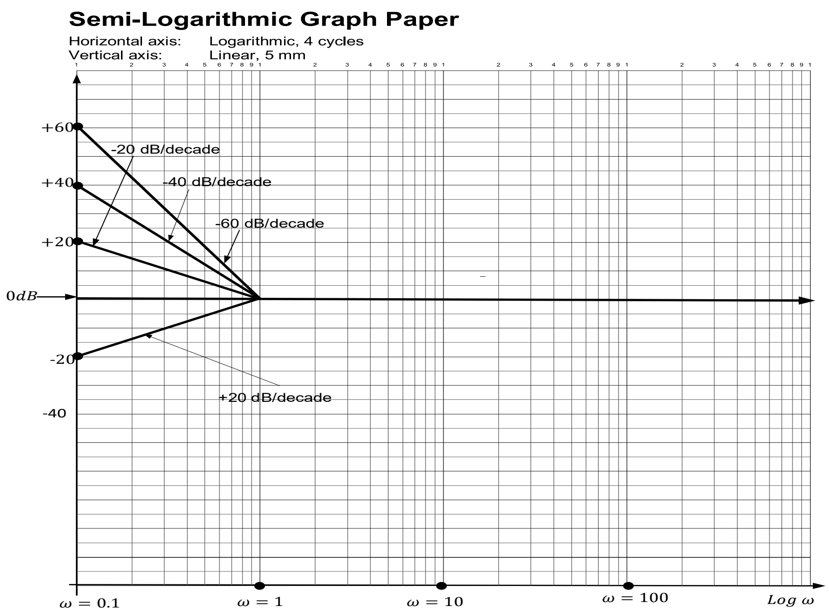



**Figure 7.15**

**Table 7.3**

| $\omega$ in rad sec | Ø due to 1 pole at origin | Ø due to simple pole $\tan^{-1} 0.1\omega$ | $\text{Ø}_R$ Resultant |
|---|---|---|---|
| 0.1 | $-90^o$ | $-0.57^o$ | $-90.57^o$ |
| 0.5 | $-90^o$ | $-2.86^o$ | $-92.86^o$ |
| 1 | $-90^o$ | $-5.7^o$ | $-95.7^o$ |
| 2 | $-90^o$ | $-11.3^o$ | $-101.3^o$ |
| 10 | $-90^o$ | $-45^o$ | $-135^o$ |
| 50 | $-90^o$ | $-78.79^o$ | $-168^o$ |

$\emptyset$ due to simple pole $= -\tan^{-1}\omega T = -\tan^{-1}0.1\,\omega$

$\emptyset$ due to 1 pole at the origin is always $-90^o$. If required, more' $\omega$' values may be selected to draw the smooth curve. Let us combine all the things on semilog paper to complete the Bode plot

Observe:

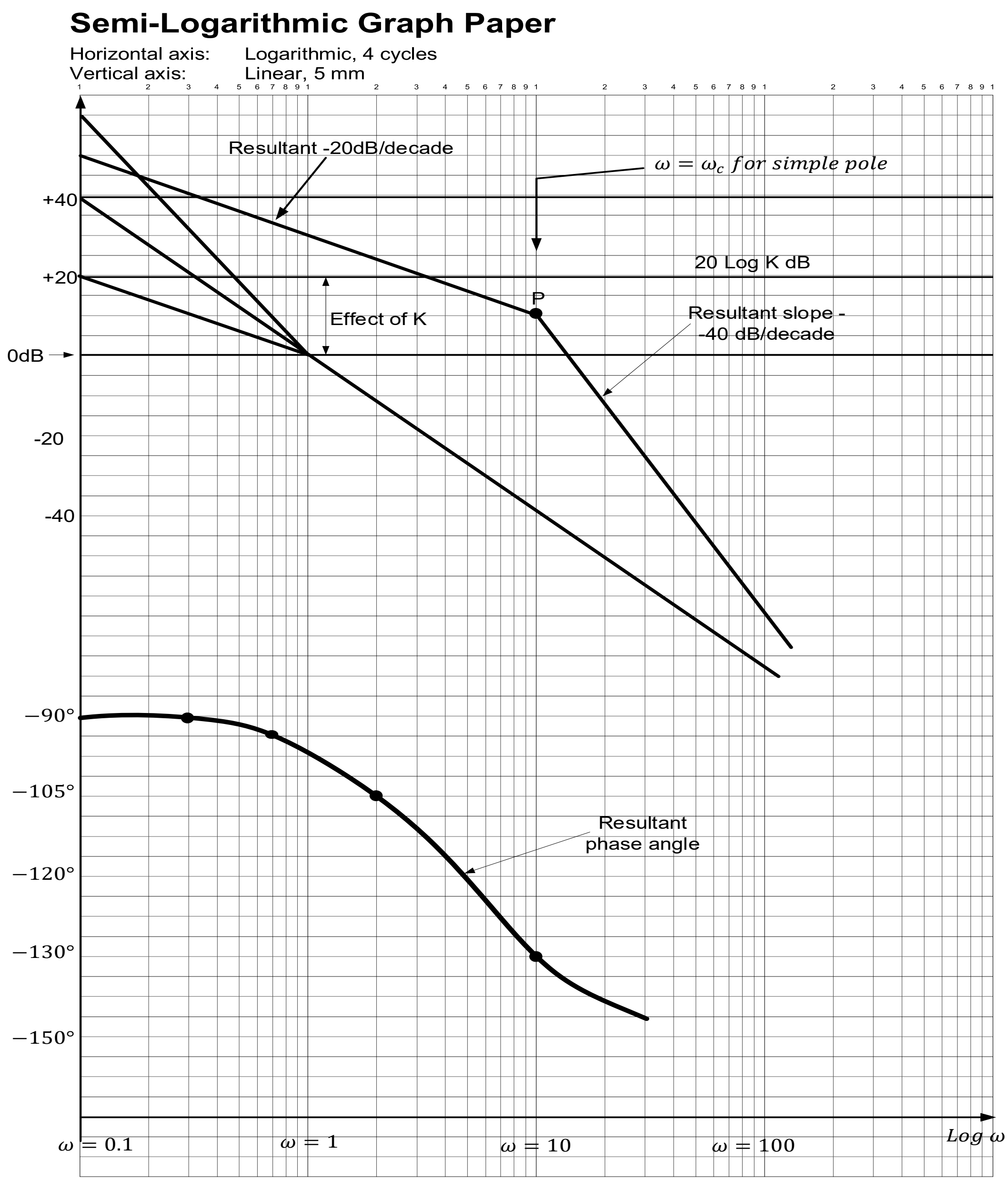



**Figure 7.16**

1. $20\ log\ K$ line
2. Line of slope — $20$ dB/decade as only 1 pole at origin.
3. Intersection point of $\omega = 1$ and 0 dB shifted on $20\ Log\ K$ line and line parallel to $-20$ dB/ decade is drawn which is resultant of K and $1/\ s$.
4. This continued till next factor becomes dominant i.e. $\omega = \omega_c = 10$. Till $\omega = \omega_c = 10$, simple pole contributes 0dB only and there is no change in the slope.

So, from intersection point $-20$ dB/decade line and $\omega = 10$ line i.e. point P shown slope is changed by $-20$ dB/decade and hence directly resultant of slope $-20 + (-20) =$ — 40 dB/ decade is drawn from point P. This is drawn parallel to $-40$ dB/decade line drawn light on semilog paper shown in Fig. 7.16.

### 7.3.4 Factor 4: Quadratic factors

Consider quadratic pole of the form,

$$G(s)\ H(s) = \frac{1}{1 + \frac{2\xi}{\omega_n} s + \frac{s^2}{\omega_n^2}} \text{ expressed in time constant form}$$

$$\therefore \quad G(j\omega)\ H(j\omega) = \frac{1}{1 + 2\xi j \left(\frac{\omega}{\omega_n}\right) + \left(\frac{j\omega}{\omega_n}\right)^2}$$

where $\omega$ is variable and $\omega_n$ is constant for that factor.

$$= \frac{1}{1 + 2\xi\ j\left(\frac{\omega}{\omega_n}\right) - \left(\frac{\omega}{\omega_n}\right)^2} \quad as\ j^2 = 1 = \frac{1}{\left\{1 - \left[\frac{\omega}{\omega_n}\right]^2\right\} + \text{j}\ 2\xi\ \left(\frac{\omega}{\omega_n}\right)}$$

$$\therefore \quad |G(j\omega)\ H(j\omega)| = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\xi^2 \left(\frac{\omega}{\omega_n}\right)^2}}$$

$$\therefore \text{ Magnitude in dB } = 20 \log \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\xi^2 \left(\frac{\omega}{\omega_n}\right)^2}}$$

$$= 20 \log \left( \sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\xi^2 \left(\frac{\omega}{\omega_n}\right)^2} \right)^{-1}$$

$$\therefore \quad \text{Magnitude in dB } = -20 \log \sqrt{\left(1 - \left[\frac{\omega}{\omega_n}\right]^2\right)^2 + 4\xi^2 \left(\frac{\omega}{\omega_n}\right)^2} \text{ dB}$$

Approximation:

For low frequency, $\omega \ll \omega_n \quad \therefore \left(\frac{\omega}{\omega_n}\right)^2 \ll 1$

$\therefore \quad$ Mag in $dB = -20 \log 1 = 0 \text{ dB}$

Thus, similar to a simple pole, quadratic pole also is negligible till its corner frequency occurs.

For high frequency, $\omega > \omega_n$ and $4\xi^2 \left(\frac{\omega}{\omega_n}\right)^2 \ll \left(\frac{\omega}{\omega_n}\right)^4$ as $\xi$ is very low.

$$\therefore \quad \text{Magnitude in dB } = -20 \log \sqrt{\left[\left(\frac{\omega}{\omega_n}\right)^2\right]^2}$$

$$= -20 \log \left(\frac{\omega}{\omega_n}\right)^2 = -40 \log \frac{\omega}{\omega_n} = -40 \log \omega + 40 \log \omega_n$$

This is equation of straight line of slope — $40 \text{ dB/decade}$.

Hence general magnitude plot for quadratic factor is $0 \text{ dB}$ line till corner frequency and then a straight line of slope — $40 \text{ dB/decade}$.

To find corner frequency $\omega_c$,

$$-40 \log \frac{\omega}{\omega_n} = 0 \, dB$$

$$\text{i.e.} \frac{\omega}{\omega_n} = 1$$

$$\therefore \quad \omega_c = \omega_n$$

So $\omega_n$ is the corner frequency for such factor

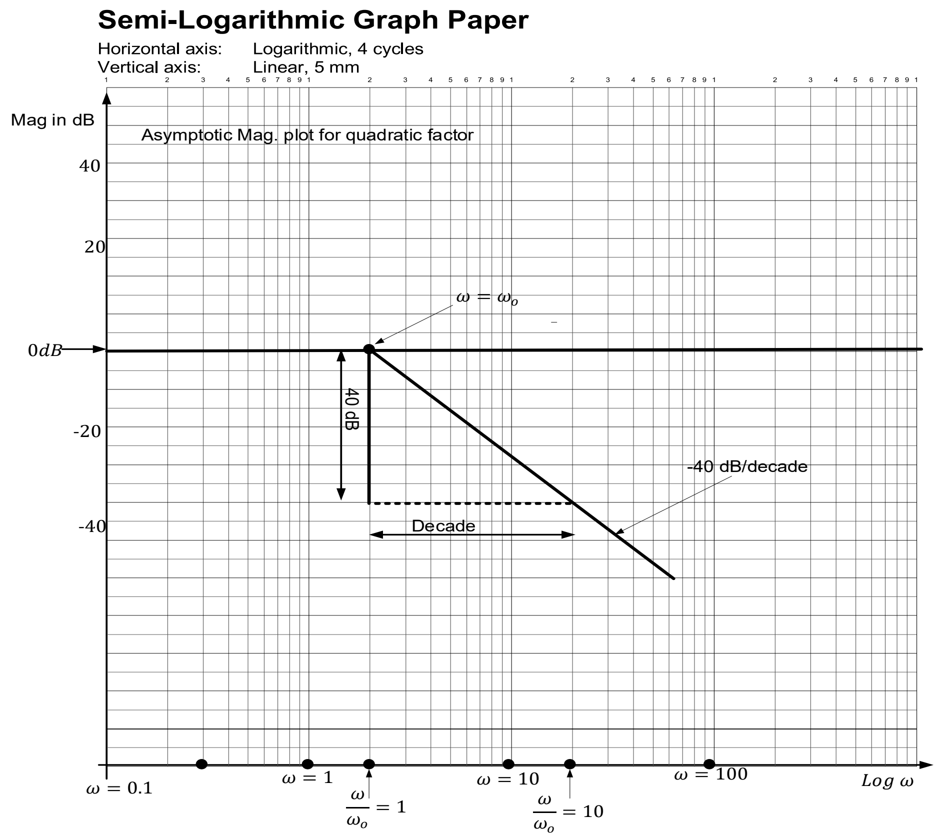



**Figure 7.17**

But the above asymptotic plot is not so accurate as the error for the quadratic factor not only depend on $\omega$ but also on the value of $\xi$ damping ratio.

Let us see the effect of variation of $\xi$ on the magnitude plot.

$$\text{Actual magnitude in dB } = -20\log\sqrt{\left(1-\left[\frac{\omega}{\omega_n}\right]^2\right)^2 + 4\xi^2\left(\frac{\omega}{\omega_n}\right)^2}$$

Now at $\omega = \omega_n \Longrightarrow \frac{\omega}{\omega_n} = 1$

Actual magnitude in dB $= -20\log\sqrt{4\xi^2}$. Let us prepare a table for various values of and corresponding error values. See Table 7.4

**Table 7.4**

| $\xi$ | Accurate magnitude in dB | Approximate magnitude in dB | error for quadratic pole |
|---|---|---|---|
| 0.1 | $+ 13.97$ | 0 | + 13.97 dB up |
| 0.2 | $+ 7.95$ | 0 | + 7.95 dB up |
| 0.3 | $+ 4.43$ | 0 | + 4.43 dB up |
| 0.4 | $+ 1.93$ | 0 | + 1.93 dB up |
| 0.7 | $- 2.92$ | 0 | 2.92 dB down |
| 0.9 | $- 5.10$ | 0 | 5.1 dB down |
| 1 | $- 6.02$ | 0 | 6 dB down |

And due to the errors calculated above the magnitude plot for quadratic pole gets modified as shown in Fig. 7.18:

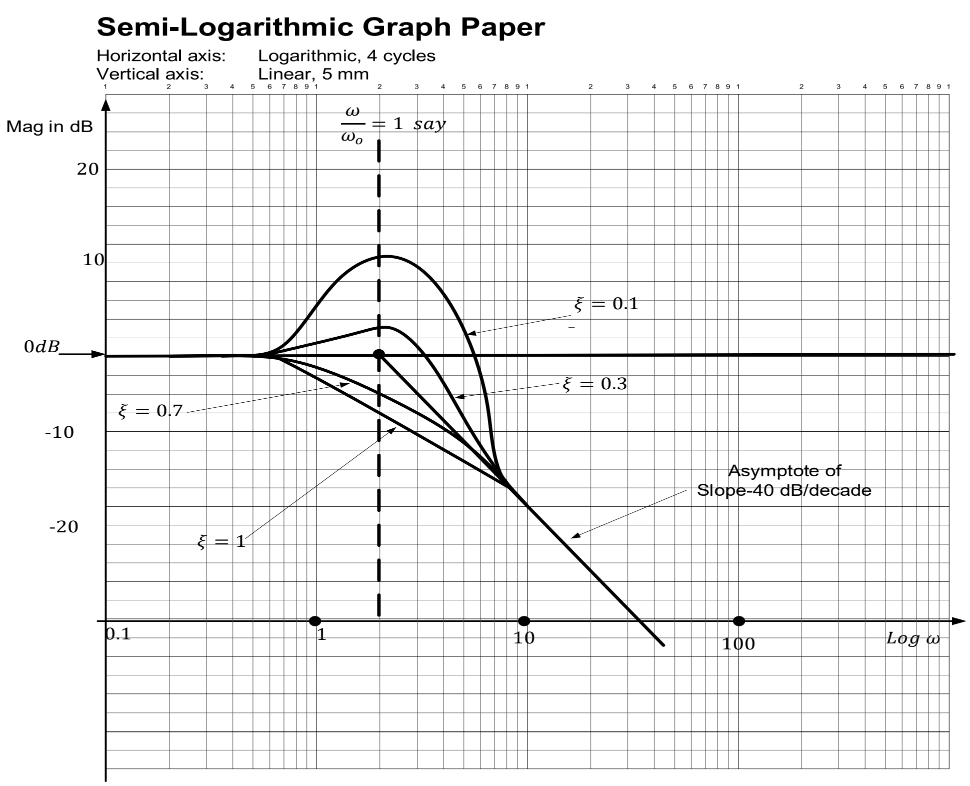



**Figure 7.18 Quadratic pole**

Hence it is necessary to modify magnitude plot for 2nd order quadratic pole as shown above at its cornet' frequency for various values of $'\xi'$

Students can use Table 7.4 to decide correction for given $\xi$ or find the correction using the formula,

Correction $= -20 \log 2\xi \;\; dB$ at $\omega = \omega_n$ of pole

Positive correction upwards and negative correction downwards.

The magnitude plot to a quadratic zero can be obtained by reversing the sign of the slop" of' basic asymptote and then by reversing the signs of the corrections at corner frequency for various values of $\xi$. Hence it looks like as shown in Fig. 7.19

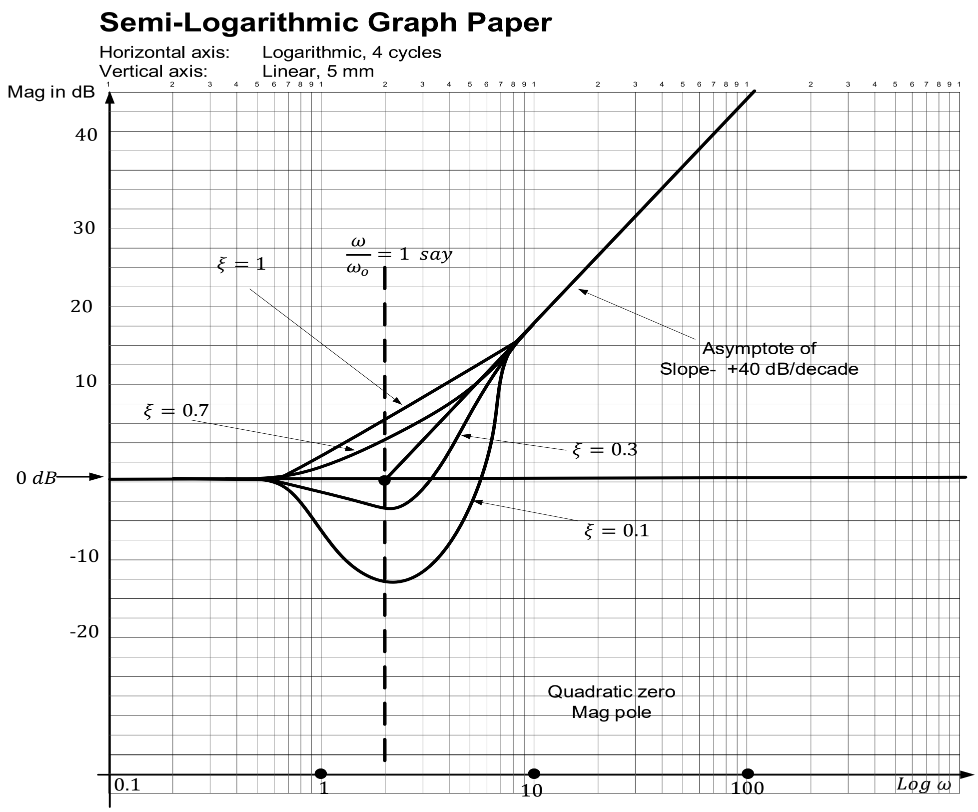



**Figure 7.19 Quadratic zero**

Let us see phase angle table:

$$G(j\omega)\,H(j\omega) = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2 + 2\xi\,\left(\frac{\omega}{\omega_n}\right)} \text{ for a quadratic pole}$$

$$\therefore \qquad \angle G(j\omega)\,H(j\omega) = \frac{0^o}{\tan^{-1}\left\{\frac{2\xi\,(\omega/\omega_n)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right\}}$$

$$\angle\, G(j\omega)\,H(j\omega) = \tan^{-1}\left\{\frac{2\xi\,(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right\}$$

The table for $\xi = 0.3$ is shown below

$$\phi = \tan^{-1}\left\{\frac{2\times 0.3\times\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right\}$$

**Table 7.5**

| $\frac{\omega}{\omega_n}$ | $\emptyset$ |
|---|---|
| 0.1 | $-3.46^o$ |
| 0.5 | $-21.8^o$ |
| 1 | $-90^o$ |

So at $\frac{\omega}{\omega_n} = 1$ i.e, $\omega = \omega_c = \omega_n$ it contributes $-90^o$ and hence must approach to $-180^o$ as $\frac{\omega}{\omega_n} \to \infty$ But according to above formula when $\frac{\omega}{\omega_n} > 1,$ $\emptyset$ becomes positive, in such case the angle contribution obtained must be considered, by subtracting $180^o$ from the positive $\emptyset$.

e.g. $\qquad \frac{\omega}{\omega_n} = 2 \quad \therefore \emptyset = -\tan^{-1}[-0.4] = -(-21.8) = +21.8^o$

But the actual angle contribution must be considered by applying correction of $-180^o$ i.e., $21.8^o - 180^o = -158.19^o$. This happens because behaviour of $\tan^{-1}$ functions for the complex quantities with real part negative or imaginary part negative cannot be identified on the calculator by using the above formula. Hence phase angle table becomes,

**Table 7.6**

| $\frac{\omega}{\omega_n}$ | $\emptyset$ |
|---|---|
| 0.1 | $-3.46^o$ |
| 0.5 | $-21.8^o$ |
| 1 | $-90^o$ |
| 2 | $+21.8-180=-158.19^o$ |
| 4 | $+10.09-180=-170.9^o$ |
| 10 | $+3.46-180=-176.53^o$ |
| ⋮ | ⋮ |
| $\infty$ | $-180$ |

For quadratic zero, the sign of the angle should be made positive.

Note: For quadratic factors make sure that its roots are complex. If roots are real, factorising it and considering its two components independently as Simple factors rather than quadratic. The above discussion is applicable only when the roots of a quadratic factor are complex conjugates of each other.

### 7.4 Steps to Sketch the Bode Plot

i. Express given $G(s)\ H(s)$ into time constant form

ii. Draw a line of $20\ Log\ K\ dB$.

iii. Draw a line of appropriate slope representing poles or zeros at the origin, passing through intersection point of $\omega = 1$ and0 dB.

iv. Shift this intersection point on $20\ Log\ K$ line and draw parallel line to the line drawn in step 3. This is in addition of the constant K and number of poles or zeros at the origin.

v. Change the slope of this line at various corner frequencies by appropriate value . i.e. depending upon which factor is occurring at corner frequency. For a simple pole, slope must be changed by — 20 dB/decade, for a simple zero by +20 dB/decade etc. Do not draw these individual lines. Change the slope of the line ' obtained in step 5 by respective value and draw line with resultant slope. Continue this line till it intersects the next corner frequency line. Change the slope and continue. Apply necessary correction for quadratic factors.

vi. Prepare the phase angle table and obtain the table of $\omega$ and resultant phase angle by $\emptyset_R$ actual calculation. Plot these points and draw the smooth curve obtaining the necessary phase angle plot.

Remember that at every corner the frequency slope of the resultant line must change.

**Example 7.2**: A feedback system has $G(s)\,H(s) = \frac{100(s+4)}{s(s+0.5)(s+10)}$

Draw the bode plot and comment on stability.

Solution:

**Step 1:** Arrange $G(s)\,H(s)$ in time constant form

$$G(s)\,H(s) = \frac{100 \times 4 \times \left(1+\frac{s}{4}\right)}{s \times 0.5 \times \left(1+\frac{s}{0.5}\right) \times 10 \times \left(1+\frac{s}{10}\right)} = \frac{80\,\left(1+\frac{s}{4}\right)}{s(1+2s)\left(1+\frac{s}{10}\right)}$$

In this problem, a simple zero is added to the previous example.

**Step 2:** Factors are

i. Constant $K = 80$

ii. 1 pole at origin, $\frac{1}{s}$

iii. Simple pole, $\frac{1}{1+2s}, T_1 = 2, \omega_{C1} = \frac{1}{T_1} = 0.5$ rad/s.

iv. Simple zero, $\left(1+\frac{s}{4}\right), T_2 = \frac{1}{4}, \omega_{C2} = \frac{1}{T_2} = 4$ rad/s.

v. Simple pole, $\frac{1}{1+\frac{s}{10}}, T_3 = \frac{1}{10}, \omega_{C3} = \frac{1}{T_3} = 10$ rad/s.

**Step 3:** Magnitude plot Analysis

i. For $K = 80, 20\log K \approx 38$ dB.

ii. For 1 pole at origin, straight line of slope — $20$ dB/decade. Passing through intersection of $\omega = 1$ and

iii. Shift intersection of $\omega = 1$ and $0\ dB$ on $20logK$ line and draw parallel to $-20$ dB/decade line representing addition of K and $1/s$. This will continue till first factor becomes dominant i.e, at $\omega_{C1} = 0.5$. So, resultant must continue only up to 0.5.

iv. At $\omega_{C1} = 0.5$, simple pole occurs, individually contributing $-20$ dB/decade hence resultant pole will have slope $-20 - 20 = -40$ dB/decade from 0.5 onwards till next corner frequency occurs i.e. $\omega_{C2} = 4$

v. At $\omega_{C4} = 4$, simple zero occurs, individually contributing $+ 20$ dB/decade hence resultant plot will have slope $-40 + 20 = -20$ dB/decadeagain from '4' onwards till next corner frequency occurs i.e., $\omega_{C3} = 10$.

vi. At $\omega_{C3} = 10$, simple pole occurs, individually contributing — 20 dB/decade hence resultant plot will have slope $-20 - 20 = -40$ dB/decade, again from'10' onwards. This will continue up to $\omega \to \infty$ as there is no other factor present in $G(s)H(s)$

**Step 4:** Phase Angle Plot

$$G(j\omega)\ H(j\omega) = \frac{80\left(1 + j\frac{\omega}{4}\right)}{j\omega(1 + j2\omega)\left(1 + j\frac{\omega}{10}\right)}$$

$$\therefore \quad \angle G(j\omega)\ H(j\omega) = \frac{\angle 80 + j0\ \ \angle 1 + j\frac{\omega}{4}}{\angle j\omega\ \angle 1 + j2\omega\ \left(\angle + j\frac{\omega}{10}\right)}$$

$$\angle 80 + j0 = 0^o, \qquad \angle 1 + j\frac{\omega}{4} = +\tan^{-1}\frac{\omega}{4}, \qquad \angle\frac{1}{j\omega} = -90^o$$

$$\angle\frac{1}{1 + j2\omega} = -\tan^{-1} 2\omega\,, \quad \angle\frac{1}{1 + j\frac{\omega}{10}} = -\tan^{-1}\frac{\omega}{10}$$

∴Phase Angle

**Table 7.7**

| $\omega$ | $\frac{1}{j\omega}$ | $-\tan^{-1} 2\omega$ | $+\tan^{-1}\frac{\omega}{4}$ | $-\tan^{-1}\frac{\omega}{10}$ | $\emptyset_R$ |
|---|---|---|---|---|---|
| 0.1 | $-90^o$ | $-11.3^o$ | $+1.43^o$ | $-0.57^o$ | $-100.4^o$ |
| 0.5 | $-90^o$ | $-45^o$ | $+7.12^o$ | $-2.86^o$ | $-130.7^o$ |
| 1 | $-90^o$ | $-63.43^o$ | $+14.03^o$ | $-5.71^o$ | $-145.1^o$ |
| 2 | $-90^o$ | $-75.96^o$ | $+26.56^o$ | $-11.3^o$ | $-150.7^o$ |
| 4 | $-90^o$ | $-82.87^o$ | $+45^o$ | $-21.8^o$ | $-149.6^o$ |
| 10 | $-90^o$ | $-87.13^o$ | $+68.19^o$ | $-45^o$ | $-153.9^o$ |
| 50 | $-90^o$ | $-89.42^o$ | $+85.42^o$ | $-78.69^o$ | $-172.6^o$ |
| ∞ | $-90^o$ | $-90^o$ | $+90^o$ | $-90^o$ | $-180^o$ |

Note that as simple zero $\left(1 + j\frac{\omega}{4}\right)$ is more dominating than simple pole $\frac{1}{\left(1+j\frac{\omega}{10}\right)}$ forany frequency $+ve$ angle contribution by zero will be more than negative angle contribution by pole. Hence resultant cannot intersect $-180^o$ line but will run parallel to $-180^o$line at the end.

Hence it is clear that $\omega_{pc} = \infty$ , hence $\omega_{gc}$ is always less than $\omega_{pc}$ and hence the system is always absolutely stable. In such a case G.M. can be said to be $+\infty$ dB. P.M can be decided from Bode Plot.

**Step 5:** Bode plot and solution.

Also note that when there is addition of simple zero in the unstable system, the GM. has to increase tremendously and the system becomes stable in nature. Addition of zero in a system makes the system relatively more stable. (See Fig. 7.20)

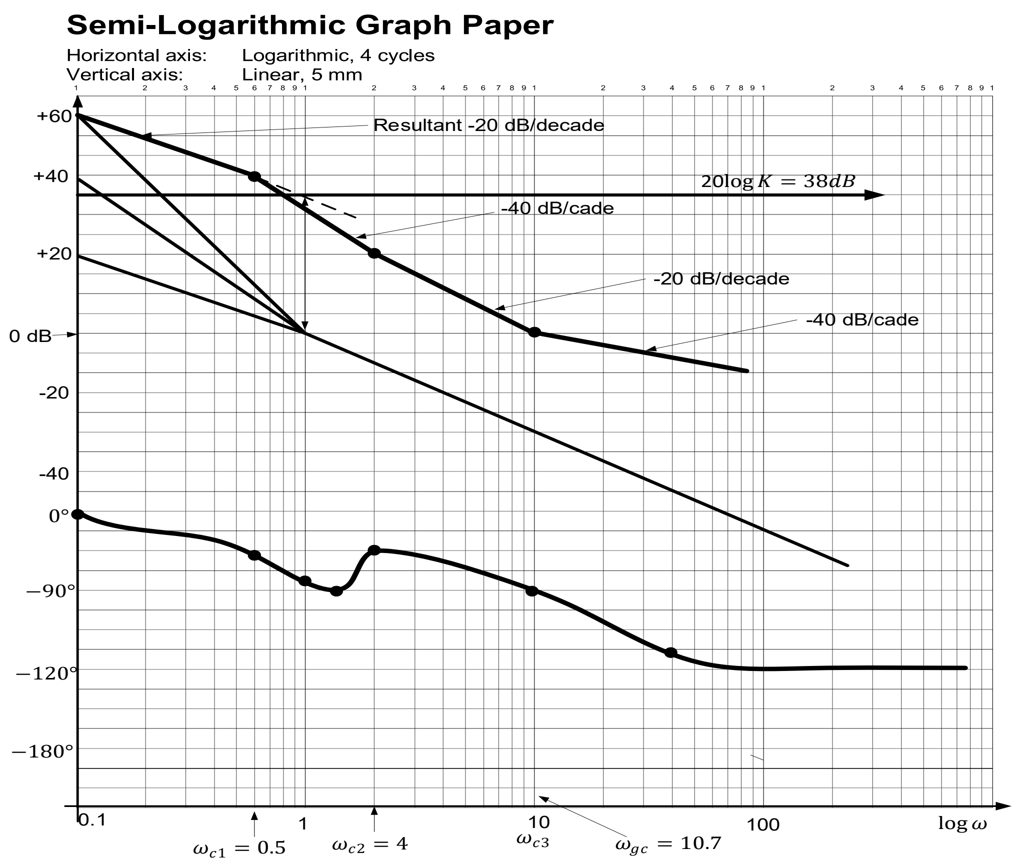



**Figure 7.20**

**Example 7.3:** For a unity feedback system $G(s) = \frac{800(s+2)}{s^2(s+10)(s+40)}$

Sketch the Bode plot, asymptotic in nature. Comment on stability,

Solution:

**Step 1**: Arrange $G(s)\ H(s)$ in time constant form.

$$G(s)H(s) = \frac{800 \times 2 \times \left(\frac{s}{2} + 1\right)}{s^2 \times 10\left(1 + \frac{s}{10}\right) \times 40 \times \left(1 + \frac{s}{40}\right)} = \frac{4\left(1 + \frac{s}{2}\right)}{s^2\left(1 + \frac{s}{10}\right)\left(1 + \frac{s}{40}\right)}$$

**Step 2**: Factors are

i. Constant $K = 4$,

ii. 2 poles at the origin, $\frac{1}{s^2}$

iii. Simple zero, $1 + \frac{s}{2},\ T_1 = \frac{1}{2},\quad \omega_{C1} = \frac{1}{T_1} = 2$ rad/s.

iv. Simple pole, $\frac{1}{1+\frac{s}{10}},\ T_2 = \frac{1}{10}, \omega_{C2} = \frac{1}{T_2} = 10$ rad/s.

v. Simple pole, $\frac{1}{1+\frac{s}{40}},\ T_3 = \frac{1}{40}, \omega_{c3} = \frac{1}{T_3} = 40$ rad/s

**Step 3**: Magnitude plot analysis

i. For $K = 4,\ 20\ log\ K = 20\ log\ 4 = 12dB$.

ii. 2 poles at the origin i.e. $\frac{1}{s^2}$ It contributes a straight line of slope -40 dB/decade passing through intersection point of $\omega = 1$ and 0 dB. So, the starting slope becomes —40 dB/decade.

iii. Shift intersection point of $\omega = 1$ and 0 dB on $20\ log\ K$ line and draw parallel line to — $40$ dB/decade. This represents addition of K and $\frac{1}{s^2}$. This resultant will continue till first corner frequency $\omega_{c1} = 2$.

iv. At $\omega_{C1} = 2$ , simple zero occurs which contributes $+20$ dB/decade individually and hence resultant slope from '2' onwards becomes $-40 + 20 = -20$ dB/decade. This continues till $\omega_{C1} = 10$.

v. At $\omega_{C2} = 10$, simple pole occurs which contributes — $20$ dB/decade individually and hence resultant slope from 10 onwards becomes $-20 - 20 = -40$ dB/decade again. This continues till $\omega_{C3} = 40$.

vi. At $\omega_{C3} = 40$, Simple pole occurs which contributes — $20\text{ dB/decade}$ individually and hence resultant slope from 40 onwards becomes $-40 - 20 = -60\text{ dB/decade}$. This continues upto $\omega \to \infty$ as there is no other factor present in $G(s)\,H(s)$.

**Step 4**: Phase Angle Plot

$$G(j\omega)H(j\omega) = \frac{4\left(1 + \frac{j\omega}{2}\right)}{(j\omega)^2\left(1 + \frac{j\omega}{10}\right)\left(1 + \frac{j\omega}{40}\right)}$$

$$\angle G(j\omega)\,H(j\omega) = \frac{\angle 4 + j0 \;\; \angle 1 + \frac{j\omega}{2}}{\angle (j\omega)^2 \;\; \angle 1 + \frac{j\omega}{10} \;\; \angle 1 + \frac{j\omega}{40}}$$

$$\angle 4 + j0 = 0^o, \qquad \angle 1 + \frac{j\omega}{2} = +\tan^{-1}\frac{\omega}{2}$$

$\angle \frac{1}{(j\omega)^2}$, 2 Poles at origin $= -180^o$ always

$$\angle \frac{1}{1 + \frac{j\omega}{10}} = -\tan^{-1}\frac{\omega}{10} \text{ and } \angle \frac{1}{1 + \frac{j\omega}{40}} = -\tan^{-1}\frac{\omega}{40}$$

Phase angle

**Table 7.8**

| $\omega$ | $\frac{1}{(j\omega)^2}$ | $+\tan^{-1}\frac{\omega}{2}$ | $-\tan^{-1}\frac{\omega}{10}$ | $-\tan^{-1}\frac{\omega}{40}$ | $\emptyset_R$ |
|---|---|---|---|---|---|
| 0.2 | $-180^o$ | $+5.7^o$ | $-1.14^o$ | $-0.28^o$ | $-175.72^o$ |
| 2 | $-180^o$ | $+45^o$ | $-11.3^o$ | $-2.86^o$ | $-149.16^o$ |
| 10 | $-180^o$ | $+78.6^o$ | $-45^o$ | $-14.03^o$ | $-160.43^o$ |
| 20 | $-180^o$ | $+84.28^o$ | $-63.43^o$ | $-26.56^o$ | $-185.71^o$ |
| 50 | $-180^o$ | $+87.7^o$ | $-78.6^o$ | $-51.3^o$ | $-222^o$ |
| 100 | $-180^o$ | $+88.85^o$ | $-84.28^o$ | $-68.19^o$ | $-243.54^o$ |
| ∞ | $-180^o$ | $+90^o$ | $-90^o$ | $-90^o$ | $-270^o$ |

**Step 5: Bode plot and Solution. (see Fig 7.21 on next page)**

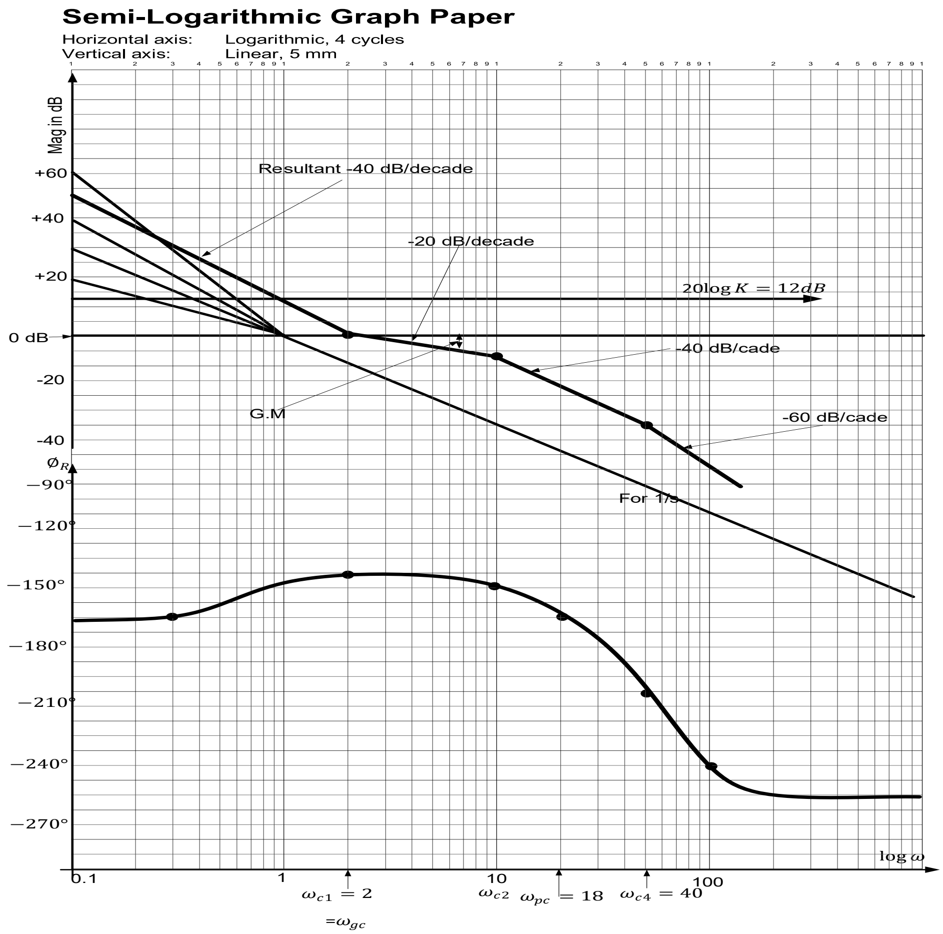



**Figure 7.21**

**Example 7.4**: For a certain feedback system $G(s)H(s) = \dfrac{3(s+1)(s+6)}{s^2(s^2+18s+400)}$

Sketch the Bode plot and comment on G.M. P.M and stability

Solution:

**Step 1**: arrange $G(s)H(s)$ in time constant form

$$G(s)\,H(s) = \frac{3(s+1)(6)(1+s/6)}{s^2(400)\left(1+\frac{18}{400}s+\frac{s^2}{400}\right)} = \frac{0.045(1+s)\left(1+\frac{s}{6}\right)}{s^2\left(1+0.045s+\frac{s^2}{400}\right)}$$

**Step 2**: Factors:

i. Constant $K = 0.045$

ii. $\frac{1}{s^2}$, 2 poles at the origin

iii. Simple zero $(1+s), T_1 = 1, \omega_{C1} = 1$

iv. Simple zero $\left(1+\frac{s}{6}\right), T_2 = \frac{1}{6}, \omega_{C2} = 6$

v. Quadratic pole, $\frac{1}{\left(1+0.045s+\frac{s^2}{400}\right)}, \omega_{C3} = \omega_n = 20$

Now compare $s^2 + 18s + 400$ with $s^2 + 2\xi\omega_n s + \omega_n^2$

$$\therefore \quad \omega_n^2 = 400, \quad \omega_n = 20, \quad \text{and } 2\xi\,\omega_n = 18, \quad \therefore \xi = 0.45$$

Its corner frequency is 20 while as $\xi = 0.45$, magnitude plot will exhibit $+2$ dB overshoot at $\omega_{C3} = 20$ (referring to the correction table given in discussion of quadratic pole).

**Step 3**

i. $K = 0.045$ $\therefore$ its contribution is $20\log K = 20\log 0.045 = -27$ dB

ii. $\frac{1}{s^2}$, 2 poles at the origin so magnitude pot is straight line of slope $-40$ dB/decade passing through intersection point of $\omega = 1$ and 0 dB. this is the starting slope of the magnitude plot.

iii. Shift intersection point of $\omega = 1$ and $0dB$ line on $20\log K$ line i.e $-27$ dB downward (as $K < 1$) and from that point draw parallel to $-40$ dB/decade line. This will represent addition of $\frac{K}{s^2}$. This will continue till first factor becomes dominant having least corner frequency i.e. $\omega_{C1} = 1$

iv. At $\omega_{C1} = 1$, simple zero occurs contributing $+20$ dB/decade individually hence resultant will have slope $-40 + 20 = -20$ dB/decade. Hence '1' onward slope of resultant will be $-20$ dB/decade contributing up to next corner frequency $\omega_{C2} = 6$.

v. At $\omega_{C2} = 6$, another simple zero occur contributing $+20$ dB/decade individually making the slope of the resultant will become $-20 + 20 = 0$ dB/decade from 6 onward i.e line parallel to x-axis till next corner frequency $\omega_{C3} = 20$

vi. At $\omega_{C3} = 20$, quadratic pole occurs contributing 40dB/decade individually hence the slope of resultant will become $0 - 40 = -40$ dB/decade from 20 onwards and will continue up to '∞' as there is no Other factor. But at $\omega_{C3} = 20$ it will show overshoot of $+2$ dB.

**Step 4**: Phase angle plot

$$G(j\omega)\,H(j\omega)=\frac{0.045(1+j\omega)\left(1+j\frac{\omega}{6}\right)}{(j\omega)^2\left(1+0.045j\omega+\frac{(j\omega)^2}{400}\right)}=\frac{0.045\;(1+j\omega)\left(1+j\frac{\omega}{6}\right)}{(j\omega)^2\left(1+0.045j\omega=\frac{\omega^2}{400}\right)}$$

$$\angle G(j\omega)\,H(j\omega)=\frac{\angle 0.045\;\;\angle 1+j\omega\angle 1+j\frac{\omega}{6}}{\angle(j\omega)^2\;\;\angle 1+0.045j\omega-\frac{\omega^2}{400}}$$

$$\angle 0.045+j0=0^o,\qquad \angle 1+j\omega=+\tan^{-1}\omega\,,\;\;\angle 1+j\frac{\omega}{6}=+\tan^{-1}\frac{\omega}{6}$$

$$\angle\frac{1}{(j\omega)^2}=-20\times 90^o=-180^o$$

As pole 2 at origin

$$\angle\frac{1}{1+0.045j\omega-\frac{\omega^2}{400}}=-\tan^{-1}\left\{\frac{0.045\omega}{1-\frac{\omega^2}{400}}\right\}$$

**Table 7.9**

| $\omega$ | $\frac{1}{(j\omega)^2}$ | $+\tan^{-1}\omega$ | $+\tan^{-1}\frac{\omega}{6}$ | $-\tan^{-1}\left\{\frac{0.045\omega}{1-\frac{\omega^2}{400}}\right\}$ | $\emptyset_R$ |
|---|---|---|---|---|---|
| 0.1 | $-180^o$ | $+5.7^o$ | $+0.95^o$ | $-0.25^o$ | $-173.6^o$ |
| 1 | $-180^o$ | $+45^o$ | $+9.46^o$ | $-2.58^o$ | $-128.1^o$ |
| 6 | $-180^o$ | $+80.53^o$ | $+45^o$ | $-16.52^o$ | $-70.9^o$ |
| 10 | $-180^o$ | $+84.28^o$ | $+59^o$ | $-30.96^o$ | $-67.6^o$ |
| 20 | $-180^o$ | $+87.13^o$ | $+73.3^o$ | $-90^o$ | $-109.57^o$ |
| 50 | $-180^o$ | $+88.85^o$ | $+83.15^o$ | $+23.19-180^o$ $=-156.8^o$ | $-164.8^o$ |
| 100 | $-180^o$ | $+89.42^o$ | $+86.56^o$ | $+10.61-180^o$ $=-169.38^o$ | $-173.4^o$ |
| $\infty$ | $-180^o$ | $+90^o$ | $+90^o$ | $-180^o$ | $-180^o$ |

As two zeros are always contributing more than a quadratic pole phase angle plot cannot cross $-180^o$ but at the end will run parallel to it.

**Step 5**: Sketch the Bode plot and obtain the solution

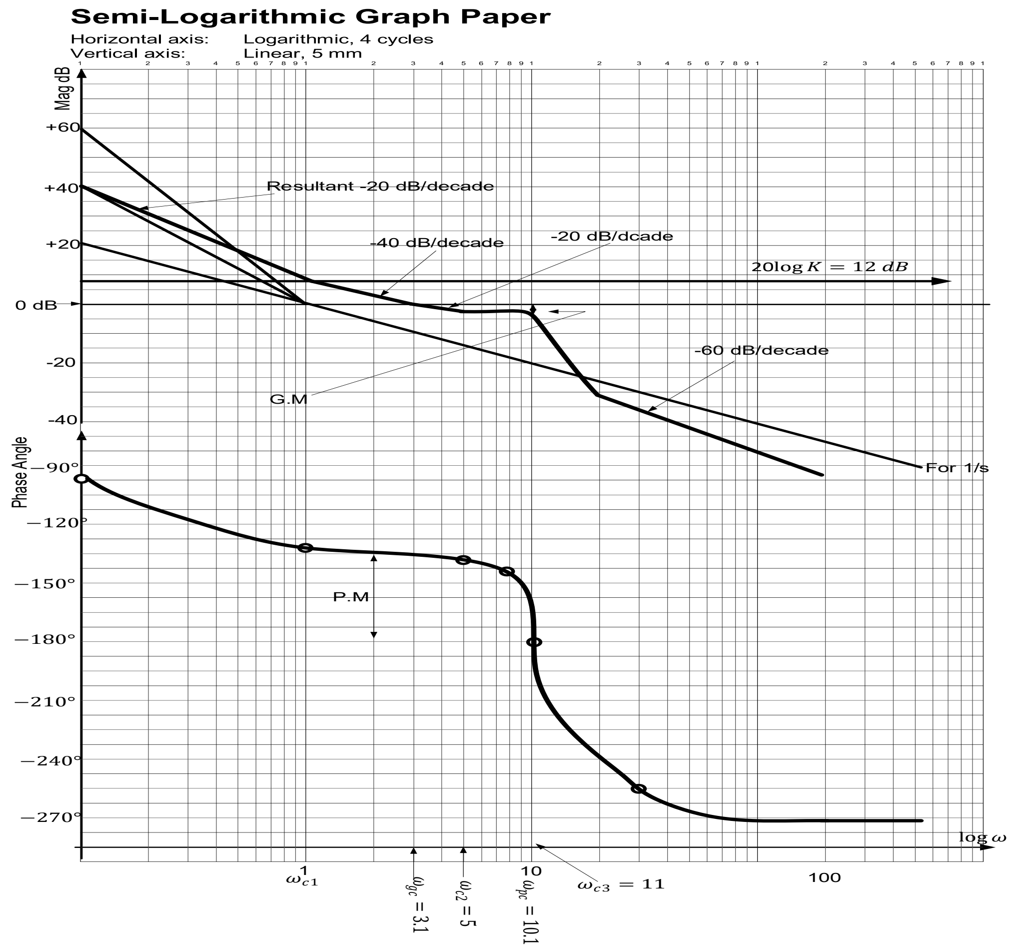



**Figure 7.22**

## 7.5 Further Examples

**1.** Draw a Bode plot for $H(s)$ equal to (a) $\frac{50}{(s+100)}$(b) $30\frac{(s+10)}{(s+100)}$ (show detailed derivations) (c) indicate the corner frequency(ies) in each of the above cases.

$(a)$

$$\frac{50}{(s+100)} = \frac{\frac{50}{100}}{1+\frac{j\omega}{100}}$$

$$20\log 50 - 20\log 100 - 20\log\sqrt{\omega^2+100^2} = 34 - 40 - 20\log\sqrt{1+\left(\frac{\omega}{100}\right)^2}$$

$K \approx -6\ dB$

For $\omega \ll 100 \Longrightarrow -6\ dB$;

For $\omega = 100 \Longrightarrow 34 - 40 - 3 \approx -9$;

For $\omega \gg 100 \Longrightarrow 34 - 40 - 20 dB/dec$

$\omega \ll 100 \Longrightarrow ang \approx 0 - 0 = 0$;

$\omega = 100 \Longrightarrow 0 - 45 = 45^o$; $\omega \gg 100 \Longrightarrow 0 - 90 = -90^o$;

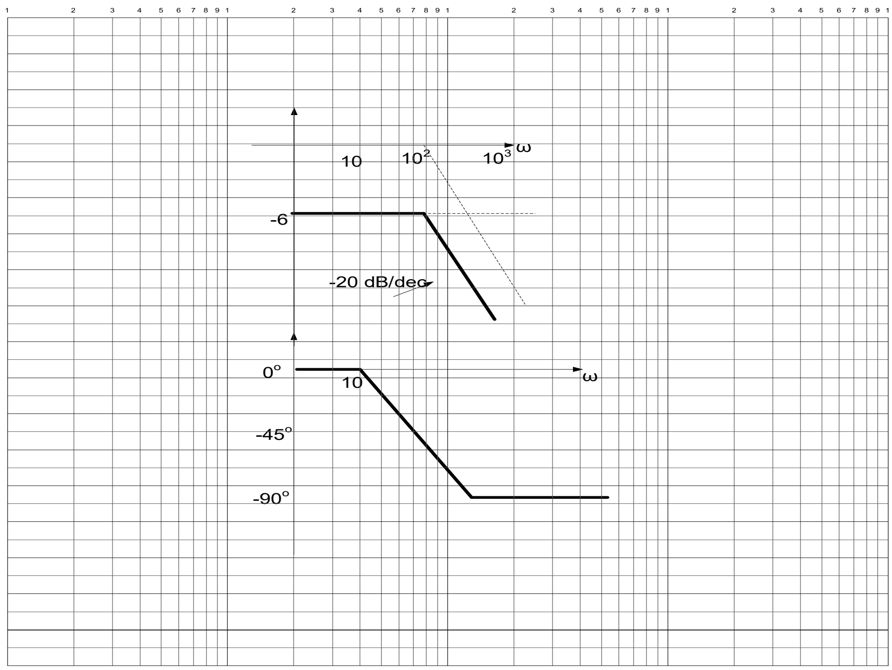



**Figure 7.23**

$$\text{(b)}\quad 30\frac{(s+10)}{(s+100)} \Longrightarrow 20\log\left|\frac{30\times(s+10)}{(s+100)}\right| = 20\log\frac{30\times\sqrt{\omega^2+100}}{\sqrt{\omega^2+10^4}}$$

$$= 20\log\frac{30\times 10\sqrt{1+\left(\frac{\omega}{10}\right)^2}}{100\sqrt{1+\left(\frac{\omega}{10^2}\right)^2}}$$

$$= 20\log 3 + 20\log\sqrt{1+\left(\frac{\omega}{10}\right)^2} - 20\log\sqrt{1+\left(\frac{\omega}{10^2}\right)^2}$$

$\omega \ll 10 \Longrightarrow K \approx 10\ dB$; $\omega = 10 \Longrightarrow 10 + 3 - 0 = 13\ dB$ $\omega = 100 \Longrightarrow 10 + 40 - 3 = 47\ dB$

$\omega \gg 10 \Longrightarrow +20\ dB/dec$

$\omega \gg 100 \Longrightarrow -20\ dB/dec$

$\omega \ll 100 \Longrightarrow ang \approx 0^o;$

$\omega = 10 \Longrightarrow ang = 45^o,\ \omega = 100 \Longrightarrow ang = -45^o$

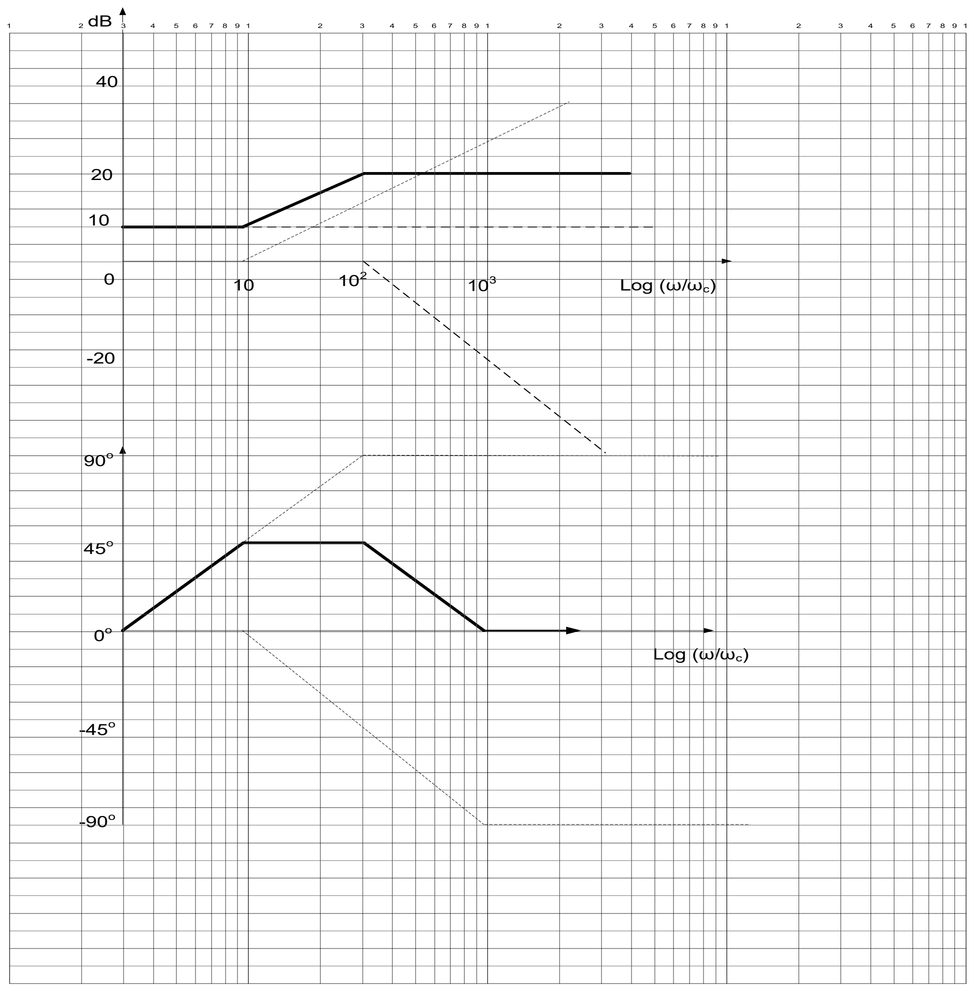



**Figure 7.24**

**2.** Draw the Bode plots [**Show detailed derivations;** indicate the **corner frequency(ies)** in each of the cases] for **H**(s) equal to: $100(1+\ s)/\ (10\ +\ s)(100+s)$

Solution: Given $\quad 100(1+\ s)/\ (10\ +\ s)(100+s)$

$$\Longrightarrow 20\log\left|\frac{100\times(s+1)}{(10\ +\ s)\times(s+100)}\right| = 20\log\frac{100\times\sqrt{\omega^2+1}}{\sqrt{\omega^2+10^2}\times\sqrt{\omega^2+10^4}}$$

$$= 20\log\frac{100\sqrt{1+\left(\frac{\omega}{1}\right)^2}}{1000\sqrt{1+\left(\frac{\omega}{10}\right)^2}\times\sqrt{1+\left(\frac{\omega}{10^2}\right)^2}}$$

$$= 20\log 10^{-1} + 20\log\sqrt{1+\left(\frac{\omega}{1}\right)^2} - 20\left[\sqrt{1+\left(\frac{\omega}{10}\right)^2} + \log\sqrt{1+\left(\frac{\omega}{10^2}\right)^2}\right]$$

$\omega \ll 1 \Longrightarrow K \approx -20\ dB;$

$\omega = 1 \Longrightarrow -20 + 3 - 0 = -17\ dB$

$\omega = 10 \Longrightarrow -20 + 40 - 3 = -17\ dB$

$\omega = 100 \Longrightarrow -20 + 80 - 40 - 3 = -17\ dB$

$\omega \gg 100 \Longrightarrow -20\ dB/dec$

$\omega \ll 100 \Longrightarrow ang \approx 0^o;$

$\omega = 1 \Longrightarrow ang = 45^o,\ \omega = 10 \Longrightarrow$

$ang = -45^o,\ \omega = 100 \Longrightarrow ang = -45^o$

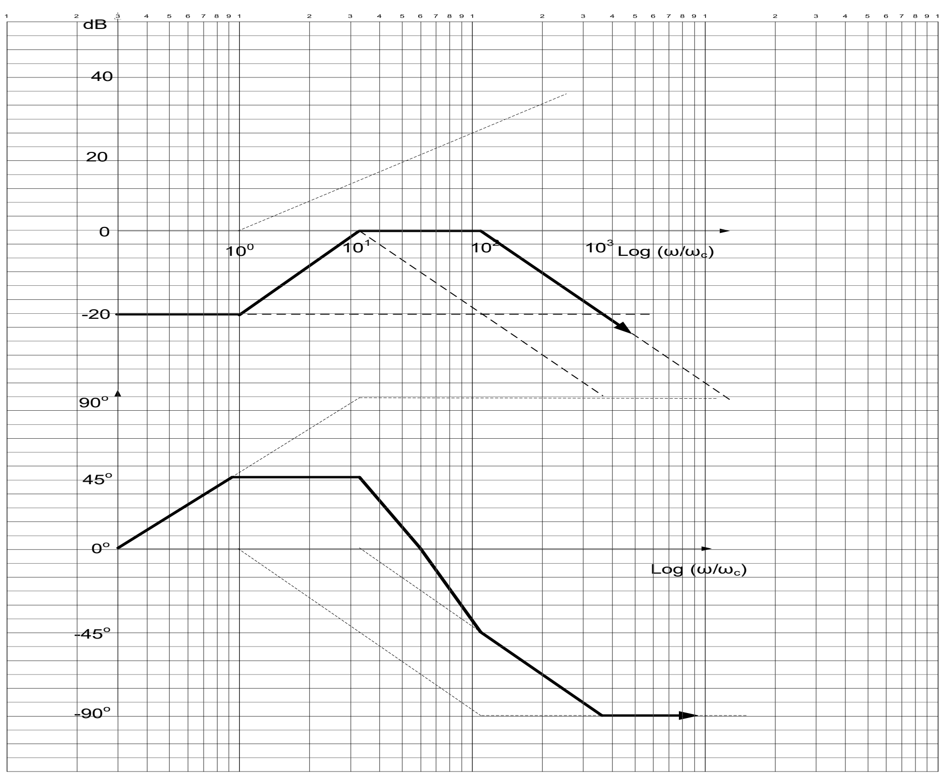



**Figure 7.25**

### 7.6 Advantages of Bode Plots

1. It shows both low and high frequency characteristics of the transfer function in a single diagram.
2. The plots can be easily constructed using some valid approximations.
3. Relative stability of the system can be studied by calculating GM. and P.M. from the Bode Plot.
4. The various other frequency domain specifications like cutoff frequency, bandwidth etc.
5. Data for constructing complicated polar and Nyquist plots can be easily obtained from Bode Plot.
6. Transfer function of the system can be obtained from Bode plot.
7. It indicates how the system should be compensated to get the desired response.
8. The value of system gain K can be designed for required specifications of GM. and P.M. from Bode Plot.
9. Without the knowledge of the transfer function the Bode Plot of a stable open loop system can be obtained experimentally.

### 7.7 Exercise

1. What are Bode Plots?
2. State the advantages of Bode Plots.
3. Explain the nature of Bode Plots for

a. Poles at origin

b. Simple pole

c. Simple zero

4. Explain the concept of gain margin and phase margin. Explain how these values help in studying relative stability.
5. Draw the Bode diagram for

$$G(s) = \frac{100(0.2s + 1)}{(s + 1)(0.1s + 1)(0.01s + 1)^2}$$

Mark the following on the Bode diagram, recording the numerical values.

1. gain crossover frequency
2. phase margin
3. phase crossover frequency

4. gain margin

a. Is the system stable?

6. Sketch the asymptotic Bode plot for the transfer function given below

$$G(s)H(s) = \frac{2(s+0.25)}{s^2(s+1)(s+0.5)}$$

From the bode plot determine

a. The phase crossover frequency

b. The gain crossover frequency

c. The gain margins

d. The phase margins

e. The system stable?

7. Determine the values of gain K for the open loop transfer function given below so that

a. The gain margin is $15dB$ and

b. Phase margin is $60^o$

$$G(j\omega)H(j\omega) = \frac{K}{j\omega\,(0.1\omega+1)(j\omega+1)}$$

8. Determine the value of K in the transfer function given below such that

a. The gain margin is 20 $dB$

b. The phase margin is $30^o$

$$G(j\omega)H(j\omega) = \frac{K}{j\omega(0.1\omega+1)(j0.05\omega+1)}$$

9. Draw a bode plot for the following and determine gain and phase crossover frequency. Also determine gain and phase margin. Using the Figure **Q**

a. $\frac{10}{s(0.1s+1)}$

b. $\frac{10}{s(0.1s+1)^2}$

# CHAPTER 8

# FILTERS

## 8.0 Passive Filters

A filter is a passive filter if it consists of only passive elements R, L, and C. It is said to be an active filter if it consists of active elements (such as transistors and op amps) in addition to passive elements R, L, and C. We consider passive filters only in the textbook. LC filters have been used in practical applications for more than eight decades. LC filter technology feeds related areas such as equalizers, impedance-matching networks, transformers, shaping networks, power dividers, attenuators, and directional couplers, and is continuously providing practicing engineers with opportunities to innovate and experiment. Besides the LC filters we study in this textbook, there are other kinds of filters—such as digital filters, electromechanical filters, and microwave filters—which are beyond the level of this textbook.

Filters separate different components which are mixed together, and in the case of an electrical filter, components of different frequencies are separated from one another.

As a frequency-selective device, a filter can be used to limit the frequency spectrum of a signal to some specified band of frequencies. Filters are the circuits used in radio and TV receivers to allow us to select one desired signal out of a multitude of broadcast signals in the environment.

To accomplish the above, inductors and capacitors are employed due to their different characteristics with regard to frequency. For instant, inductive reactance $(2\pi f L)$ increases with increasing frequency, while capacitance reactance $\left(\frac{1}{2\pi f c}\right)$ decreases with increasing frequency, and their filtering action depends on whether they are placed in series or in parallel with their load.

Before going into a detailed discussion of filters, these terms need to be understood:

1. Attenuation: a reduction in signal amplitude

2. High-pass filter: a filter that allows the higher frequency components of the applied voltage to develop appreciable output voltage while attenuating or altogether dominating the lower frequency components.

3. Low-pass filter: does the opposite of the above

4. Band-pass filter: passes only a specific band of frequencies from its input to its output

5. Band-stop filter: blocks or severely attenuates only a specific band of frequencies, while passing others of lower or higher frequencies.

6. Cut-off frequency is a frequency at which the attenuation of a filter reduces the output amplitude to 70.7% of its value in the passband. In other words, the frequency cut which the output voltage is reduced to 70.76% of its maximum value.

## 8.1 AC and DC Components

In dc insertion, a direct current voltage is coupled or put in series with alternating current voltage, to provide a pulsating (non-steady) direct current voltage. The effect is to provide an output that does not change in polarity, either positive or negative depending on the actual value of the d.c input. For a positive d.c component, the output fluctuates in amplitude but remains on the positive side. For a negative d.c component, the output fluctuates in amplitude but remains on the negative side. To filter out the a.c component while blocking the d.c a transformer with a separate secondary winding, or a capacitor, is used to block the (steady) d.c voltage. These procedures are known as transformer or capacitor coupling. In the latter, which is common in amplifier circuits, the coupling connects the output of one circuit to the input of the next, with the requirement to include all frequencies in the desired signal while rejecting undesired components. The result is that a specific d.c level is maintained for the operation of the amplifier.

## 8.2 Low-Pass Filters

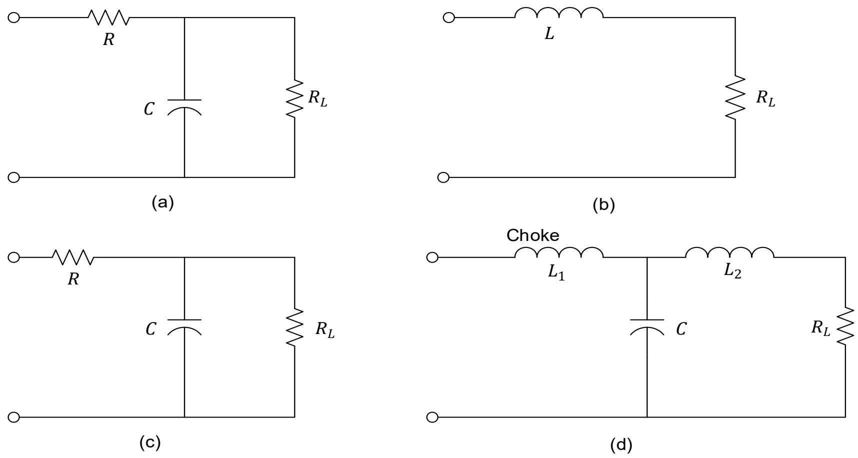



**Figure 8.1 Lowpass Filters (a) input = low and high frequencies, (b) input same as in (a), (c) Bypass capacitor parallel with the load resistor, with input same as in (a), (d) inductor in series with the load resistor, with input same as in (a).**

Capacitor depends on the internal resistance of the generator supplying input voltage to the filter. A low-resistance generator needs the T-filter so that the choke can provide high series impedance for the bypass capacitor, otherwise the latter must have extremely large value to short-circuit the low resistance generator at high frequencies.

When the input capacitors are effective as a bypass, the $\pi$ −types are more suitable with a high resistance generator. The L-type can also have the shunt by-pass either in the input for a high resistance generator or across the output for a low resistance generator.

For a balanced filter circuit, the series component can be connected on both sides of the live without having any effect on the filtering action. In all situations, however, the series choke can be connected either in the high side of the line, or in series in the opposite side of the line

## 8.3 High-Pass Filters

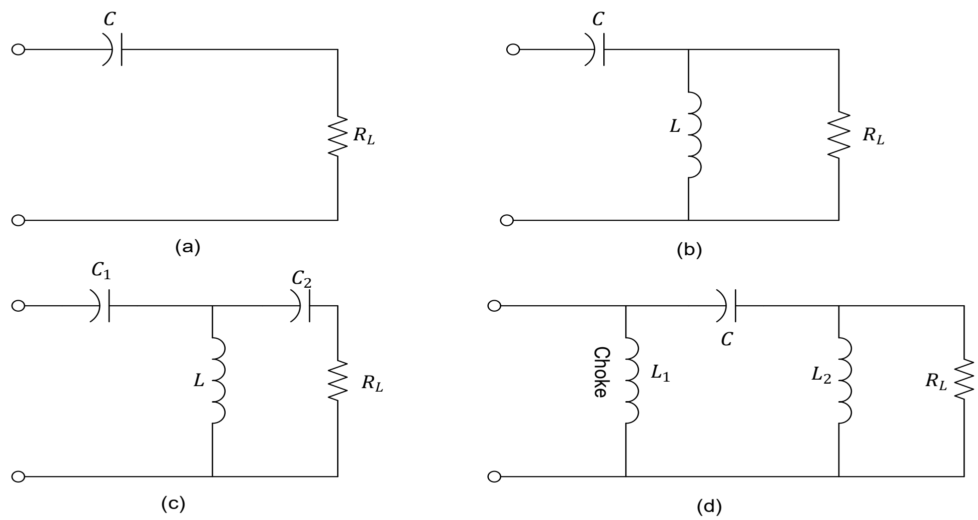



**Figure 8.2 High-pass Filters (a) input = high & low frequencies, (b) input same as in (a), the choke is parallel with the load resistor (c) Passband capacitor parallel with the load impedance, with input same as in (a), (d) inductor in parallel with the load resistor, with input same as in (a).**

High pass filter passes to the load all frequencies higher than the cut-off frequency, designated $f_c$, while voltages at lower frequencies are severely attenuated. Depicted above are the different configurations that high-pass filters can take. Notice that the roles of c and L are interchanged with respect to the low-pass filter considered earlier. This is because of the dissimilar behaviours of the inductor and the capacitor in the presence of an alternating current. Whereas capacitive reactance $\left(\frac{1}{2\pi f C}\right)$ decreases with increasing frequencies, inductive reactance $(2\pi f L)$ does just the opposite and these peculiar behaviours are employed in reproducing signals at certain frequencies while attenuating or completely rejecting them at other frequencies. Note also, that exchanging the placements of R and C, or R and L, would turn a filter from a low/high pass to high/low pass. The high-pass filters pass to the load all frequencies above the cut-off frequency, $f_c$ in the graph below, whereas at lower frequencies, below $f_c$ appreciable voltage cannot be developed across the load. Frequencies below the cutoff

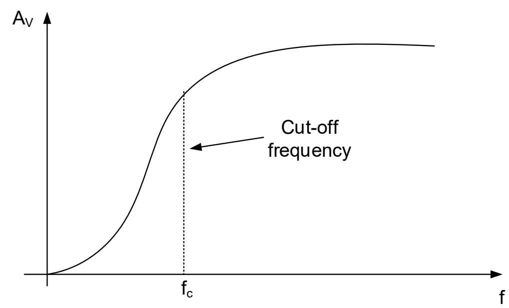



**Figure 8.3 Cut-off frequency**

Frequencies are referred to as stop-band. Note the different RC, RL, RLC arrangements in Fig. 8.2, and it's always desirable to arrange them to cause a sharper response curve which means narrower bandwidth. At the cut-off frequency, the amplitude is $70.7\% \left(\frac{1}{\sqrt{2}}\right)$ of its value (maximum) at resonance. Here, impedance is minimum because capacitive and inductive reactance cancel out each other, as learnt in an earlier circuit theory course.

## 8.4 Band-Pass Filter

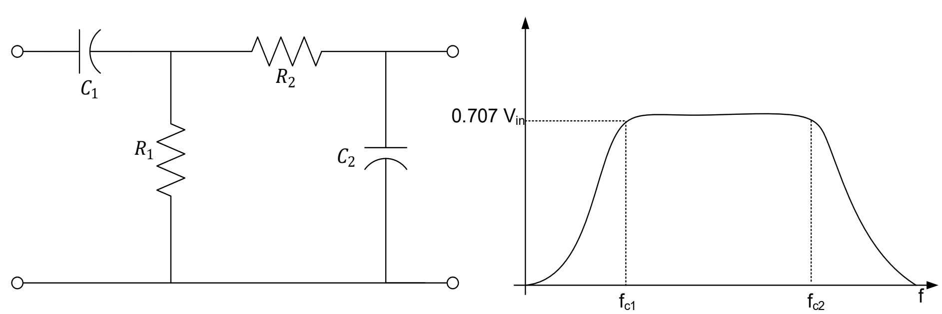



**Figure 8.4**

In its simplest form, a series placement of back-to-back C/R and R/C configuration gives rise to a band-pass filter, going by the two already previously discussed. The response curve is predictably as appearing in Fig. 8.4 with the result that the two different corner (cut-off) frequencies are affected. However, at these cut-off frequencies the response is 70.7% of the (maximum) input. $R_2$ is usually made much higher (at least 10 times) than the value of $R_1$ this to ensure that the low-pass section does not become a load to the high-pass section!

### 8.5 Band-Stop Filter

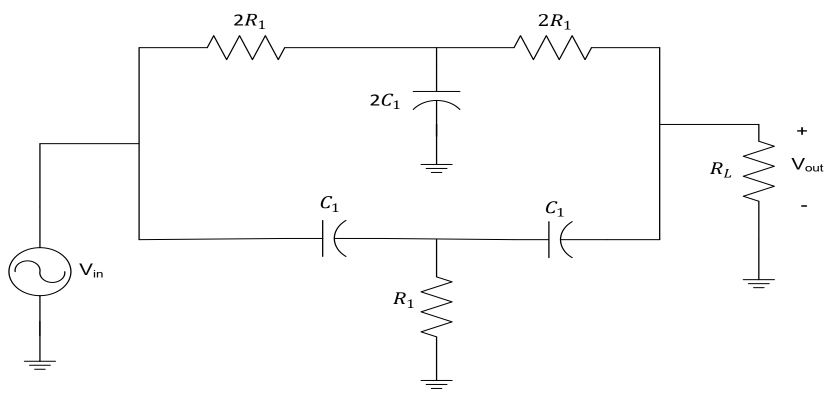



**Figure 8.5 Band-stop circuit**

Fig. 8.5 is a band-stop filter, affected by the parallel combination of a low-pass and high-pass filter. The $2R_1 - 2C_1$ components make up the low-pass and the $R_1 - C$, components make up the high-pass filters. The response curve is as shown in Fig. 8.6, with the low-pass section on is in Fig. 8.6(b), while

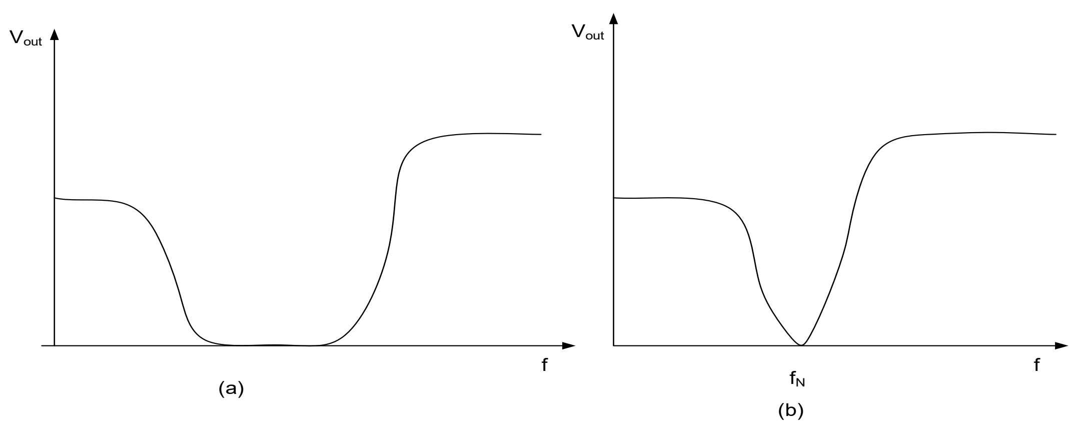



**Figure 8.6 Response curve of a Band-stop**

Fig 8.6(a) constitutes the high-pass section. The response curve of Fig 8.6 (a) is a special type of band-pass filter, called the notch filter where a single frequency, called the notch frequency $f_N$, constitutes the pass band. A greater circuit loss by the series resistances

$(2R_1)$ than the series capacitances $(C_1)$ is the reason for the greater maximum signal response in Fig. 8.6(a), than on Fig. 8.6(b).

There are other types of filters each one specialized for one purpose or another, namely: Resonant filter and L-type resonant filter.

## 8.6 Resonant Filters

Where a turned circuit is used in filtering band of radio frequencies with small values of L and C at resonance. Fig. 8.7 are two different configurations of resonant filters. Fig. 8.7(a) is a parallel resonant.

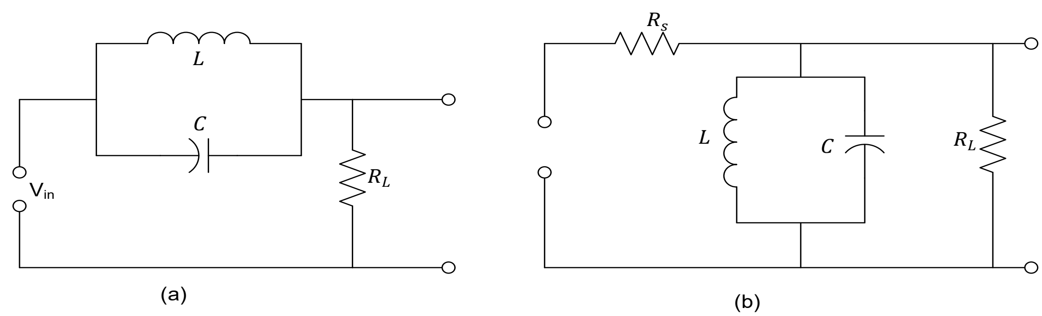



**Figure 8.7 Resonant filter circuits**

Filter in series with the load $R_L$ and shown in Fig 8.7(b)is a parallel resonant filter that is in shunt with the load. The series arrangement constitutes a band-stop filter, whereas the shunt configuration gives rise to a band-pass filter.

## 8.7 L-Type Resonant Filter

For L-type resonant filter a special inverted version can be employed to affect

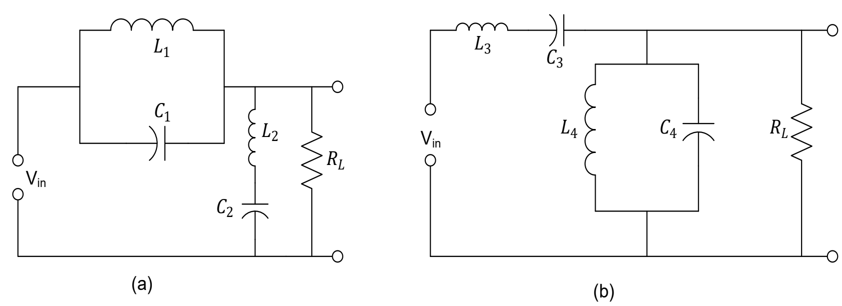



**Figure 8.8 L-type Resonant circuit**

Either a band-pass or band-stop filter. In the Fig. 8.8(a), the parallel resonant $L_1C_1$ circuit is in series with the $R_L$ load, whereas in the same setup, series resonant $L_2C_2$ circuit is in parallel with the same. This is an inverted L, band-stop filter.

Fig 8.9(b) is of an opposite (dual) configuration from the one on Fig. 8.8(b). This time the series $L_3C_3$ circuit is in series with the load, while the parallel resonant $L_4C_4$ circuit is in shunt with the same load, constituting a band-pass filter.

There are others such as crystal filters, where a thin slice of quartz brings about effects of resonance by mechanical vibration at a specified frequency; interference filters, used to rid power lines of interference; power-time filters; television antenna filters etc.

Analysis of filters means analysis of transfer functions, usually written as a function of complex frequencies, encountered earlier on, and in earlier courses. Just as time-dependent responses of a circuit can be determined, responses that are frequency-dependent can be analysed. In circuits where the response is not determined at the same pair of terminals as the input, an expression that relates two quantities is called a transfer function. So, transfer function relates two quantities taken at different pairs of terminals, e.g., ratios of voltages to currents, of voltage to voltage, or ratio of current to voltage, or current to current.

## 8.8 Application of Bode Plot to Filters

### 8.8.1 Frequency Response

Frequency response can be defined in 4 different ways:

1. $H_V(s) = \frac{V_L(s)}{V_s(s)}$ (Voltage amplitude-phase response) (voltage gain) 8.1a

   where $V_L(s), V_s(s)$ are the load voltage, source voltage, respectively.

2. $H_I(s) = \frac{I_L(s)}{I_s(s)}$ (Current amplitude-phase response) (current gain) 8.1b

   where $I_L(s), I_s(s)$are the load current source current, respectively.

3. $H_Z(s) = \frac{V_L(s)}{I_s(s)}$ (Impedance response). 8.1c

4. $H_Y(s) = \frac{I_L(s)}{V_s(s)}$ (Admittance response). 8.1d

Note that in each of these cases, the expression relates the signal at the load to that at the source. Analysis of the working of filters, therefore, means analysis of transfer functions. For instances for a low-pass filter earlier described the transfer function is (as shown in Eq 8.2)

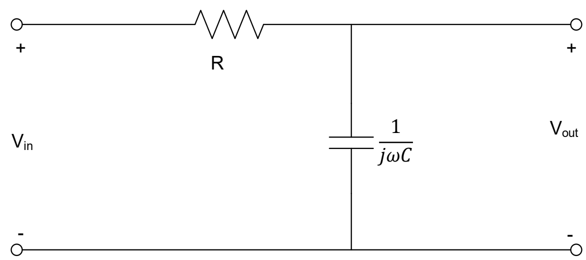


**Figure 8.9 RC low-pass filter**

$$\frac{V_0}{V_{in}} = \frac{\frac{1}{sC}}{R + \frac{1}{Sc}} \qquad 8.2$$

by voltage divider rule, where the complex frequency is purely imaginary as the neper frequency in this case is zero, meaning undamped sinusoid. The above expression can be rationalized by multiplying throughout by

$$sC \Longrightarrow \frac{V_0}{V_{in}} = \frac{1}{(sCR + 1)} = \frac{1}{(1 + j\omega CR)} \qquad 8.3$$

The zero of the Eq 8.3 transfer function, $s = \infty$, is the value of s that "vanishes" the function, and the process of same, is (are) that (those) value(s) that makes(make) it increases without bounds (infinite). That would be when the denominator is equal to zero, or $sCR + 1 = 0 \Longrightarrow -\frac{1}{RC}$, a single pole, i.e, the single factor of the denominator. To investigate how both the amplitude and phase response to changes in frequency, a technique known as Bode diagram or Asymptotic plot, has been developed, and named after its investor, Henry Bode. Before going further to analyse the above low-pass filter (output taken across the capacitor), let's analyse the simplest transfer function:

$$H(s) = 1 + \frac{s}{a} \qquad 8.4$$

$|H(j\omega)|$ in decibels (Db) is defined as:

$$H_{\mathrm{dB}} = 20\log|H(j\omega)|$$

Note that this is strictly a definition therefore, no proof is required.
For the reverse, if $H_{\mathrm{dB}}$ is known then:

$$|H(j\omega)| = 10^{\left(\frac{H_{dB}}{20}\right)} \qquad 8.5$$

For example, $|H(j\omega)| = 1 \Longrightarrow H_{dB} = 20\log 1 = 0$

$$|H(j\omega)| = 10 \Longrightarrow H_{dB} = 20\log 10 = 20$$

In general, $|H(j\omega)| = 10^n \Longrightarrow H_{dB} = 20\log 10^n = n \times 20 = 20n$

$\Longrightarrow$From Eq 8.5 therefore,

$$|H(s)| = |H(j\omega)| = 20\log\left|\left(1+\frac{j\omega}{a}\right)\right| = 20\log\sqrt{(1+\frac{\omega^2}{a^2}}$$

With $\omega \ll a$ [no (omega) for less than a, i.e $\omega \leq 0.1a$],

$$H_{dB} \approx 20\log 1 = 0$$

$$\omega \gg a\ (\omega \geq 10a) \Longrightarrow H_{dB} \approx 20\log\left(\frac{\omega}{a}\right),$$

Since '1' can be ignored by comparison. This represents increase of 20dB per decade (ten times the previous value) since, for instance, for an increase in no form a to 10a, $H_{dB}$ increases from $20\log 1 = 0$ to $20\log 10 = 20$ with a normalized to 1 or gradually,

$$20\log 10a - 20\log a$$

$$= (20\log 10 + 20\log a) - 20\log a = 20\log 10 = 20\times 1 = 20$$

For $10a$ to $100a$, $H_{dB}$ increases by

$$(20\log 100 + 20\log a) - (20\log 10 + 20\log a) = 40 - 20 = 20, etc$$

The asymptotic plot is as shown in Fig. 8.10:

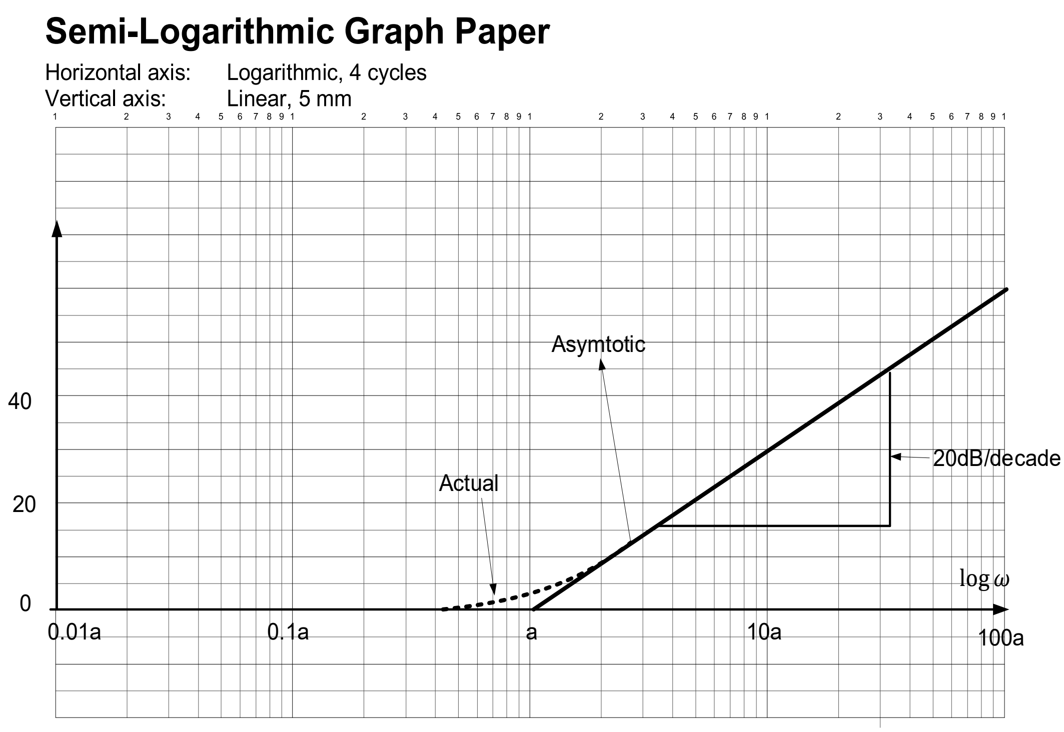



**Figure 8.10**

### 8.8.2 Comments on the Bode diagram (Asymptotic plot)

The scales of both axes (abscissa and ordinate) are logarithmically ruled, rather than linear, as this presents better information on the amplitude response vis-a-vis variations in the frequency. The abscissa is this normalized ("linearized") by taking the log of the quotient $\omega/a$ (omega over a) as the scale rather than merely $\omega$. $a$ is known as the corner frequency, also called the cut-off frequency, break frequency, half power frequency [because at $\omega_c(a)$ the power delivered to the load is half of what it was (maximum) at resonance $(\omega_0)$] or 3dB frequency, so called because 3dB (Actual $20\log\sqrt{1+1} = 20\log 2^{1/2}$) is what has been approximated out by the asymptotic plot, that is the difference between the actual value (3dB) and the asymptotic value (0). This is also true for any amplitude response for asymptotic plots with multiple breakpoints.

The entire plot can be smoothened by determining the actual value of the response around the break point. For instance, at $\omega = 0.5a$,

$$H_{dB} = 20\log(1+0.5^2)\frac{1}{2} = \frac{1}{2}\times 20\log(1+0.25) = 10\log 1.25 \approx 1dB$$

And at $1.5a$,

$$H_{dB} = 20\log\sqrt{1+1.5^2} = 10\log 3.25 \approx 5,\ etc$$

At a decade lower or higher than the corner frequency (0.1a and 10a respectively), the actual and asymptotic corresponds.

### 8.8.3 Phase Response

The phase angle of the function $H(j\omega) = 1 + \frac{j\omega}{a}$ is $\angle H(j\omega) = \tan^{-1}\frac{\omega}{a}$ 8.6

For $\omega \ll a$,

$$\angle H(j\omega) = \phi \approx \tan^{-1} 0 = 0$$
$$\omega = a \Longrightarrow \phi = \tan^{-1} 1 = 45^o$$
$$\omega \gg a \Longrightarrow \phi = \tan^{-1}\infty = 90^o$$

So, between about $\omega = 0.1a$ and $\omega = 10a$ ($a$ rise of 2 decades from $0.1a$ to $10a\left[= \log\left(\frac{10a}{0.1a}\right)\right] = \log 100 = 2$ , $\phi$ goes from zero to $90^o$, a rise of $45^o$ per decade.

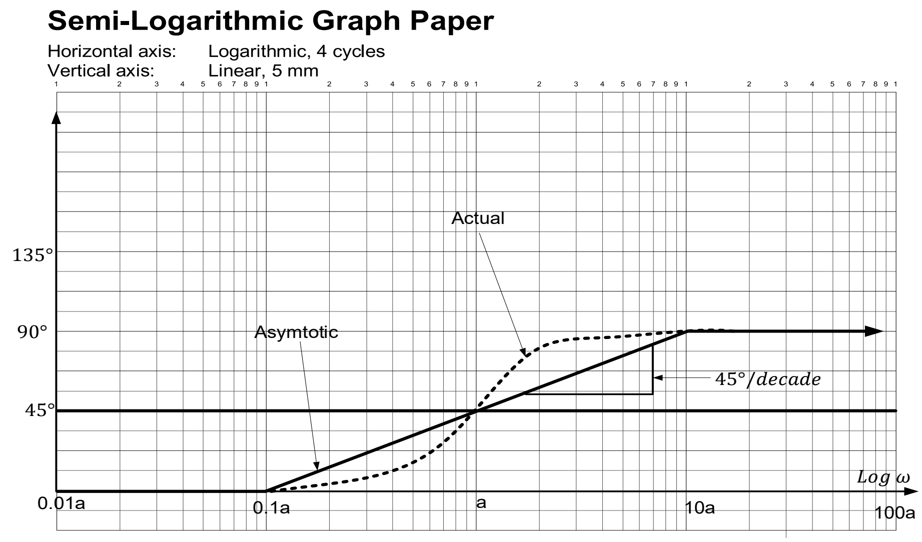



**Figure 8.11**

It's possible to plot both the amplitude and phase responses, on the same graph. For the transfer function $H(s) = \frac{1}{\left(1+\frac{s}{a}\right)}$, a reciprocal of the one we just dealt with in Eq 8.4, both the amplitude and phase responses are just a reflection of that for $H(s) = 1 + \frac{s}{a}$. This is because, for instance, if

$20 \log x = a$, then $20 \log\left(\frac{1}{x}\right) = -a$ 8.7

A detailed analysis as the one close previously would show this to be so:

$$H_{dB} = 20 \log\left|1/\left(1+\frac{j\omega}{a}\right)\right| = 20\log 1 - 20 \log\left|1+\frac{j\omega}{a}\right| = 0 - 20\log\left|1+\frac{j\omega}{a}\right| \quad 8.8$$

So, every value is the negative of the one obtained previously.

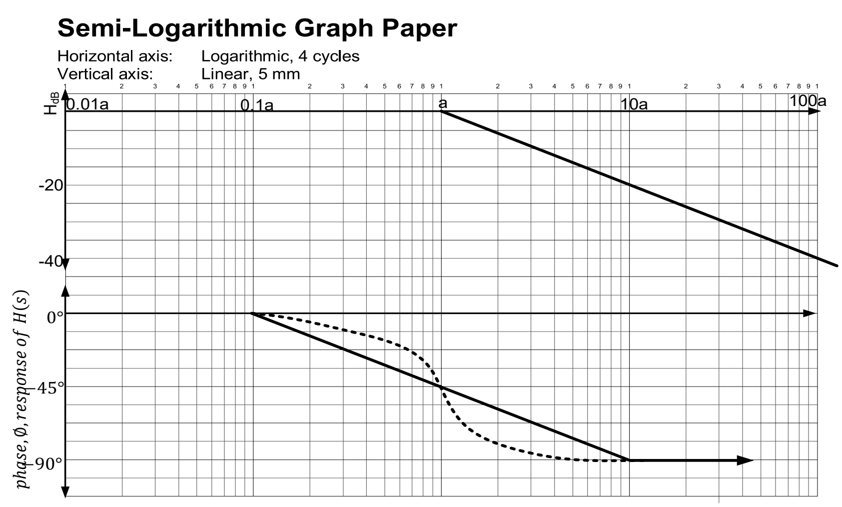



**Figure 8.12**

## 8.9 Analysis of Filters-Transfer Functions with Bode Plot

### 8.9.1 Low-Pass Filter

In the filter of Fig. 8.9, the output is taken across the capacitor.
By voltage-divider rule,

$$\frac{V_o}{V_{in}} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = H_v \angle\phi = \frac{1}{j\omega CR + 1} \qquad 8.9$$

Where voltage gain $H_V = \left|\frac{V_o}{V_{in}}\right|$

$$H_V = \left|\frac{1}{j\omega RC + 1}\right| \qquad 8.10.1$$

$$= \frac{1}{\sqrt{(\omega CR)^2 + 1}}$$

$$= \frac{1}{\sqrt{\left(\frac{\omega}{\omega_c}\right)^2 + 1}} \qquad 8.10.2$$

Where $\omega_c = \frac{1}{RC}$ is called the corner (or cut off) frequency
By definition
$H_{dB} = 20\log H_V$, where log is understood to be $log_{10}$
So, for the low-pass filter under consideration,

$$H_{dB} = 20\log\frac{1}{\sqrt{\left(\frac{\omega}{\omega_0}\right)^2 + 1}} \qquad 8.11$$

These cases are to be considered with respect to the gain in amplitude (amplitude response): $\omega \ll \omega_c$ (frequencies far less than the corner frequency):

$$H_{dB} \approx 20\log\frac{1}{1} = 20\log 1 = 0$$

$$\omega = \omega_c:$$

$$H_{dB} = 20\log 1/\sqrt{2} = 20\log 2^{-\frac{1}{2}} = -\frac{1}{2}\times 20\log 2 = -3dB$$

And this represents the largest error (at the corner frequency) in the straight-line approximation (see Fig.8.13) which assumes zero response up to $\omega \gg \omega_c$.
$\omega \gg \omega_c$ (Far greater):

$$H_{dB} \approx 20\log\frac{1}{\sqrt{\left(\frac{\omega}{\omega_c}\right)^2}} = 20\log\frac{\omega_c}{\omega} = 20\log\omega_c - 20\log\omega \qquad 8.12$$

The first term is a constant that depends on the value of the corner frequency, while the second term, $-\,20\log\omega$, in Eq 8.12, is the rate of fall of 20Db (decibel) per frequency decade (i.e., per energy factor of 10)

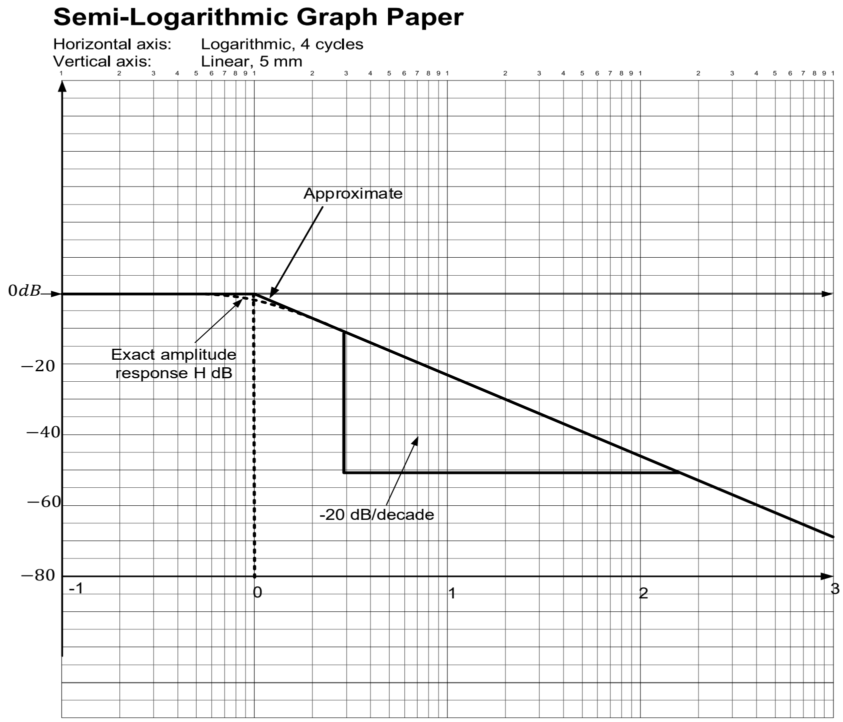



**Figure 8.13**

### 8.9.1.1 Phase Response (phase shift)

Recall from Eq 8.10.1, $H_V \angle\emptyset = \frac{V_o}{V_{in}} = \frac{1}{j\omega CR+1}$

$$\Longrightarrow \quad \emptyset = 0 - \tan^{-1}\omega RC = 0 - \tan^{-1}\left(\frac{\omega}{\omega_c}\right)$$

For $\omega \ll \omega_c$,

$$\emptyset \approx -\tan^{-1}0 = 0\,;$$

$$\omega = \omega_c$$

$$\emptyset = 0 - \tan^{-1}1 = -45^o$$

So that the phase shift is midway between the maximum and minimum values at the corner frequency $\omega \gg \omega_c$,

$$\emptyset \approx 0 - \tan^{-1}\infty = -90^o$$

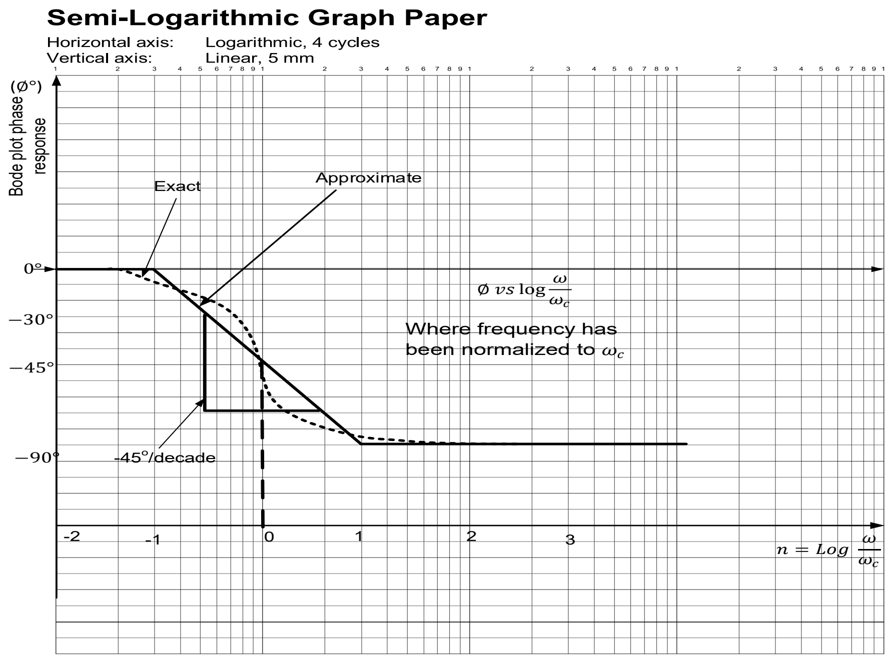



**Figure 8.14**

## 8.9.2 High-Pass Filter

Here the resistor and capacitor are interchanged, as should be expected intuitively:

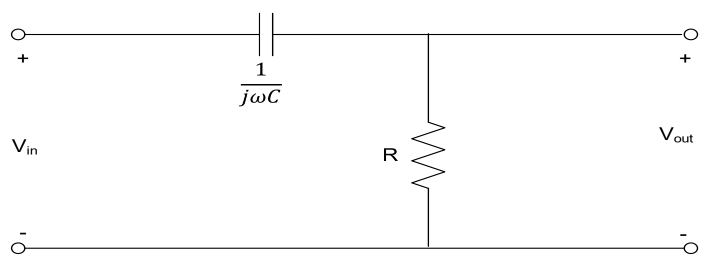



**Figure 8.15**

The output is now taken across the resistor:

$$\frac{V_0}{V_{in}} = \frac{R}{\frac{1}{j\omega C} + R} = \frac{1}{\frac{1}{j\omega CR} + 1} = \frac{1}{1 - \frac{j\omega C}{\omega}} \qquad 8.13$$

$$H_V\ \angle\emptyset = \frac{V_0}{V_{in}} \Longrightarrow H_V = \left|\frac{V_0}{V_{in}}\right| = \left|\frac{1}{1 - \frac{j\omega_c}{\omega}}\right|$$

Whereas before, $\omega_c = \frac{1}{RC}$

$$\Longrightarrow \quad H_{dB} = 20\log H_V = 20\log\left|\frac{1}{1-\frac{j\omega_c}{\omega}}\right|$$

$$= 20\log\frac{1}{\sqrt{1+\left(\frac{\omega_c}{\omega}\right)^2}}$$

For $\omega \ll \omega_c$,

$$H_{dB} \approx 20\log\frac{\omega}{\omega_c} = 20\log\omega - 2 = \log\omega_c$$

Where $-20\log\omega_c$ is as in the case of low-pass filter a constant while $20\log\omega$ now represents a rise of 20 decibels per decade.

$\omega = \omega_c$,

$$H_{dB} = 20\log\frac{1}{\sqrt{2}} = 20\log 2^{-\frac{1}{2}} = -10\log 2 = -3dB$$

And represents the largest error between the approximate and exact graphs

$\omega \gg \omega_c$, see Fig. 8.16

$$H_{dB} \approx 20\log 1 = 0$$

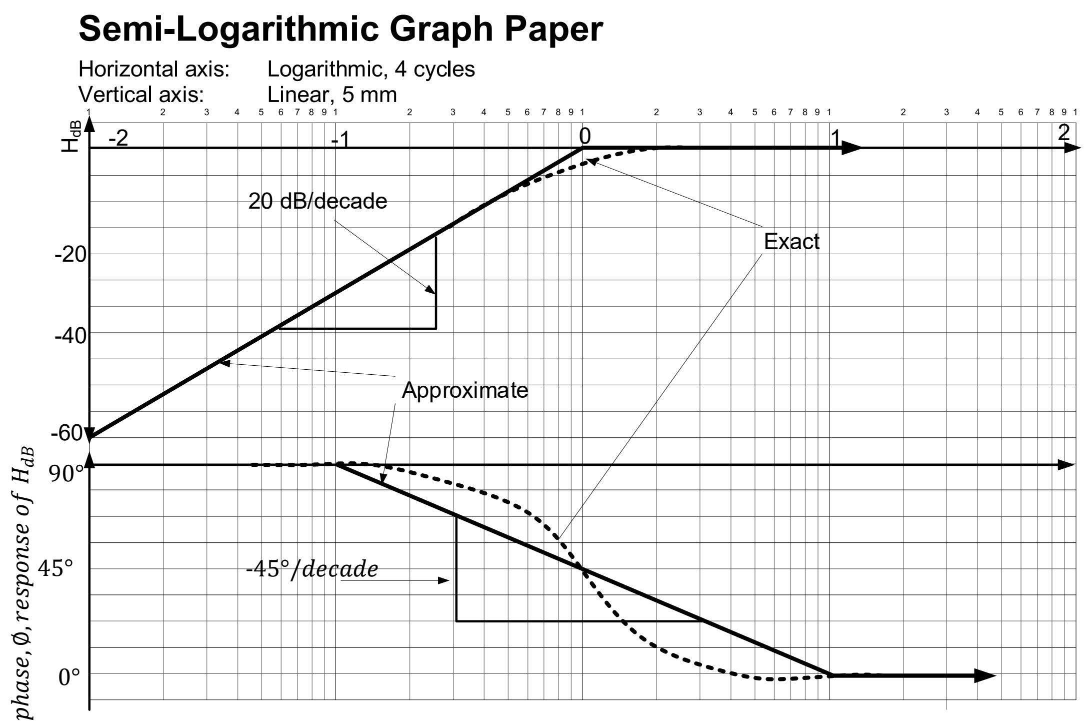



**Figure 8.16**

$$\emptyset = 0 - \left[-\tan^{-1}\left(\frac{\omega_c}{\omega}\right)\right] = \tan^{-1}\left(\frac{\omega_c}{\omega}\right)$$

For $\omega \ll \omega_c$,

$$\emptyset \approx \tan^{-1}\infty = 90^o$$

$\omega = \omega_c$,

$$\emptyset = \tan^{-1}1 = 45^o$$

$\omega \gg \omega_c$,

$$\emptyset \approx \tan^{-1}0 = 0$$

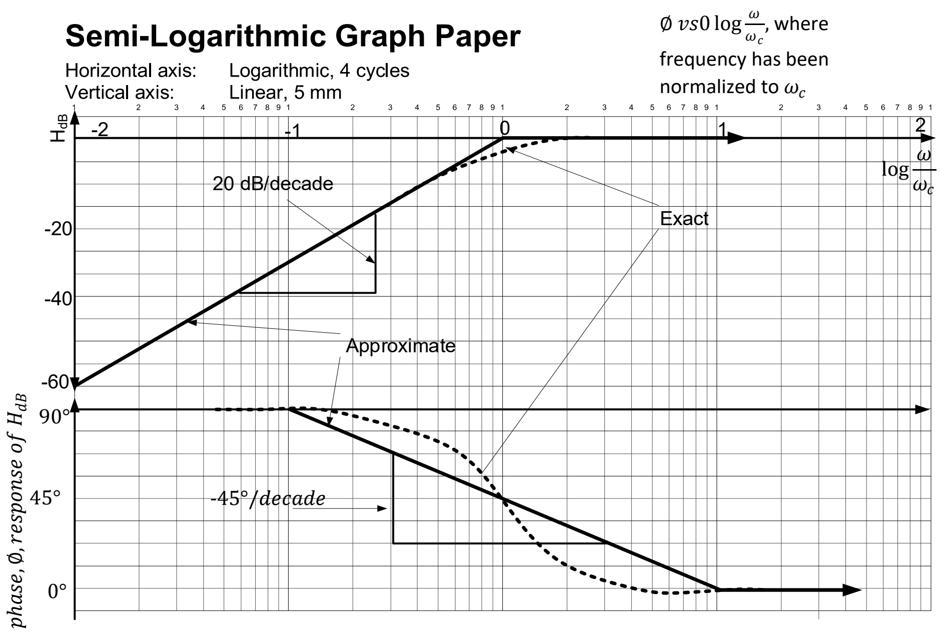



**Figure 8.17**

### 8.9.3 Circuit with Two Corner Frequencies

It's possible to obtain more than one breakpoint with R-L or R-C circuits.

**Example 8.1**: Consider the circuit of Fig. 8.18, with the resistor in series with an inductor at the output.

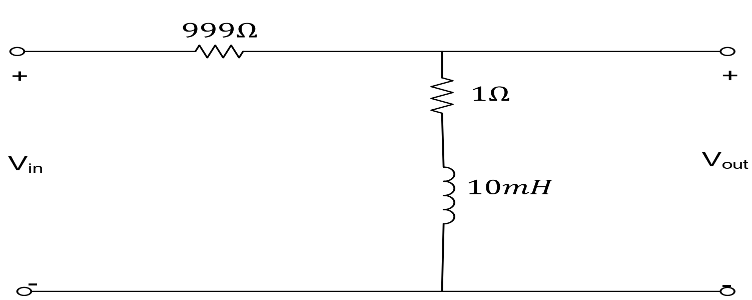


**Figure 8.18**

The 1 Ω represent the internal resistance of the inductor coil.
Voltage divider rule gives:

$$\frac{V_o}{V_{in}} = \frac{1 + j10 \times 10^{-3}\omega}{999 + 1 + j10 \times 10^{-3}\omega}$$

$$= \frac{1 + j0.01\omega}{1000 + j0.01\omega} = \frac{100 + j\omega}{100000 + j\omega}$$

$$= \frac{100\left[1 + j\left(\frac{\omega}{10^2}\right)\right]}{100000\left[1 + j\left(\frac{\omega}{10^5}\right)\right]}$$

$$H_V = \left|\frac{10^2\left[1 + j\left(\frac{\omega}{10^2}\right)\right]}{10^5\left[1 + j\left(\frac{\omega}{10^5}\right)\right]}\right|$$

Two corner frequencies occur at $\omega = 10^2$ rad/s and at $\omega = 10^5$ rad/s

$$H_{dB} = 20\log H_V = 20\log\frac{10^2\sqrt{\left[1 + \left(\frac{\omega}{10^2}\right)^2\right]}}{10^5\sqrt{\left[1 + (\frac{\omega}{10^5})^{\,2}\right]}}$$

$$= 20\log\left\{10^{-3}\sqrt{\frac{1 + \left(\frac{\omega}{10^2}\right)^2}{1 + \left(\frac{\omega}{10^5}\right)^2}}\right\}$$

$$= 20\log 10^{-3} + 20\log\left[1 + \left(\frac{\omega}{10^2}\right)^2\right]^{\frac{1}{2}} - 20\log[1 + (\omega + 10^5)^2]$$

$$= -3 \times 20\log 10 + \frac{1}{2} \times\ 20\log\left[1 + \left(\frac{\omega}{10^2}\right)^2\right] - 20\log\left[1 + \left(\frac{\omega}{10^5}\right)^2\right]$$

$$= -60 + 10\log\left[1 + \left(\frac{\omega}{10^2}\right)^2\right] - 10\log\left[1 + \left(\frac{\omega}{10^5}\right)^2\right] \qquad \text{(i)}$$

When $\omega \ll 10^2$

$$H_{dB} \approx -60 + 0 - 0 = -60$$

$\omega = 10^2$; (ii)

$$H_{dB} \approx -60 + 10\log 2 = -60 + 3 = -57dB;$$

$10^2 < \omega < 10^5$;

From Eq (ii), $\omega = 10^2$ rad/s, the second term of Eq (i) will starts to increase at 20dB/decade Eq (ii) $10\log\left[1 + \left(\frac{\omega}{10^2}\right)^2\right] \approx 10\log\left(\frac{\omega}{10^2}\right)^2 = 20\log\left(\frac{\omega}{10^2}\right)$

$\omega > 10^5$, the last term of Eq (i) starts to decrease at $-20\,dB$/decade [from $-10\log\left(\frac{\omega}{10^5}\right)^2 = -20\log\left(\frac{\omega}{10^5}\right)$.

These decreases would now cancel out the increase of the second term, leaving the overall response of:

$-60 + 60$ (increase of 20dB /decade for 3 decades between $10^2$ and $10^5$) $+ 0 = 0dB$

$\emptyset = \emptyset_N - \emptyset_D$, where $\emptyset_N, \emptyset_D$ denote the phase angles of the numerator, denominator, respectively

$$\emptyset = \tan^{-1}\left(\frac{\omega}{100}\right) - \tan^{-1}\left(\frac{\omega}{10^5}\right)$$

For $\omega \ll 100$

$$\emptyset \approx 0 - 0 = 0$$

$$\omega = \frac{100}{10} = 10\text{rad/s}$$

$\emptyset \approx \emptyset_N$ starts to increase at $45^o$/decade while $\emptyset_D$ remains approximately zero.

At the first corner frequency ($\omega = 100$ rad/s),

$$\emptyset = \tan^{-1}\left(\frac{100}{100}\right) - 0 = \tan^{-1} 1 = 45^o\,;$$

At $\omega = 1\ k\text{rad/s}$ (10×first corner frequency),

$$\emptyset \approx \tan^{-1}\infty - 0 = 90^o$$

$\emptyset$ remains $90^o$ up to $10k$ rad/s (second corner frequency $\div 10$) and $\emptyset_D$ starts to increase at $45^o$/decade, this causing $\emptyset = \emptyset_N - \emptyset_D$ to fall at $45^o$/decade.

At $\omega = 10^6(10 \times$ second comer frequency),

$$\emptyset_D = 90^o \Longrightarrow \emptyset \approx 90^o - 90^o = 0$$

Note: the unit of the abscissa is $\log\omega$, unlike in the previous cases, where it was normalized to the one comer frequency.

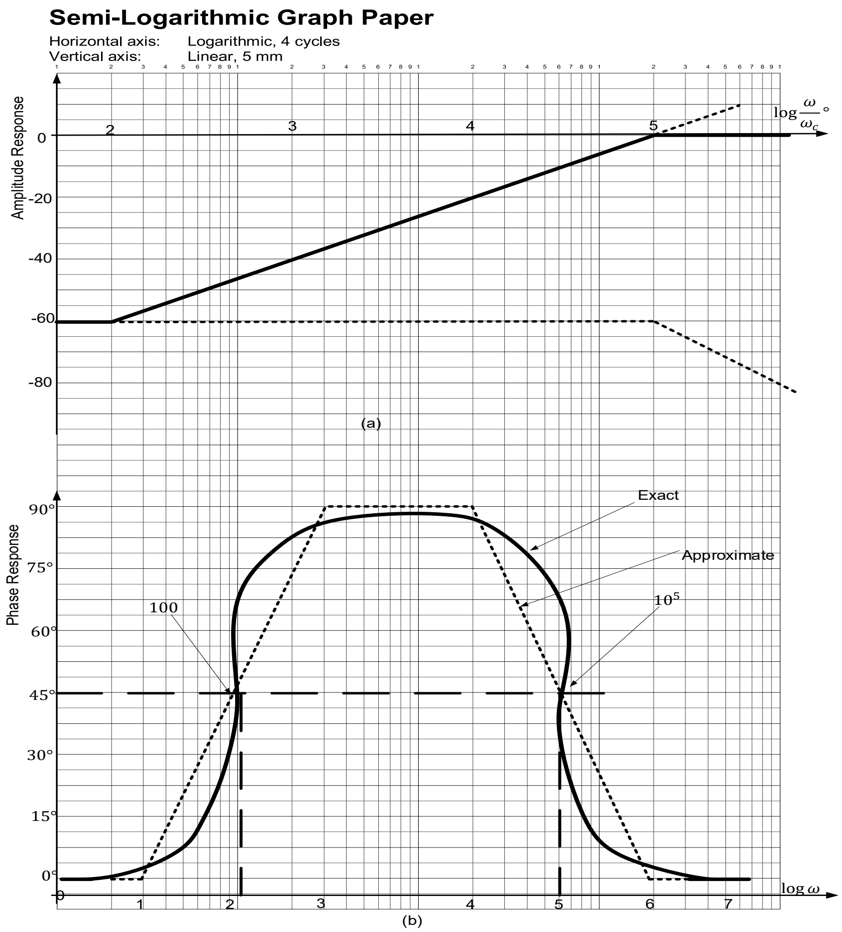



**Figure 8.19**

**Example 8.2**: Obtain the transfer function of the Operational amplifiers (OP Amps) design for a lowpass filter.

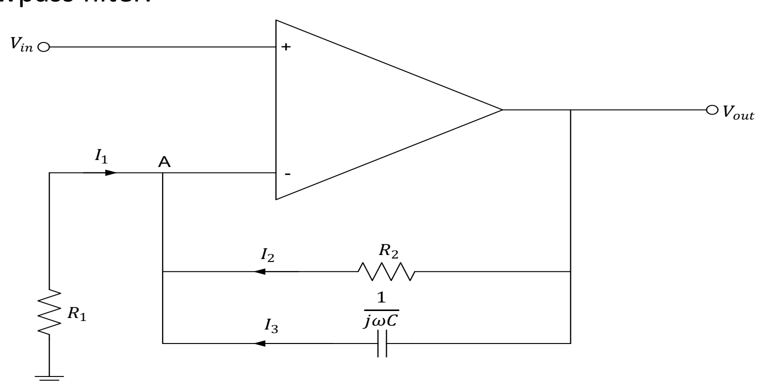



**Figure 8.20 Low pass filter with Op Amp**

KCL at A: $I_1 + I_2 + I_3 = 0$ 8.14a

$$\Rightarrow \; -\frac{V_{in}}{R_1} + \frac{(V_{out} - V_{in})}{R_2} + (V_{out} - V_{in})j\omega C = 0 \qquad 8.14b$$

$$V_{out}\left(\frac{1}{R_2} + j\omega C\right) = V_{in}\left(\frac{1}{R_1} + \frac{1}{R_2} + j\omega C\right)$$

$$\frac{V_{out}(1 + j\omega C R_2)}{R_2} = \frac{V_{in}(R_2 + R_1 + j\omega C R_1 R_2)}{R_1 R_2}$$

$$\frac{V_{out}}{V_{in}} = \frac{R_2 + R_1 + j\omega C R_1 R_2}{(1 + j\omega C\, R_2)R_1}$$

$$= \frac{\frac{1}{R_1} + \frac{1}{R_2} + j\omega C}{\frac{1}{R_2} + j\omega C} = \frac{\frac{1}{R_1 C} + \frac{1}{R_2 C} + j\omega}{\frac{1}{R_2 C} + j\omega} \qquad 8.15$$

$$= \frac{\omega_1 + j\omega}{\omega_2 + j\omega} \qquad 8.16$$

$$\text{Where } \omega_1 \equiv \frac{1}{R_1 C} + \frac{1}{R_2 C} \; and \; \omega_2 \equiv 1/R_2 C \qquad 8.17$$

For $R_1 = 1.01\ k\Omega,\ R_2 = 100\ k\Omega, C = 1\ \mu\text{F}$

$$\omega_1 = \frac{1}{(1010 \times 10^{-6})} + \frac{1}{(10^5 \times 10^{-6})} = 990 + 10 = 1000,$$

$$\omega_2 = \frac{1}{(10^5 \times 10^{-6})} = 10\ \text{rad/s}$$

$$\frac{V_{out}}{V_{in}} = \frac{1000 + j\omega}{10 + j\omega} = \frac{1000\left[1 + j\left(\frac{\omega}{1000}\right)\right]}{10\ \left[1 + j\left(\frac{\omega}{10}\right)\right]}$$

$$= \frac{100\left[1 + j\left(\frac{\omega}{1000}\right)\right]}{1 + j\left(\frac{\omega}{10}\right)}$$

$$H_{dB} = 20\log\left|\frac{V_{out}}{V_{in}}\right| = 20\log 100 + 20\log\left[1 + \left(\frac{\omega}{1000}\right)^2\right]^{\frac{1}{2}} - 20\log\left[1 + \left(\frac{\omega}{10}\right)^2\right]^{\frac{1}{2}}$$

$$= 40 + 10\log\left[1 + \left(\frac{\omega}{1000}\right)^2\right] - 10\log\left[1 + \left(\frac{\omega}{10}\right)^2\right]$$

### 8.9.4 Bandpass Filter

The RLC series resonant circuit provides a bandpass filter when the output is taken off the resistor as shown in Fig. 8.21. The transfer function is

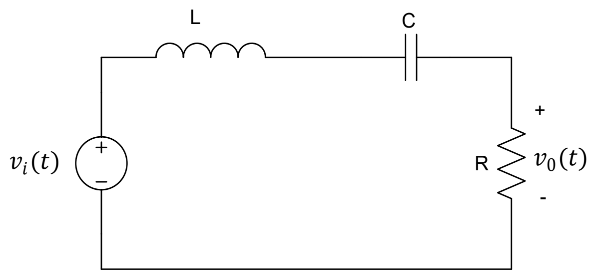



**Figure 8.21**

$$H(\omega) = \frac{V_o}{V_i} = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)} \qquad 8.18$$

We observe that $H(0) = 0, H(\infty) = 0$. Fig. 8.22 shows the plot of $|H(\omega)|$The bandpass filter passes a band of frequencies $(\omega_1 < \omega < \omega_2)$ centered on $\omega_0$, the center frequency, which is given by

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad 8.19$$

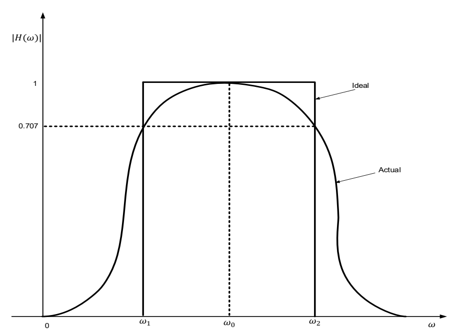



**Figure 8.22**

*A bandpass filter is designed to pass all frequencies within a band of frequencies,* $\omega_1 < \omega < \omega_2$

Since the bandpass filter in Fig. 8.21 is a series resonant circuit, the half-power frequencies, the bandwidth, and the quality factor are determined. A bandpass filter can also be formed by cascading the lowpass filter (where $\omega_2 = \omega_c$) in Fig. 8.15 with the highpass filter (where $\omega_1 = \omega_c$.) in Fig. 8.15. However, the result would not be the same as just adding the output of the lowpass filter to the input of the highpass filter, because one circuit loads the other and alters the desired transfer function.

### 8.9.5 Band-Stop Filter

A filter that prevents a band of frequencies between two designated values ($\omega_1$ and $\omega_2$) from passing is variably known as a bandstop, bandreject, or notch filter. A bandstop filter is formed when the output RLC series resonant circuit is taken off the LC series combination as shown in Fig. 8.23. The transfer function is

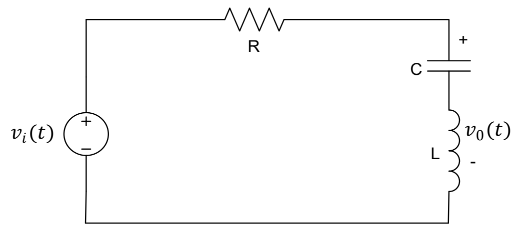



**Figure 8.23**

$$H(\omega) = \frac{V_0}{V_i} = \frac{j\left(\omega L - \frac{1}{\omega C}\right)}{R + j\left(\omega L - \frac{1}{\omega C}\right)} \qquad 8.20$$

Notice that $H(0) = 1, H(\infty) = 1$. Fig. 8.24 shows the plot of Again, the center frequency is given by.

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad 8.21$$

while the half-power frequencies, the bandwidth, and the quality factor are calculated for a series resonant circuit. Here, $\omega_0$ is called the frequency of rejection, while the corresponding bandwidth ($B = \omega_2 - \omega_1$) is known as the bandwidth of rejection. Thus,

A bandstop filter is designed to stop or eliminate all frequencies within a band of frequencies, $\omega_1 < \omega < \omega_2$

Notice that adding the transfer functions of the bandpass and the bandstop gives unity at any frequency for the same values of R, L, and C. Of course, this is not true in general but true for the circuits treated here. This is due to the fact that the characteristic of one is the inverse of the other.

In concluding this section, we should note that:

1. From Eqs. (8.9), (8.13), (8.18), and (8.20), the maximum gain of a passive filter is unity. To generate a gain greater than unity, one should use an active filter as the next section shows.
2. There are other ways to get the types of filters treated in this section.

3. The filters treated here are the simple types. Many other filters have sharper and complex frequency responses.

Table 8.1

Summary of the characteristic of ideal filters:

| Type of filter | $H(0)$ | $H(\infty)$ | $H(\omega_c)$ or $H(\omega_0)$ |
|---|---|---|---|
| Lowpass | 1 | 0 | $\frac{1}{\sqrt{2}}$ |
| High pass | 0 | 1 | $\frac{1}{\sqrt{2}}$ |
| Bandpass | 0 | 0 | 1 |
| Bandstop | 1 | 1 | 0 |

$\omega_c$ is the cutoff frequency for lowpass and highpass filters: $\omega_0$ is the center frequency for bandpass and bandstop filters.

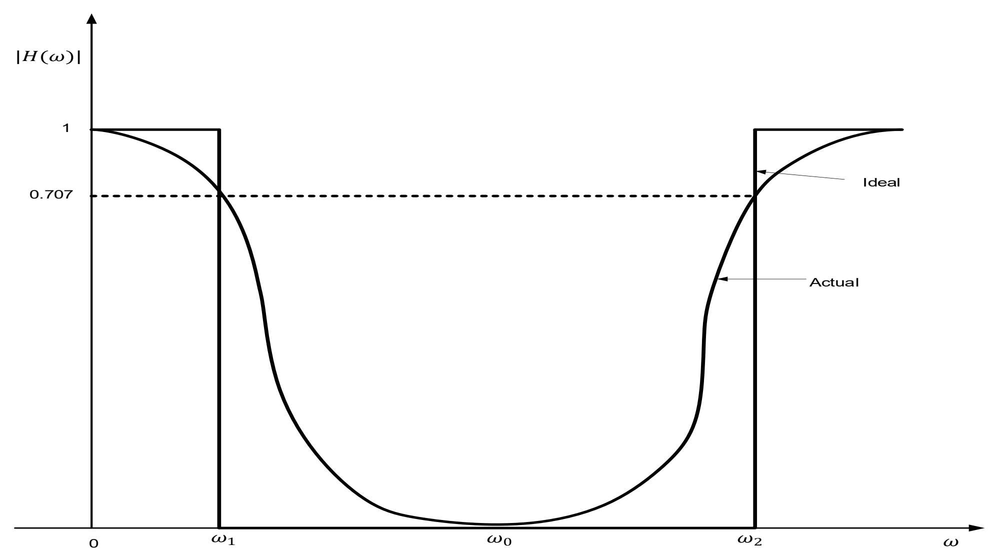



**Figure 8.24**

**Example 8.3**: Determine what type of filter is shown in Fig. 8.25. Calculate the corner or cutoff frequency. Take $R = 2\ k\Omega, L = 2$ H and $C = 2\ \mu$F.

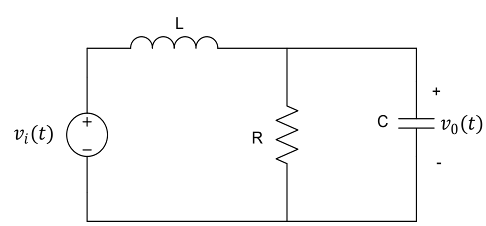



**Figure 8.25**

Solution:

The transfer function is

$$H(s) = \frac{V_o}{V_i} = \frac{R \parallel \frac{1}{sC}}{sL + R \parallel \frac{1}{sC}} \quad s = j\omega \qquad 8.22.1$$

$$R \left\| \frac{1}{sC} = \frac{\frac{R}{sC}}{R + 1/sC} = \frac{R}{1 + sC} \right.$$

Substituting this into Eq 8.22.1 gives

$$H(s) = \frac{\frac{R}{(1+sRC)}}{sL + \frac{R}{(1+sRC)}} = \frac{R}{s^2RLC + sL + R} \quad s = j\omega$$

Or

$$H(\omega) = \frac{R}{-\omega^2 RLC + j\omega L + R} \qquad 8.22.2$$

Since $H(0) = 1$ and $H(\infty) = 0$, we conclude from Table 8.1 that the circuit in Fig. 8.25 is a second-order lowpass filter. The magnitude of $H$ is

$$H = \frac{R}{\sqrt{(R - \omega^2 RLC)^2 + \omega^2 L^2}} \qquad 8.22.3$$

The corner frequency is the same as the half-power frequency, i.e., where H is reduced by a factor of $\frac{1}{\sqrt{2}}$. Since the dc value of $H(\omega)$ is l, at the corner frequency, Eq. 8.22.3 becomes after squaring

$$H^2 = \frac{1}{2} = \frac{R^2}{\sqrt{(R - \omega_c^2 RLC)^2 + \omega_c^2 L^2}}$$

Or

$$2 = (1 - \omega_c^2 LC)^2 + \left(\frac{\omega_c L}{R}\right)^2$$

Substituting the values of $R, L$, and $C$ we obtain

$$2 = (1 - \omega_c^2 4 \times 10^{-6})^2 + (\omega_c 10^{-3})^2$$

Assuming that $\omega_c$ is in krad/s,

$$2 = (1 - 4\omega_c^2)^2 + \omega_c^2 \quad or \quad 16\omega_c^4 - 7\omega_c^2 - 1 = 0$$

Solving the quadratic equations in $\omega_c^2$ we get $\omega_c^2 = 0.5509$ and $-0.1134$. since $\omega_c$ is real

$$\omega_c = 0.742 \text{ krad/s} = 742 \text{ rad/s}$$

**Example 8.4:** If the bandstop filter in Fig.8.23 is to reject a 200 Hz sinusoid while passing other frequencies, calculate the values of L and C. Take $R = 150\ \Omega$ and the bandwidth as 100 Hz.

Solution:

We use the formulas for a series resonant circuit which states that:

$$B = 2\pi(100) = 200\pi \text{ rad/s}$$

But

$$B = \frac{R}{L} \implies L = \frac{R}{B} = \frac{150}{200\pi} = 0.2387 \text{ H}$$

Rejection of the $200$ Hz sinusoid means that $f_0$ is $200$ Hz so that $\omega_0$ in Fig. 8.24 is

$$\omega_0 = 2\pi f_0 = 2\pi(200) = 400\pi$$

Since $\omega_0 = 1/\sqrt{LC}$

$$C = \frac{1}{\omega_0^2 L} = \frac{1}{(400\pi)^2(0.2387)} = 2.653\ \mu\text{F}$$

## 8.10 Limitations of Passive Filter

There are three major limitations to the passive filters considered in the previous section. First, they cannot generate gain greater than l; passive elements cannot add energy to the network. Second, they may require bulky and expensive inductors. Third, they perform poorly at frequencies below the audio frequency range $(300\text{ Hz} < f < 3{,}000\text{ Hz})$. Nevertheless, passive filters are useful at high frequencies.

Active filters consist of combinations of resistors, capacitors, and op amps. They offer some advantages over passive RLC filters. First, they are often smaller and less expensive, because they do not require inductors. This makes feasible the integrated circuit realizations of filters. Second, they can provide amplifier gain in addition to providing the same frequency response as RLC filters. Third, active filters can be combined with buffer amplifiers (voltage followers) to isolate each stage of the filter from source and load impedance effects. This isolation allows designing the stages independently and then cascading them to realize the desired transfer function. (Bode plots, being logarithmic, may be added when transfer functions are cascaded.) However, active filters are less reliable and less stable. The practical limit of most active filters is about 100 kHz—most active filters operate well below that frequency.

Filters are often classified according to their order (or number of poles) or their specific design type.

**Example 8.5**: Design a bandpass filter in the using an Operational amplifier to pass frequencies between 250 Hz and 3,000 Hz and with K = 10. Select $R = 20\ k\Omega$.

Solution

1. Define. The problem is clearly stated and the circuit to be used in the design is specified.

2. Present. We are asked to use the op amp circuit specified to design a bandpass filter. We are given the value of R to use $(20\ k\Omega)$. In addition, the frequency ranges of the signals to be passed is 250 Hz to 3 kHz.

3. Alternative. We will use the equations developed so far in this section to obtain a solution. We will then use the resulting transfer function to validate the answer.

4. Attempt. Since $\omega_1 = \frac{1}{RC_2}$, we obtain

$$C_2 = \frac{1}{R\omega_1} = \frac{1}{2\pi f_1 R} = \frac{1}{2\pi \times 250 \times 20 \times 10^3} = 31.83\text{ nF}$$

Similarly, since $\omega_2 = \frac{1}{RC_1}$

$$C_1 = \frac{1}{R\omega_2} = \frac{1}{2\pi f_2 R} = \frac{1}{2\pi \times 3{,}000 \times 20 \times 10^3} = 2.65\text{ nF}$$

From Equations in Example 8.3,

$$\frac{R_f}{R_i} = K\ \frac{\omega_1 + \omega_2}{\omega_2} = K\frac{f_1 + f_2}{f_2} = \frac{10(3{,}250)}{3{,}000} = 10.83$$

If we select $R_i = 10\ k\Omega$, then $R_f = 10.83\ R_i \simeq 108.3\ k\Omega$

5. Evaluate. The output of the first op amp is given by

$$\frac{V_i - 0}{20} + \frac{V_1 - 0}{20} + \frac{s2.65 \times 10^{-9}(V_1 - 0)}{1}$$

$$= 0 \rightarrow\ V_1 = -\frac{V_i}{1 + 5.3 \times 10^{-5}s}$$

The output of the second op amp is given by

$$\frac{V_1 - 0}{20 + \frac{1}{31.83s}} + \frac{V_2 - 0}{20} = 0 \rightarrow$$

$$V_2 = -\frac{6.366 \times 10^{-4}}{1 + 6.366 \times 10^{-4}}$$

$$= \frac{6.366 \times 10^{-4}sV_i}{(1 + 6.366 \times 10^{-4}s)(1 + 5.3 \times 10^{-5}s)}$$

The output of the third op amp is given by

$$\frac{V_2 - 0}{10} + \frac{V_o - 0}{108.3} = 0 \rightarrow\ V_o = 10.83V_2 \rightarrow j2\pi\ \times 25^o$$

$$V_0 = -\frac{6.894 \times 10^{-3}\, sV_i}{(1 + 6.366 \times 10^{-4}s)(1 + 5.3 \times 10^{-5}s)}$$

Let $j2\pi \times 25^o$ and solve for the magnitude of $V_o/V_i$

$$\frac{V_o}{V_i} = \frac{-j10.829}{(1 + j1)(1)}$$

$\left|\frac{V_o}{V_i}\right| = (0.7071)10.829$, which is the lower corner frequency point.

Let $s = j2\pi \times 3000 = j18.849\ k\Omega$. we then get

$$\frac{V_o}{V_i} = \frac{-j129.94}{(1 + j12)(1 + j1)}$$

$$= \frac{129.94\ \angle 90^o}{(12.042\angle 85.24^o)(1.4142\angle 45^o)} = (0.7071)10.791\angle - 18.61^o$$

Clearly this is the upper corner frequency and the answer checks.

6. Satisfactory? We have satisfactorily designed the circuit and can present the results as a solution to the problem.

### 8.11 Exercise

**1.** (a) Explain the basic function of a high-pass filter, with the appropriate graphical sketch and relevant circuit diagram. (b) for a low-pass filter, what are meant by the terms (i) passband (ii) stopband

**2.** In Fig. 1 is a circuit diagram for a non-inverting operational amplifier (op amp). Determine the output voltage $V_{out}$ in terms of the input voltage $V_{in}$, i.e a gain which is a quotient of the two voltages. (bi) Shunt $R_f$ and let $R_1$ increase without bound. Analyse and determine the type of an amplifier that results thereby, by again finding $V_{out}$ in terms of $V_{in}$. (ii) By what other name is this type of amplifier known?

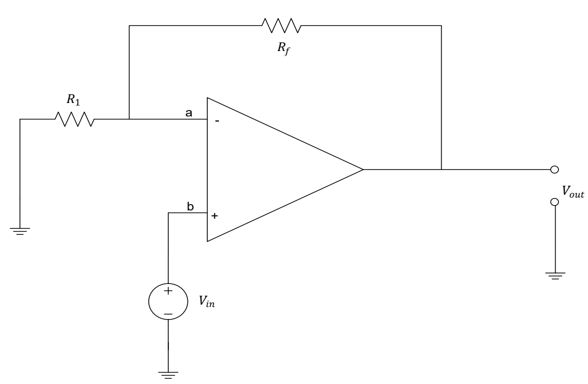



**Fig. 1**

# CHAPTER 9

# TRANSMISSION LINES

## 9.0 Introduction

There are means of relaying signals (also power) from one point to another, usually a pair of electrical conductors, with coaxial cables and twisted pair cable being some of the examples. Having said this, I must point out that the lines are not merely “wire” or cables in their simplest form, but rather are intricate cascades of electrical circuits! Bearing in mind costs, convenience and ease of calculations that involve the properties of the transmission line, they are then arranged in definite geometric patterns.

The goal of the transmission is to transport a typical signal with minimal loss. Loss there must be when we’re dealing with physical realities, but the idea behind any design is to minimize such.

Up to this point in your circuit theory series, we’ve dealt with the more familiar low-frequency circuit where the wires that connect devices are justifiably assumed to have zero resistance, and phase delays are absent across wires. Furthermore, short circuited lines always yield zero resistance. Not so in high frequency transmission lines where the above does not obtain and we have to expect the unexpected! For example, short circuits can actually possess infinite impedance, and open circuits (the idealized model of an infinite impedance) can actually behave like short circuited wires!

For low frequency signals and d.c signals, transportation normally involves very low losses, but high frequency ones in the range of radio waves, losses are quite pronounced and the objective of the design engineer is to eliminate or minimize such. So, here, attention is focused on high frequency applications whereby the length of the line is of at least the same order of magnitude as the least the same order of magnitude as the wavelength of the signal under consideration. This is strictly with regard to systems of conductors having a forward and return path.

Areas of application include communication engineering where study is made to determine the most efficient use of power and equipment available to transfer for example, as much power as possible from the feeder line into the antenna. To avoid power wastage, a receiving antenna must be correctly matched to the line that connects it to the receiver.

To eliminate losses, we resort to "matching" the line to the load, by making the factor known as the characteristic impedance of the line, designated $Z_0$, equal or very close to the load impedance $(Z_L)$. In d.c and low frequency a.c circuits earlier referred to, the characteristic impedance of parallel wires is usually insignificant and can therefore be ignored in analyzing circuit behavior. Here the phase difference between the sending and the receiving and is negligible, the period of propagation is very small compared to the period of the waveform under consideration. It can be practically assumed that the voltage along all the respective points (of a low frequency, two conductor line) are equal and in-phase with each other at any given point in time.

An idealized transmission line has an "infinite" length, this way all the energy is absorbed and more is reflected back to the source, because the characteristic (natural) impedance of the line is now matched to the frictions load impedance $(Z_L)$

To investigate low voltage or current changes along transmission lines, the following assumptions are made and the following parameters must be borne in mind, so that circuit analysis can be employed.

The line is made up of continuous conductors with constant cross-sectional configuration, and therefore indicating even distribution of the parameters, the problem is tracked by considering a very short length of the line that would imply a very discreet distribution of the parameters. The problem is tackled by considering a very short length of the line that would imply discrete distribution of the **parameters** which are:

1. Resistance (R): The resistance of the conductors to the flow of current.
2. Inductance (L): Associated with the time varying signal, and depends on the geometry of the cross-section of the conductors.
3. Conductance (G): Leakage current passes through the dielectric material that holds the line in position.
4. Capacitance (C): A capacitive reactance to a time-varying signal due to capacitor form from conductors and the dielectric in-between.

So, for a two-wire line, we deal with series inductance and resistance, and parallel (shunt) capacitance and conductance, because any conductor (coil) possess "natural" resistance and there is always capacitance formed wherever two conductors come close to each other!

The totality of these parameters is obtained by multiplying by the length of the line, since they are given on a per-length basis. Continuous distribution is approximated by its representation as a cascade of network of elements, with each element of length $\delta z$, (delta $z$).

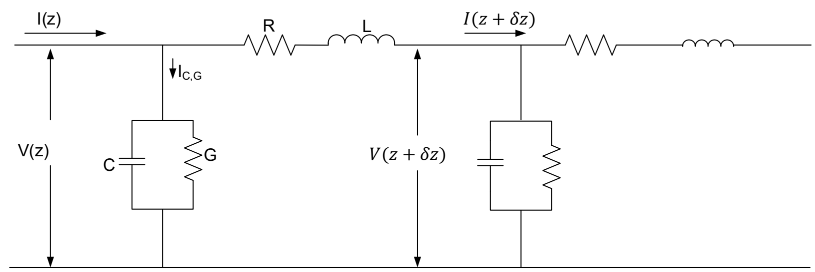



**Figure 9.1 A 2-cascade representation of transmission line**

**Using telegrapher's equation**

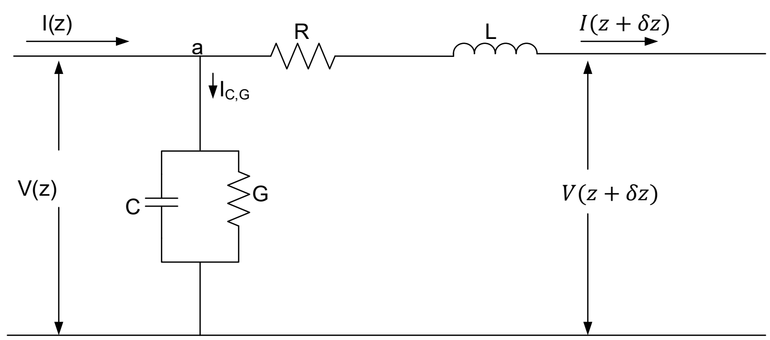



**Figure 9.1a One section of the transmission line**

Consider one section of the transmission line for the derivation of the characteristic impedance. The voltage on the left would be $V$ and on the right side would be $V(z + \delta z)$. Fig. 9.1a is to be used for both the derivation methods.

The differential equations describing the dependence of the voltage and current on time and space are linear, so that a linear combination of solutions is again a solution. This means that we can consider solutions with a time dependence and the time dependence will factor out, leaving an ordinary differential equation for the coefficients, which will be phasors depending on space only. Moreover, the parameters can be generalized to be frequency-dependent.

Taking KCL at point (a) of Fig. 9.1a, the current through the parallel combination of the capacitance and admittance elements is:

$$I_{CG} = I(z) - I(z+\delta z) = C\delta z \frac{\partial V(z)}{\partial t} + G\delta z\, V(z),$$

with $\delta z$ indicating per unit length basis, and with the partial derivatives noted. Voltage drops across the series combination of the resistor and inductor by KVL:

$$V_{RL} = V_R + V_L = [V(z) - V(z+\delta z)]$$

$$= R\delta z I\,(z+\delta z) + L\delta z \frac{\partial I(z+\partial z)}{\partial t}$$

Recall from first principles

$$\lim_{\delta x \to 0}\left[\frac{f(x+\delta x) - f(x)}{\delta x}\right] = \frac{df(x)}{dx}$$

So, $$\lim_{\delta z \to 0}\left[\frac{I(z+\delta z) - I(z)}{\delta z}\right] = \frac{\partial I(z)}{\partial z}$$

So that, $$I(z) - I(z+\delta z) \approx -\frac{\partial I(z)}{\partial z}\delta z$$

Similarly, $$V(z) - V(z+\delta z) \approx -\frac{\partial V(z)}{\partial z}\delta z$$

$$\Longrightarrow \quad \frac{-\partial I(z)}{\partial z}\delta z = C\delta z \frac{\partial V(z)}{\partial t} + G\delta z V(z)$$

$$\frac{\partial I(z)}{\partial z} = -\left(G + C\frac{\partial}{\partial t}\right)V(z)$$

Similarly, $$\frac{\partial V(z)}{\partial z} = -\left(R + L\frac{\partial}{\partial t}\right)I(z+\delta z)$$

$$\approx -\left(R + L\frac{\partial}{\partial t}\right)I(z) \qquad \text{for } \delta z \text{ small}$$

For sinusoidal signals, dependence on line is expressed by $e^{j\omega t}$ and derivative $\partial t$ expressed by $j\omega$, $\left(\frac{d}{dt}e^{j\omega t} = j\omega e^{j\omega t}, \text{recall}\right)$, and partial derivatives then become total derivatives.

$$\frac{dI}{dz} = -(G + j\omega C)V \qquad 9.1$$

$$\frac{dV}{dz} = -(R + j\omega L)\, I \qquad 9.2$$

Taking the second derivatives of $V$, from Eq (9.2),

$$\frac{d^2V}{dz^2} = -(R + j\omega L)\frac{dI}{dz} = (R + j\omega L)\,(G + j\omega C)V \;\; = \gamma^2 V \qquad 9.3$$

### 9.1 Propagation Constant $'\boldsymbol{\gamma}'$

Where $\gamma^2$ (gamma squared) $= (R + j\omega L)(G + j\omega C)$

Eq. (9.3) above has as its solution,

$$V = V_1 e^{-\gamma z} + V_2 e^{\gamma z} \qquad 9.4$$

Where $\quad \gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \qquad 9.5$

In general, $\gamma$ is a complex quality, and can therefore be represented by

$$\gamma = \alpha + j\beta$$

Substituting this is the expression for $V$,

$$V = V_1 e^{-(\alpha + j\beta)z} + V_2 e^{(\alpha + j\beta)z} \qquad 9.6$$

By a similar analysis, current is expressed with $I's$ replacing the $V's$ so, voltage at some point $z$ down the transmission line is made up of two components, namely:

a. $V_1 e^{-(\alpha+j\beta)z} = V_1 e^{-\alpha z} e^{-j\beta z}$ whose amplitude decreases (is attenuated) as it travels down the line with $z$ as $e^{-\alpha z}$, while $e^{-j\beta z}$ is just a phase term with no effect on the amplitude. Therefore, this component is known as the forward, or incident wave.

b. $V_2 e^{(\alpha+j\beta)z} = V_2 e^{\alpha z} e^{j\beta z}$ increases with increasing $z$, but since voltage must be attenuated as it travels along the line, $z$ must then decrease to accommodate this fact, therefore making this component to be known as the backward, or reflected, wave, caused by a mismatch between the transmission line and the load.

So, the voltage at any point on the line a distance $z$ from the sending end is the shin of the voltages of the incident and reflected waves at the said point.

Line parameters, $\alpha$ and $\beta$ are determined by the **line characteristics**:

1. $\alpha$ is known as attenuation coefficient, and the negative/positive exponential of this is the rate at which the forward/backward wave is attenuated, and is a function of R, L, G and C, with the unit being $\mathrm{dB/m}$ (decibels per metre) or repers/m.
2. $\beta$ is the phase constant and shows the phase dependence of both the incident and the reflected waves with distance $z$

$\beta\lambda = 2\pi \Longrightarrow \beta = \frac{2\pi}{\lambda}$, where $\lambda$ (Greek alphabet lambda) is the signal wavelength.

3. $\gamma$ (Gamma, Greek third alphabet) is the propagation constant, and is the complex sum of the attenuation coefficient and phase constant, where the former is the real part, and the letter the imaginary part. $\gamma$ determines how the voltage (or by implication the current) along the line changes with $z$

## 9.2 Characteristic Impedance

From the Eq. (9.2), $\frac{dV}{dz} = -(R + j\omega L)I,$

$$I = -\frac{I}{R + j\omega L} \times \frac{dV}{dz}$$

Differentiating Eq. (9.4)

$$\frac{dV}{dz} = -\gamma V_1 e^{-\gamma z} + \gamma V_2 e^{\gamma z} = \gamma[V_2 e^{\gamma z} - V_1 e^{-\gamma z}]$$

And substituting in the above for

$$I = -\frac{1}{R + j\omega L} \times \gamma[V_2 e^{\gamma z} - V_1 e^{-\gamma z}] = \frac{\gamma}{R + j\omega L} \times [V_1 e^{-\gamma z} - V_2 e^{\gamma z}]$$

Substituting from Eq. (9.5) for $\gamma$,

$$\mathrm{I} = \frac{\sqrt{(R + j\omega L)(G + j\omega C}}{(R + j\omega L)} \times [V_1 e^{-\gamma z} - V_2 e^{\gamma z}]$$

$$\Longrightarrow \qquad I = \sqrt{\left(\frac{G + j\omega C}{R + j\omega L}\right)} \times [V_1 e^{-\gamma z} - V_2 e^{\gamma z}] \qquad 9.7$$

By analogy with Ohm's law, $\frac{G+j\omega C}{R+j\omega L}$, is an admittance. Therefore, $\frac{R+j\omega L}{G+j\omega C}$ its reciprocal, is an impedance called the characteristic impedance of the transmission line, determined by the line parameters R, L, G & C.

$$\boxed{Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}} \qquad 9.8$$

Characteristic impedance $Z_o$ can be variously described as:

1. The value the load impedance must have to match the load to the line (to either eliminate power loss, or at least minimize same), or
2. The impedance seen from the sending end of an infinitely long line, or
3. The impedance seen looking towards the load at any point on a matched line, i.e moving along the line produces no change in the impedance towards the load.

The transmission line is idealized as follows:

1. The line is uniform, straight and homogenous,
2. Line parameters R, L, G and C do not vary with atmospheric conditions like temperature and humidity.
3. Line parameters do not depend on frequency,
4. The analysis is applicable only between the junctions on the line because the circuit model on Fig. 9.2 (one of the cascades) is invalid across a junction

The above assumptions may be occasionally taken into consideration as we analyze transmission line.

### 9.3 Reflection from the Load

Shown in Fig. 9.2 where, $V_1 e^{-\gamma l}$ is the incident wave, while $V_2 e^{\gamma l}$ is the reflected or backward, wave on a line with total length of $l$. If the load has an impedance equal to the characteristic impedance $Z_0$, therefore say that the line is matched, and there is no reflected wave (theoretically speaking) as the incident wave is totally absorbed by the load. It, however, the load is of a value different from $Z_0$, then some of the incident wave would be reflected, and the amount of reflection by the load. Is expressed in terms of voltage reflection coefficient, designated by the Greek letter $\rho$ $(rho)$, and defined as the ratio of the reflected voltage to the incident voltage at the load terminals.

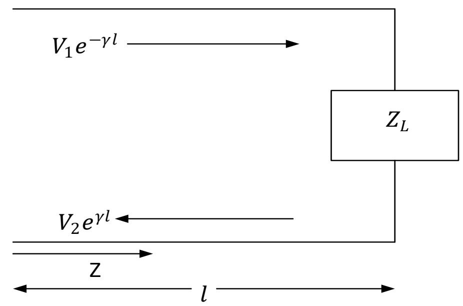



**Figure 9.2 Incident wave $(V_1e^{-\gamma l})$ and Reflected wave $(V_2e^{\gamma l})$**

Given that the load is at the position $z = l$,

$$V_L = V_1e^{-\gamma l} + V_2e^{\gamma l} \qquad 9.9$$

$$\rho = \frac{V_2e^{\gamma l}}{V_1e^{-\gamma l}} = \left(\frac{V_2}{V_1}\right)e^{2\gamma l} = |\rho|e^{j\psi} \qquad 9.10$$

Where the last indicates that $\rho$ in general would be a complex quantity that can be expressed in polar form with $|\rho|$ as the magnitude and $\psi$ as the phase angle of the reflection coefficient.

From Eqs. (9.7) and (9.8)

$$I_L = \left(\frac{V_1}{Z_0}\right)e^{-\gamma l} - \left(\frac{V_2}{Z_0}\right)e^{\gamma l} \qquad 9.11$$

$Z_L$ (Load impedance)= $\frac{V_L}{I_L}$

And from Eqs. (9.9) and (9.11)

$$Z_L = \frac{V_L}{I_L} = \frac{V_1e^{-\gamma l} + V_2e^{\gamma l}}{\left[\left(\frac{V_1}{Z_0}\right)e^{-\gamma l} - \left(\frac{V_2}{Z_0}\right)e^{\gamma l}\right]}$$

Dividing through by $V_1e^{-\gamma l}$ and multiplying by $Z_0$

$$Z_L = Z_0\left(\frac{\left[1 + \left(\frac{V_2}{V_1}\right)e^{2\gamma l}\right]}{\left[1 - \left(\frac{V_2}{V_1}\right)e^{2\gamma l}\right]}\right)$$

The term in the (inner) parenthesis namely $\left(\frac{V_2}{V_1}\right)e^{2\gamma l}$, is simply the voltage reflection coefficient $e$, leading to

$Z_L = Z_0\left(\frac{1+\rho}{1-\rho}\right)$, or rearrange

$$\boxed{\rho_v = \frac{Z_L - Z_0}{Z_L + Z_0}} \qquad 9.12$$

For $Z_L = 0$ (indicating short circuit load),

$$\rho = -\frac{Z_0}{Z_0} = -1 \Longrightarrow |\rho| = 1, \qquad \mu = \pi$$

Note that $\mu \Longrightarrow \psi$ as in Eq 9.10. so, in place of $\psi$, we can use $\mu$

For $Z_L = \infty$ (open circuit load):

$$\rho = \frac{Z_L}{Z_L} = 1 \Longrightarrow |\rho| = 1, \mu = 0$$

By a similar analysis, current reflection coefficient is given by

$$\rho_I = \frac{(Z_0 - Z_L)}{Z_0 + Z_L} = -\rho_V$$

Where $\rho_v$ stands for voltage reflection coefficient.

$$Z_L = 0 \Longrightarrow \rho_I = \frac{Z_0}{Z_0} = 1, \Longrightarrow |\rho| = 1, \psi = 0$$

Showing quality between the incident and reflected waves with no change in phase (with KCL taken at the no-load terminal).

$$Z_L = \infty \text{ (open circuit)} \Longrightarrow \rho_I = -\frac{Z_L}{Z_L} = -1 \Longrightarrow |\rho_I| = 1, \psi = \pi$$

**Example 9.1:** If $Z_L = 75 + j50\ \Omega$, $Z_0 = 25\ \Omega$, find the reflected coefficient

$$\rho = \frac{(75 + j50 - 25)}{(75 + j50 + 25)} = \frac{50 + j50}{100 + j50} = \frac{1 + j}{2 + j}$$

$$= \frac{(1 + j)(2 - j)}{2^2 + 1} = \frac{3 + j}{5} = \frac{\sqrt{10}}{5} \angle \tan^{-1}\frac{1}{3} = 0.63\angle 18.43^{\circ}$$

**Example 9.2**: The lossless transmission line has characteristic impedance of 75 Ω and phase constant of 3rad/m at 100 MHz. Find inductance and capacitance of line/meter.

$$\text{Solution: } Z_0 = \sqrt{\frac{L}{C}}$$

$$\gamma = \beta = \omega\sqrt{LC}$$

$$\frac{Z_0}{\beta} = \frac{\sqrt{\frac{L}{C}}}{\omega\sqrt{LC}} = \frac{1}{\omega C}$$

$$\frac{75}{3} \times 2\pi f = \frac{1}{C}$$

$$\Rightarrow \quad 25 \times 2\pi f = \frac{1}{C}$$

$$\Rightarrow \quad C = \frac{1}{25 \times 6.28 \times 10^8}$$

$$\Rightarrow \quad \boxed{C = 63.69 \text{ pF/m}}$$

$$Z_0{}^2 C = L$$

$$\Rightarrow \quad \boxed{L = (75)^2 \times 63.69 \times 10^{-12} = 358 \text{ nH/m}}$$

**Example 9.3**: A lossless transmission is 80 cm long and operates at a frequency of 600 MHz the line parameters are $L = 0.25\ \mu\text{H/m}$ and $C = 100\ \text{pF/m}$. Find the characteristic impedance, the phase constant, and the phase velocity.

Solution:

Since the line is lossless, both R and G are zero. The characteristic impedance is

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.25 \times 10^{-6}}{100 \times 10^{-12}}} = 50\ \Omega$$

Since $$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$= j\omega\sqrt{LC} \qquad \text{we see that}$$

$$\beta = \omega\sqrt{LC} = 2\pi\ (600 \times 10^6)\ \sqrt{(0.25 \times 106^{-6})\ (100 \times 10^{-12})} = \boxed{18.85 \text{ rad/m}}$$

Also,

$$V_p = \frac{\omega}{\beta} = \frac{2\pi\,(600 \times 10^6)}{18.85} = 2 \times 10^8\ \text{m/s}$$

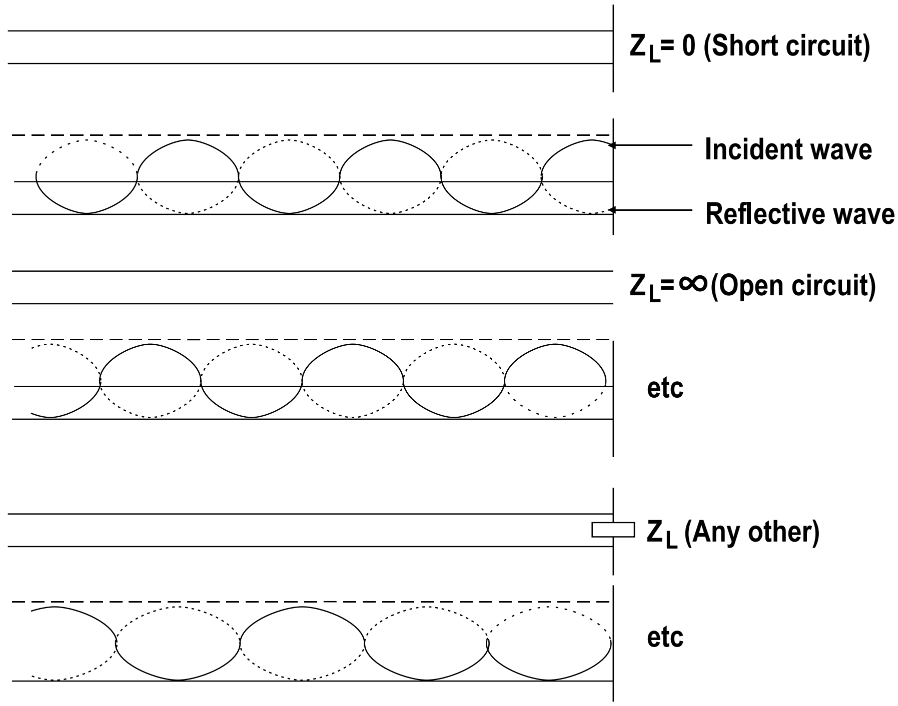



**Figure 9.3 combination of 'Short circuit Impedance A ', 'Open circuit impedance B' and when the line impedance equals the load impedance C**

## 9.4 Distortionless Line $\left(\frac{R}{L} = \frac{G}{C}\right)$

Distortionless line is the one in which attenuation constant '$\alpha$' is frequency independent while phase constant is linearly dependent on frequency.

(a) $$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)\left(\frac{RC}{L} + j\omega C\right)} \quad 9.13$$

$$= \sqrt{\frac{C}{L}}\ (R + j\omega L)$$

$\Longrightarrow$ $$\alpha = R\sqrt{\frac{C}{L}}\ and\ \ \beta = \omega\sqrt{LC} \quad 9.14$$

(b) $$V_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \quad 9.15$$

(c) $$Z_0 = R_0 + jX_0 = \sqrt{\frac{R\left(1+\frac{j\omega L}{R}\right)}{G\left(1+\frac{j\omega C}{G}\right)}} = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}} \qquad 9.16$$

$$\Rightarrow \qquad R_0 = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}; \qquad X_0 = 0 \qquad 9.17$$

A lossless line is also distortionless line, but a distortionless line is not necessarily lossless

**Example 9.4:** A 60 $\Omega$ distortionless transmission line has a capacitance of 0.15 $\mathrm{nF/m}$ . The attenuation on the line is 0.01 dB/m. Calculate

a. the line parameters: resistance, inductance and conductance per meter of line
b. velocity of propagation
c. voltage at a distance of 1 km and 4 km with respect to sending end voltage.

Solution:
For a distortionless line,

$$\boxed{\frac{R}{L} = \frac{G}{C}}$$

$$Z_0 = R_0 = \sqrt{\frac{L}{C}} = 60\ \Omega$$

and

$$\alpha = R\sqrt{\frac{C}{L}} = 0.01\frac{\mathrm{dB}}{\mathrm{m}} = \frac{0.01}{8.69}\mathrm{Np/m} = 1.15\times10^{-3}\ \mathrm{Np/m}$$

Line parameters:

$$R = \alpha R_0 = (1.15\times10^{-3})\times60 = 0.069\ \Omega/\mathrm{m}$$

$$L = CR_0^2 = 0.15\times10^{-9}\times60^2 = 0.54\ \mu\mathrm{H/m}$$

$$G = \frac{RC}{L} = \frac{R}{{R_0}^2} = \frac{0.059}{60^2} = 19.2\ \mu\mathrm{S/m}$$

(b). $V = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.54\times10^{-6}\times0.15\times10^{-9}}} = 1.11\times10^{8}\ \text{m/s}$

(c). The ratio of two voltages at a distance x apart along the line

$$\boxed{\frac{V_2}{V_1} = e^{-\alpha x}}$$

At 1 km

$$\frac{V_2}{V_1} = e^{-1000\alpha} = e^{-1.15} = 0.317\ or\ 31.7\%$$

At 4 km

$$\frac{V_2}{V_1} = e^{-4000\alpha} = e^{-4.6} = 0.01\ or\ 1\%$$

## 9.5 Low-Loss Dielectric

A low-loss dielectric is a good but imperfect insulator with a non-zero equivalent conductivity such that $t"\ \ll \in'$ or $\frac{\sigma}{\omega\in} \ll 1$. Under this condition $\gamma$ can be approximated by using binomial expansion.

$$\gamma = \alpha + j\beta \equiv j\omega\sqrt{\mu\ \in'}\left[1 - \frac{j\ \in'}{2\ \in'} + \frac{1}{8}\left(\frac{\in"}{e'}\right)^2\right]$$

From which we can say

$$\alpha \cong \frac{\omega \in "}{2}\sqrt{\frac{\mu}{\in}}\left(\frac{N_p}{m}\right)\text{attenuation constant}$$

$$And \quad \beta \cong \omega\sqrt{\mu \in}\left[1 + \frac{1}{8}\left(\frac{\epsilon"}{\epsilon'}\right)^2\right]\left(\frac{\text{rad}}{\text{m}}\right)\text{phase constant}$$

$$\boxed{\begin{array}{c}'\alpha'\text{for low} - \text{loss dielectric is a positive quantity and}\\ \text{is approximately directly proportional}\\ \text{to frequency. }\beta\text{ deviates only very slightly from value }2\sqrt{\mu \in}\ \text{(lossless dielectric)}\end{array}}$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}}\left(1 - j\ \frac{\in"}{\in'}\right)^{-\frac{1}{2}} \simeq \sqrt{\frac{\mu}{\in'}}\left(1 + j\frac{\in"}{2\ \in'}\right)(\Omega)\ \rightarrow \text{intrinsic impedance}$$

We can say that $\frac{E_x}{H_y} = \eta$ and here the electric and magnetic field intensities in lossy dielectric are not in time phase as in lossless medium.

$$V_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu \in'}}\left[1 - \frac{1}{8}\left(\frac{\in''}{\in'}\right)^2\right] \text{m/s} \qquad \text{phase velocity}$$

## 9.6 Equivalent Circuit in Terms of Primary and Secondary Constants

**Equivalent T-section of a line of length $\delta$**

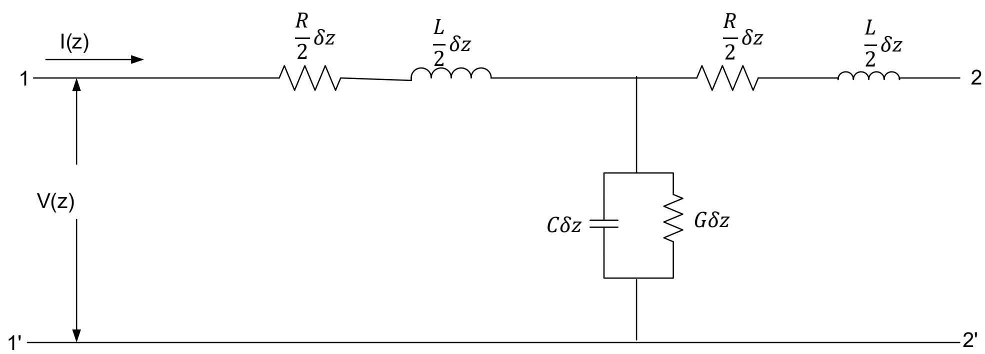



**Figure 9.3 Equivalent $'T'$ Transmission Line Circuit**

**Equivalent $\pi$ −section of a line of length $\delta$**

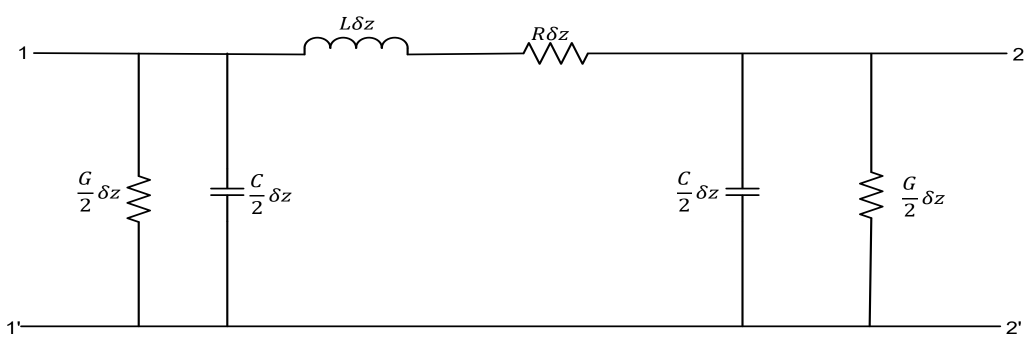



**Figure 9.4 Equivalent $'\pi'$ Circuit**

Here, $z = (R + j\omega L)\ \Omega;\ \ y = (G + j\omega C)\ \mho$

Secondary constants of line

a. The input impedance of line is called its characteristics impedance

$$Z_0 = \sqrt{\frac{z}{y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

b. $\gamma = \alpha + j\beta$ **(Propagation constant)**

- Real part $\alpha$ of $\gamma$ is measured of charge in magnitude of current or voltage in each $\tau$-section and called attenuation constant.
- Imaginary part $\beta$ of $\gamma$ equal difference in phase angle between the input current and the output current or the corresponding voltages and called phase shift constant.

$$\gamma = \sqrt{zy} = \sqrt{(R + j\omega L)(G + j\omega C)}$$

c. The phase shift constant or wavelength constant $\beta$ indicates the amount by which the phase of an input current changes in a unit distance. In a distance equal to one wavelength $\lambda$, the phase shift is $2\pi$ radians, $\lambda = \frac{2\pi}{\beta}$, wavelength.

d. The phase velocity of propagation is

$$v_p = f\lambda = \frac{\omega}{\beta}$$

**Example 9.5:** An open wire transmission line has $R = 5\ \Omega/\text{m}$, $L = 5.2 \times 10^{-8}\ \text{H/m}$, $G = 6.2 \times 10^{-3}\ \Omega/\text{m}$, $C = 2.13 \times 10^{-13}\ \text{F/m}$, frequency = 4 GHz. Find $Z_0, \gamma\ and\ \text{v}_\text{f}$.

Solution:

$$\text{v}_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{5.2 \times 10^{-8} \times 2.13 \times 10^{-10}}}$$

$$= 0.3 \times 10^{9} = 0.3 \times 10^{8}\ \text{m/s}$$

$$\omega = 2\pi f = 2\pi \times 4 \times 10^{9} = 8\pi \times 10^{9} = 2.512 \times 10^{16}\ \text{rad}$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

$$R + j\omega L = 5 + j2.512 \times 10^{10} \times 5.2 \times 10^{-8}$$

$$= 5 + j1306.24 = 1306.25 < 89.78^{\circ}$$

$$G + j\omega L = 6.2 \times 10^{-3} + j2.512 \times 10^{10} \times 2.13 \times 10^{10}$$

$$= 6.2 \times 10^{-3} + j5.35 = 8.18 < 40.79^{\circ}$$

$$\boxed{Z_0 = 12.64 < 24.49^{\circ}}$$

$$\gamma = \sqrt{(R + j\omega L)\,(G + j\omega C)}$$

$$\boxed{\gamma = 103.37 < 65.23^{\circ}}$$

**Example 9.6:** A typical transmission line has a resistance of 8 Ω/km, impedance of 2 mH/km, a capacitance of 0.002 μF/km and a conductance of 0.07 μs/km. Calculate the characteristic impedance, attenuation constant, phase constant of the transmission line at a frequency of 2 kHz. If a signal of 2 V is applied and the line terminated by its characteristic impedance, calculate the power delivered to load

Solution:

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

$$= \sqrt{\frac{8 + j4\pi \times 2 \times 10^{-3} \times 10^3}{0.007 \times 10^{-6} + j4\pi \times 0.002 \times 10^{-6} \times 10^3}}$$

$$= 1.024 < -8.75^o \times 10^3\ \Omega$$

$$= (1012.1 - j155.72)\ \Omega$$

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)\,(G + j\omega C)}$$

$$\sqrt{(8 + j4\pi \times 2 \times 10^{-3} \times 10^3)(0.007 \times 10^{-6} + j4\pi \times 0.002 \times 10^{-6} \times 10^3)}$$

$$= 0.02574 < 81.09 = 0.003987 + j0.02543$$

$$\Longrightarrow \qquad \alpha = 0.003987\ \text{Np/km}$$

$$\beta = 0.02543\ \text{rad/km}$$

Input voltage $V_s = 2\ \text{V}$; $l = 500\ \text{km}$; $Z_0 = 1012.1\ \Omega$ (real part)

Since line is terminated in its characteristic impedance, $Z_{in} = Z_0 = Z_L$

$$\text{I}_s = \frac{V_s}{Z_{in}} = \frac{2}{1024 < -8.75^o \times 10^3} = \frac{2}{1024 < -8.75^o} = 1.953 < 8.75^o\ \text{mA}$$

$$I_l = I_s\, e^{-\gamma l} = (1.953 < 8.75^o) e^{(-0.003987 + j0.02543) \times 500}$$

$$|I_l| = 1.953 \times e^{-1.9935} = 0.2669\ \text{mA}$$

$$P = |I_l|^2\ \text{Real}\ (Z_0) = 1012.1 \times (0.2669)^2 = 72.1\ \mu\text{W}$$

$$V_p = \frac{\omega}{\beta} = \frac{4\pi \times 10^3}{0.02543} = 494.22\ \text{km/s}$$

**Example 9.7:** A 600 Ω lossless transmission line is fed by a 50 Ω generator. If the line is 200 m long and terminated by load of 500 Ω, determine in $dB's$.

(i) Reflection loss
(ii) Transmission loss
(iii) Return loss.

Solution:

$$\rho = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{500-600}{500+600} = \frac{-100}{1100} = \frac{-1}{100} = \frac{-1}{11}$$

i. Reflection loss $= 10\log_{10}\frac{1}{1-|\rho|^2} = 10\log_{10}\frac{1}{1-\left(\frac{1}{121}\right)} = 0.036\ \text{dB}$

ii. Transmission loss = Attenuation loss + Reflection loss

$$= \text{lossless} + 0.036$$
$$= 0 + 0.036 = 0.036\ \text{dB}$$

iii. Return loss $= 10\log_{10}|\rho| = 10\log_{10}\left(\frac{1}{11}\right) = -10.414\ \text{dB}$

## 9.7 Sending-End Impedance

To determine the degree of mismatch between the source and line, we have to know the impedance that the combination of transmission line and load presents to the source. Sending end impedance is that looking into the line from the source:

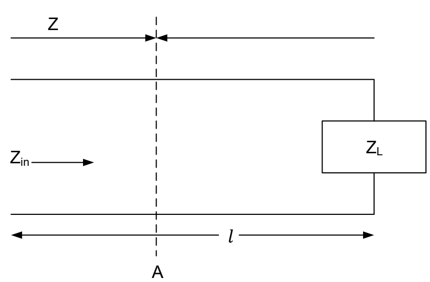



**Figure 9.5 Sending end Impedance and Load Impedance**

from Eqs. (9.4), (9.7) and (9.8)

$$Z_A = \frac{V_A}{I_A} = Z_0\,\frac{V_1 e^{-\gamma z} + V_2 e^{\gamma z}}{V_1 e^{-\gamma z} - V_2 e^{\gamma z}}$$

From Eq. (9.10), $\frac{V_2}{V_1} = e^{-2\gamma l}$

$$\Rightarrow \qquad Z_A = Z_0 \left(\frac{e^{-\gamma z} + \rho\, e^{-2\gamma l} e^{\gamma z}}{e^{-\gamma z} - \rho\, e^{-2\gamma l}\, e^{\gamma z}}\right)$$

After dividing through by $V_1$

Multiplying through by $e^{\gamma l}$

$$Z_A = Z_0 \left(\frac{e^{\gamma l} e^{-\gamma z} + \rho\, e^{\gamma l - 2\gamma l} e^{\gamma z}}{e^{\gamma l} e^{-\gamma z} - \rho\, e^{\gamma l - 2\gamma l}\, e^{\gamma z}}\right) = Z_0 \left(\frac{e^{\gamma (l-z)} + \rho e^{-\gamma (l-z)}}{e^{\gamma (l-z)} - \rho e^{\gamma (l-z)}}\right)$$

$l - z = x$ from Fig.9.5

$$\Rightarrow \qquad Z_A = Z_0 \left(\frac{e^{\gamma x} + \rho e^{-\gamma x}}{e^{\gamma x} - \rho e^{-\gamma x}}\right)$$

For $\quad \rho_V = \dfrac{(Z_L - Z_0)}{(Z_L + Z_0)}$

$$Z_A = Z_0 \left[\frac{e^{\gamma x} + \left[\frac{(Z_L - Z_0)}{(Z_L + Z_0)}\right] e^{-\gamma x}}{e^{\gamma x} - \left[\frac{(Z_L - Z_0)}{(Z_L + Z_0)}\right] e^{-\gamma x}}\right]$$

Multiplying through by $(Z_L + Z_0)$

$$Z_A = Z_0 \left[\frac{(Z_L + Z_0) e^{\gamma x} + (Z_L - Z_0) e^{-\gamma x}}{(Z_L + Z_0) e^{\gamma x} - (Z_L - Z_0) e^{-\gamma x}}\right]$$

Factorizing,

$$Z_A = Z_0 \left[\frac{Z_L (e^{\gamma x} + e^{-\gamma x}) + Z_0 (e^{\gamma x} - e^{-\gamma x})}{Z_L (e^{\gamma x} - e^{-\gamma x}) + Z_0 (e^{\gamma x} + e^{-\gamma x})}\right]$$

Dividing through by 2 to give hyperbolic functions

$$Z_A = Z_0 \left[\frac{Z_L \cosh \gamma x + Z_0 \sinh \gamma x}{Z_L \sinh \gamma x + Z_0 \cosh \gamma x}\right]$$

Dividing through by $\cosh \gamma x$,

$$Z_A = Z_0 \left[\frac{Z_L + Z_0 \tanh \gamma x}{Z_L \tanh \gamma x + Z_0}\right]$$

Putting $x = l,\ Z_A$ becomes $Z_{in}$ (sending-end impedance)

$$\Longrightarrow \qquad Z_{in} = \left[\frac{Z_0\, Z_L + {Z_0}^2 \tanh \gamma l}{Z_L \tanh \gamma l + Z_0}\right] \qquad 9.18a$$

When normalized to the characteristic impedance $Z_0$,

$$z_{in} = \frac{Z_{in}}{Z_0} = \left[\frac{Z_L + Z_0 \tanh \gamma l}{Z_L \tanh \gamma l + Z_0}\right]$$

Normalized, load impedance $z_L = \frac{Z_L}{Z_0}$

$$\Longrightarrow \qquad z_{in} = \left[\frac{\left(\frac{Z_L}{Z_0}\right) + \tanh \gamma l}{\left(\frac{Z_L}{Z_0}\right) \tanh \gamma l + 1}\right] \qquad 9.18b$$

$$z_{in} = \frac{z_L + \tanh \gamma l}{z_L \tanh \gamma l + 1} \qquad 9.18c$$

## 9.8 Low Loss Lines

Eq. (9.5): $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$

Factoring out $j\omega L$ and $j\omega C$,

$$\gamma = \sqrt{(j\omega L)\,(j\omega C)\left(\frac{R}{j\omega L} + \frac{j\omega L}{j\omega L}\right)\left(\frac{G}{j\omega C} + \frac{j\omega C}{j\omega C}\right)}$$

$$= j\omega\sqrt{LC}\left(1 + \frac{R}{j\omega L}\right)^{\frac{1}{2}}\left(1 + \frac{G}{j\omega C}\right)^{\frac{1}{2}}$$

Binomial series expansion of $\gamma$ gives:

$$\gamma = j\omega\sqrt{LC}\left(1 + \frac{R}{2j\omega L} - \frac{1}{4}\,\frac{R^2}{(j\omega L)^2}\right) \times \left(1 + \frac{G}{2j\omega C} - \frac{G^2}{4(j\omega C)^2}\right)$$

For low-loss lines, R and G are very small, and can therefore be ignored:

$$\Longrightarrow \qquad \gamma \approx j\omega\sqrt{LC}\,\left(1 + \frac{R}{2j\omega L}\right) \times \left(1 + \frac{G}{2j\omega C}\right)$$

$$= j\omega\sqrt{LC}\,\left(1 + \frac{R}{2j\omega L} + \frac{G}{2j\omega C} - \frac{RG}{(2j\omega)^2 LC}\right)$$

$$= j\omega\sqrt{LC}\left(1 + \frac{G}{2j\omega C} + \frac{R}{2j\omega L} - \frac{RG}{4\omega^2 LC}\right)$$

$$= j\omega\sqrt{LC}\left(1 - \frac{RG}{4\omega^2 LC} - \frac{jR}{2\omega L} - \frac{jG}{2\omega C}\right)$$

$$\gamma = \alpha + j\beta = \omega\sqrt{LC}\left(j - \frac{jRG}{4\omega^2 LC} - \frac{j^2 R}{2\omega L} - \frac{j^2 G}{2\omega C}\right)$$

$$\alpha + j\beta = \omega\sqrt{LC}\left[\left(\frac{R}{2\omega L} + \frac{G}{2\omega C}\right) + j\left(1 - \frac{RG}{4\omega^2 LC}\right)\right]$$

$$\Rightarrow \qquad \alpha \approx \omega\sqrt{LC}\left(\frac{R}{2\omega L} + \frac{G}{2\omega C}\right), \quad \beta \approx \omega\sqrt{LC}\left(1 - \frac{RG}{4\omega^2 LC}\right)$$

$$\boxed{\alpha \approx \frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}}} \qquad 9.19$$

$$\beta \approx \omega\sqrt{LC}\left(1 - \frac{RG}{4\omega^2 LC}\right)$$

R and G very small, so at high frequencies:

$$\boxed{\beta \approx \omega\sqrt{LC}} \qquad 9.20$$

Similarly,

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{j\omega L}{j\omega C} \times \frac{j\omega C}{j\omega L} \times \left(\frac{R + j\omega L}{G + j\omega C}\right)}$$

$$= \sqrt{\frac{j\omega L}{j\omega C}\left(\frac{R}{j\omega L} + 1\right)\left(\frac{G}{j\omega C} + 1\right)} = \sqrt{\frac{L}{C}} \times \left(1 + \frac{R}{j\omega L}\right)^{\frac{1}{2}} \times \left(1 + \frac{G}{j\omega C}\right)^{\frac{1}{2}}$$

$$\approx \sqrt{\frac{L}{C}} \times \left(1 + \frac{R}{2j\omega L}\right) \times \left(1 - \frac{G}{2j\omega C}\right)$$

By binomial expansion, with terms in $R^2, G^2$ neglected

$$Z_0 = \sqrt{\frac{L}{C}} \times \left(1 - \frac{G}{2j\omega C} + \frac{R}{2j\omega L} - \frac{RG}{4j^2\omega^2 LC}\right)$$

$$\approx \sqrt{\frac{L}{C}} \times \left(1 - \frac{jR}{2\omega L} + \frac{jG}{2\omega C}\right)$$

$$\frac{R}{\omega L}, \frac{G}{\omega C} \text{ very small } \Longrightarrow \boxed{Z_0 = \sqrt{\frac{L}{C}}} \qquad 9.21$$

Plugging Eq. 9.21 in 9.19

$$\alpha = \boxed{\frac{R}{2Z_0} + \frac{GZ_0}{2}} \qquad 9.22$$

## 9.9 Lines of Zero Loss

For a relatively short line and operating at very high frequencies, it is reasonable to assume zero attenuation, i.e., lossless line

$$\Longrightarrow \qquad \alpha = 0 = \frac{R}{2Z_0} + \frac{GZ_0}{2} \Longrightarrow {Z_0}^2 = -\frac{R}{G}$$

$$\Longrightarrow \qquad Z_0 = j\sqrt{\frac{R}{G}}$$

In this case $\gamma = \alpha + j\beta = 0 + j\beta = j\beta$

Replacing $\gamma$ by $j\beta$ in Eq. 9.18b

$$z_{in} = \frac{z_L + \tanh j\beta l}{1 + z_L \tanh j\beta l} \Longrightarrow \boxed{\frac{z_L + j\tan\beta l}{1 + jz_L \tan\beta l}} \qquad 9.23$$

## 9.10 Quarter Wave Transformer

For a lossless line $(\alpha = 0)$ and replacing $\gamma$ by $j\beta$ in Eq. (9.18a)

$$\Longrightarrow \qquad Z_{in} = Z_0 \left(\frac{Z_L + j\, Z_0 \tan\beta l}{Z_0 + j\, Z_L \tan\beta l}\right)$$

$$\Longrightarrow \qquad Z_{in} = Z_0 \left(\frac{Z_L/\tan\beta l + jZ_0}{Z_0/\tan\beta l + jZ_L}\right)$$

$d$ = quarter wavelength long $\Longrightarrow \tan\beta l = \tan\frac{\pi}{2} = \infty$

$$Z_{in} = \lim Z_0 \left( \frac{\frac{Z_L}{x} + j\, Z_0}{\frac{Z_0}{x} + jZ_L} \right) = Z_0 \left( \frac{jZ_0}{j\, Z_L} \right) = \frac{Z_0}{Z_L}$$

$$\Longrightarrow \quad \boxed{{Z_0}^2 = Z_{in}\, Z_L} \qquad 9.24$$

For matching a given load to a given input impedance, a quarter wave section of lossless line is used with characteristics impedance of

$$Z_0 = \sqrt{Z_{in}\, Z_L}$$

**Example 9.8:** A 50 W lossless line has a length of $0.4\lambda$. The operating frequency is 300 MHz. A load $Z_L = 40 + j30\ \Omega$ is connected at $Z = 0$, and the Thevenin equivalent source at $Z = -1$ is $12 < 0^o$ V in series with $Z_{Th} = 50 + j0\ \Omega$. Find (a) ρ; (b) S; (c) $Z_{in}$.

Solution:

Using 9.23,

$$Z_{in} = \frac{(Z_L + jZ_0 \tan \beta l)}{(Z_0 + jZ_L \tan \beta l)}$$

Putting $Z_L = \infty \left(\text{we know that} \frac{1}{\infty} = 0\right)$ and dividing entire by $Z_L$ we get

$$\text{So,} \quad Z_{in} = Z_0 \frac{(1+0)}{(0 + j \tan \beta l)} = \frac{Z_0}{j \tan \beta l} = \frac{Z_0}{j \tan \beta l} = -j\, \frac{Z_0}{\tan \beta l}$$

***Ans:***

(a) $0.333 < 90^o$

(b) $2.00$

(c) $25.5 + j5.90\ \Omega$

**Example 9.9:** Calculate the characteristic impedance of a quarter-wave transformer if a 120 Ω load is to be matched to a 75 Ω line.

Solution:

$$Z_0 = \sqrt{Z_L\, Z_{in}}$$

$$\Longrightarrow \qquad \frac{{Z_0}^2}{Z_{in}} = Z_0 = \sqrt{120 \times 75} = 95\Omega$$

## 9.11 Stubs

Eq. (9.23) $\Longrightarrow z_{in} = \frac{z_L + j\tan\beta l}{1 + j\, z_L \tan\beta l}$ shows the variation of input impedance with the length of the line and this property can be used in stubs (short lengths of line) for matching applications. These are terminated in either short circuit or open circuit load.

$$\text{Open} - \text{circuit load} \Longrightarrow z_L = \infty = Z_{in} = \frac{z_L}{j z_L \tan\beta l}$$

$$\boxed{z_{in} = \frac{1}{j\tan\beta l} = -j\cot\beta l} \qquad 9.25$$

$$\text{Short circuit load } \Longrightarrow z_{in} = \frac{0 + j\tan\beta l}{1 + 0} = j\tan\beta l \qquad 9.26$$

For lossless line:

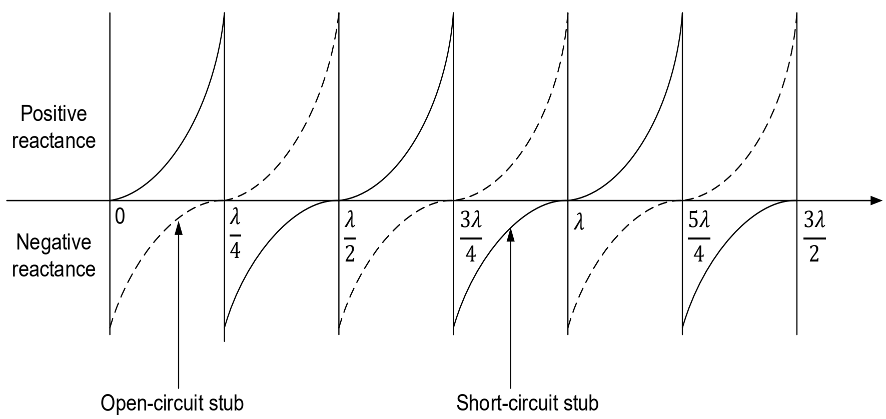



**Figure 9.6 Stubs**

**Example 9.10:** An ideal lossless $\frac{\lambda}{4}$ extension of line $Z_0 = 60\ \Omega$ is terminated with $Z_L$. Find $Z_{in}$ of extension when

(i) $Z_L = 0$
(ii) $Z_L = \infty$
(iii) $Z_L = 60\ \Omega$

Solution

i. $Z_{in} = Z_0$ where $\beta l = \frac{2\pi}{\infty} \times \frac{\lambda}{4}$

$Z_{in} = j\, Z_0 \tan\beta l = \infty$ for $Z_L = 0$

ii. $Z_{in} = \frac{Z_0}{j \tan \beta l} = 0 \quad \text{for} \qquad Z_{in} = \infty$

iii. $Z_L = 60\ \Omega$

$$= 60 \left( \frac{60 + j60 \tan\left(\frac{\pi}{2}\right)}{60 + j60 \tan\left(\frac{\pi}{2}\right)} \right) = 60\ \Omega$$

### 9.12 Standing Waves

For a lossless line ($\alpha = 0$), the total voltage at a point z from the sending end:

$$\Longrightarrow \qquad V = \left(V_1 e^{-j\beta z} + V_2 e^{j\beta z}\right) e^{j\omega t}$$

Where $e^{j\omega t}$ indicates the time dependence.

From $\rho = \left(\frac{V_2}{V_1}\right) e^{2\gamma l} \quad [\text{Eq.} (9.10)]$

$$V = V_1 e^{j\omega t} \left[ e^{-j\beta(l-x)} + \left(\frac{V_2}{V_1}\right) e^{j\beta l}\, e^{-j\beta x} \right]$$

For lossless line, $\rho = \left(\frac{V_2}{V_1}\right) e^{2\gamma l} = \left(\frac{V_2}{V_1}\right) e^{j2\beta l}$

Since, $\alpha = 0 \Longrightarrow \boxed{V = V_1 e^{j\omega t}\, e^{-j\beta l} \left[ e^{j\beta x} + e^{-j\beta x} \right]}$ 9.27

This is the equation representing voltage standing wave (VSWR), made up of two component waves, one of forward direction, and the other of backward direction reflected from the load.

For a short circuit load ($\rho = -1$) and without the time dependence,

$$V = j2V_1\, e^{-j\beta l} \left[ e^{j\beta x} - e^{-j\beta x} \right] = \boxed{R_e V_1 e^{-j\beta l} \sin \beta x = V} \qquad 9.28$$

The real part of the absolute value (modulus) of Eq. (9.28) is

$$|V| = R_e \left| \left[ j2V_2 e^{-j\beta l} \sin \beta x \right] \right|$$

$$= R_e |[j2V_1 (\cos \beta l - j \sin \beta l)] \sin \beta x|$$

$$= R_e\, |[2V_1 (j \cos \beta l + \sin \beta l)] \sin \beta x|$$

$\boxed{|V| = 2V_1\, |\sin \beta x| \sin \beta l}$ 9.29

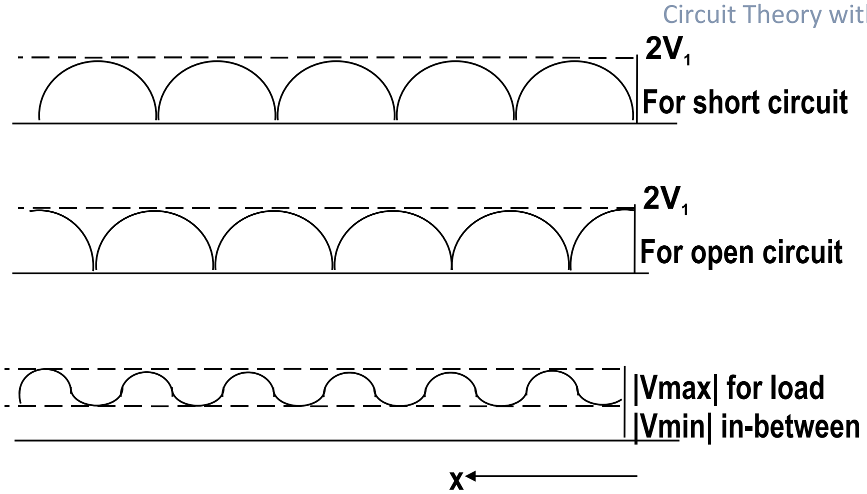



**Figure 9.7 Standing Waves**

For an open-circuit load $(\rho = 1)$ under the same conditions,

$$V = V_1 e^{-j\beta l}\left(e^{j\beta x} + e^{-j\beta x}\right) = 2V_1 e^{-j\beta l}\cos\beta x \Longrightarrow |V| = 2V_L|\cos\beta x|$$

For a load in between short and open circuit, say $e = 0.6 + j0.3$,

$$V = V_1 e^{-j\beta l}\left[e^{j\beta x} + (0.6 + j0.3)e^{-j\beta x}\right]$$

$$|V| = R_e\left[V_1 e^{-j\beta l}\left|e^{j\beta x} + (0.6 + j0.3)e^{-j\beta x}\right|\right]$$

From Eq. (9.29), for $(n-1)\pi = \beta x, \sin\beta x = 0$, and the next minimum occurs at

$$\frac{\pi}{\beta} = \left(\frac{\pi}{\frac{2\pi}{\lambda}}\right) = \frac{\lambda}{2}$$

It could be discerned that minima for short circuits occur at maxima for open circuit, and vice versa. Both the adjacent minima and maxima are separated by half a wavelength with the first minimum occurring at the load terminals for short circuit (maximum for open circuit). For a load in between, the minima and maxima the between zero and $2V_1$, but with adjacent minima and maxima still half a wavelength apart.

Voltage standing wave ratio (VSWR)

$$\text{By definition,} \quad \text{VSWR} \Longrightarrow S = \frac{|V_{max}|}{|V_{min}|}$$

$1 \leq S \leq \infty$ and depends on the degree of mismatch at the load (reflection coefficient).

From Eq. (9.27), plugging in $\rho = |\rho| e^{j\psi}$

$$\boxed{V = V_1 e^{-j\beta l}\left[e^{j\beta x} + \rho e^{-j\beta x}\right]} \qquad 9.30$$

$$\boxed{V = V_1 e^{-j\beta}\ (l - x)\left[1 + |\rho| e^{j(\psi - 2\beta x)}\right]} \qquad 9.31$$

$$\boxed{|V_{max}| = V_1\ (1 + |\rho|)} \qquad 9.32$$

When $(\psi - 2\beta x) = 2(m-1)\pi, m = 1,2,3, \ldots,$ i.e when $2\ (m-1)$ is a positive even number, making $\cos(\psi - 2\beta x)$ positive unity,

$$\boxed{|V_{min}| = V_1(1 - |\rho|)} \qquad 9.33$$

When $\psi - 2\beta x) = (2m-1), m = 1, 2, 3, \ldots ..$ i.e when $2m - 1$ is a positive odd number, making $\cos(\psi - 2\beta x)$ negative unity,

$$S = \frac{V_1(1 + |\rho|)}{V_1(1 - |\rho|)} = \boxed{\frac{1 + |\rho|}{1 - |\rho|} = S} \qquad 9.34$$

$$\Longrightarrow \qquad \boxed{|\rho| = \frac{S - 1}{S + 1}} \qquad 9.36$$

From Eq. (9.31) at the first voltage minimum, at $x = x_{min}$ from the load,

$$\psi - 2\beta x = \pi$$

$$\psi = 2\beta x_{min} + \pi \Longrightarrow Z = Z_{min} = \left(\frac{V}{I}\right) x_{min}$$

$$Z_{min} = \frac{V_{min}}{I_{min}} = \frac{V_1 e^{-j\beta(l - x_{min})}\left[1 + |\rho| e^{j(\psi - 2\beta x_{min})}\right]}{\left(\frac{V_1}{Z_0}\right) e^{-j\beta(l - x_{min})} - \left(\frac{V_2}{Z_0}\right) e^{j\beta(l - x_{min})}}$$

$$= \frac{V_1 e^{-j\beta(l - x_{min})}\left[1 + |\rho| e^{j(\psi - 2\beta x_{min})}\right]}{V_1 e^{-j\beta(l - x_{min})}\left[1 - |\rho| e^{j\psi - j2\beta x_{min}}\right]} Z_0$$

$$= \frac{V_1 e^{-j\beta(l - x_{min})}\left[1 + |\rho| e^{j(\psi - 2\beta x_{min})}\right]}{V_1 e^{-j\beta(l - x_{min})}\left[1 - |\rho| e^{j(\psi - j2\beta x_{min})}\right]} Z_0$$

$$= Z_0 \times \frac{1 + |\rho| e^{j\pi}}{1 - |\rho| e^{j\pi}}$$

But from trigonometry (Euler's identity), $e^{j\pi} = \cos\pi + j\sin\pi = -1$

$$= Z_0 \times \frac{1-|\rho|}{1+|\rho|} = \boxed{\frac{Z_0}{S} = Z_{min}} \qquad 9.37$$

Normalized to the characteristic impedance,

$$\frac{Z_{min}}{Z_0} = z_{min} \qquad 9.37a$$

$$z_{min} = \frac{1}{S} \qquad 9.37b$$

Similarly,

$$Z_{max} = Z_0 \times \frac{1+|\rho|}{1-|\rho|}$$

$$Z_{max} = Z_0\, S \qquad 9.38$$

$$S = z_{max} \qquad 9.38a$$

**Example 9.11:** A $50\ \Omega$ lossless transmission line is terminated by a load impedance,

$Z_L = 50 - j75\ \Omega$. If the incident power is $100\ \mathrm{mW}$. Find the power dissipated by the load.

Solution:

The reflection coefficient $\Longrightarrow \rho = \frac{Z_L - Z_0}{Z_L + Z_0}$

$$\rho = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 - j75 - 50}{50 - j75 + 50} = 0.36 - j\,0.48 = 0.60\ e^{-j93}$$

Then, $\langle P_t \rangle = (1 - |\rho|^2)\langle P_i \rangle = [1 - (0.60)^2](100) = 64\ \mathrm{mW}$

Impedance at a voltage minimum/maximum

**Example 9.12:** A lossless transmission line of $Z_0 = 100\ \Omega$ is terminated by an unknown impedance. The termination is found to be at a maximum of the voltage standing wave and the VSWR is 5. What is the value of terminating impedance?

Solution:

We know that $Z_{max} = Z_0$.(VSWR) as the termination is at maximum of the voltage standing wave.

$$\boxed{Z_{max} = 100 \times 5 = 500\ \Omega}$$

## 9.13 Load Impedance an a Lossless Line

This can be determined if the VSWR, wavelength $(\lambda)$ and distance from the load to the nearest voltage minimum are known.

Equation: $V = \left(V_1 e^{-j\beta(l-x)}\right)\left[1 + |\rho| e^{j(\psi - 2\beta x)}\right]$

$$\psi - 2\beta x = (2m-1)\pi, \qquad m = 1, 2, 3, \ldots$$

$$m = 1 \Rightarrow x = x_{min} \Rightarrow \psi - 2\beta x = \pi$$

$$\Rightarrow \qquad \boxed{\psi = 2\beta x_{min} + \pi}$$

$$Z_L = Z_0 \frac{1+\rho}{1-\rho} = Z_0 \frac{1 + |e|\, e^{j\psi}}{1 - |e| e^{j\psi}} \qquad 9.39$$

From Eq (9.37)

$$Z_L = Z_0 \times \left[\frac{1 + \left[\frac{(S-1)}{(S+1)}\right] e^{j\psi}}{1 - \left[\frac{(S-1)}{(S+1)}\right] e^{j\psi}}\right]$$

From Eq. (9.36)

$$Z_L = Z_0 \times \left[\frac{1 + \left[\frac{(S-1)}{(S+1)}\right] e^{j(2\beta x_{min} + \pi)}}{1 - \left[\frac{(S-1)}{(S+1)}\right] e^{j(2\beta x_{min} + \pi)}}\right]$$

$$e^{j\pi} = \cos\pi + j\sin\pi = -1 \Rightarrow Z_L = Z_0 \times \left[\frac{1 + \left[\frac{(S-1)}{(S+1)}\right] e^{j2\beta x_{min}}}{1 - \left[\frac{(S-1)}{(S+1)}\right] e^{j2\beta x_{min}}}\right]$$

From Eq. (9.39)

$$Z_L = \frac{(S+1) + (S-1)\left(-e^{j2\beta x_{min}}\right)}{(S+1) - (S-1)\left(-e^{j2\beta x_{min}}\right)} \times Z_0$$

$$= Z_0 \left[\frac{S\left(1 - e^{j2\beta x_{min}}\right) + \left(1 + e^{j2\beta x_{min}}\right)}{S\left(1 + e^{j2\beta x_{min}}\right) + \left(1 - e^{j2\beta x_{min}}\right)}\right]$$

Dividing both the numerator and denominator by $e^{j\beta x_{min}}$

$$Z_L = Z_0 \left[ \frac{S\left(e^{j\beta x_{min}} - e^{j\beta x_{min}}\right) + \left(e^{-j\beta x_{min}} + e^{j\beta x_{min}}\right)}{S(e^{-j\beta x_{min}} + e^{j\beta x_{min}}) + (e^{-j\beta x_{min}} - e^{j\beta x_{min}})} \right]$$

$$= Z_0 \frac{S(-2j \sin \beta x_{min}) + 2 \cos \beta x_{min}}{S(2 \cos \beta x_{min}) - j2 \sin \beta x_{min}}$$

Dividing through by 2 cos $\beta x_{min}$,

$$Z_L = Z_0 \left[ \frac{-Sj \tan \beta x_{min} + 1}{S - j \tan \beta x_{min}} \right]$$

$$\boxed{Z_L = Z_0 \left[ \frac{1 - jS \tan \beta x_{min}}{S - j \tan \beta x_{min}} \right]} \qquad 9.40$$

Normalized load impedance, $\frac{Z_L}{Z_0}$,

$$z_L = \frac{Z_L}{Z_0} = \boxed{\frac{1 - S \tanh j\beta x_{min}}{S - \tanh j\beta x_{min}}} \qquad 9.41$$

**Example 9.13:** A $100\ \Omega$ line feeding the antenna has $\text{VSWR} = 2$ and the distance from load to the first minima is 10 cm. Design a single stub matching to make $VSWR = 1$. Given $f = 150\ \text{MHz}$

Solution: $\text{VSWR} = 2$

$$|\rho| = \frac{\text{VSWR} - 1}{\text{VSWR} + 1} = \frac{1}{3} = 0.33$$

$$\text{F} = 150\ \text{MHz}$$

$$\lambda = \frac{C}{f} = 2\ \text{m}$$

We know that

$$\psi - 2\beta\ d_{min} = \pi$$

$$2\beta\ d_{min} = \psi - \pi = 2 \times \frac{2\pi}{2} \times 0.1 = 0.2\pi$$

The position of stub

$$l_\psi = \frac{\lambda}{4\pi}\ (\cos^{-1}(\rho) - 2\beta\ d_{min})$$

$$|l_\psi| = \frac{\lambda}{4\pi}(0.39\pi - 0.2\pi) = \frac{0.1}{4\pi} \times (0.19\pi) = 4.75 \text{ mm}$$

$$\text{Length of stub} = l_t = \frac{\lambda}{2\pi}\tan^{-1}\left(\frac{\sqrt{1-|\rho|^2}}{2|\rho|}\right) = \frac{\lambda}{2\pi}\tan^{-1}\left(\frac{\sqrt{1-|0.33|^2}}{2(0.33)}\right) = 15 \text{ mm}$$

**Example 9.14:** A UHF transmission line operating at 1 GHz is connected to $Z_L$ producing reflection coefficient $0.5\angle 30^\circ$. Design single stub matching. Find VSWR.

Solution:

$$\text{f} = 1 \text{ GHz}$$

$$\lambda = \frac{3 \times 10^8}{1 \times 10^9} = 0.3 \text{ m}$$

$$|\rho| = 0.5$$

$$\text{VSWR} = \frac{1+|\rho|}{1-|\rho|} = \frac{1.5}{0.5} = 3$$

$$\boxed{\psi = 30^\circ = \frac{\pi}{6} \text{ rad}}$$

$$l_s = \frac{\lambda}{4\pi}(\psi + \pi - \cos^{-1}(|\rho|)) = \frac{\lambda}{4\pi}\left(\frac{\pi}{6} + \pi - \cos^{-1}(0.5)\right)$$

$$= \frac{\lambda}{4\pi}\left(\frac{7\pi}{6} - \frac{\pi}{3}\right) = \frac{\lambda}{4\pi} \times \frac{5\pi}{6} = \frac{5\lambda}{24} = \frac{1.5}{24} = 6.25 \text{ cm}$$

$$\boxed{\text{Length of stub} = \text{l}_\text{t} = \frac{\lambda}{2\pi}\tan^{-1}\left(\frac{\sqrt{1-|\rho|^2}}{2|\rho|}\right)}$$

$$= \frac{\lambda}{2\pi}\tan^{-1}\left(\frac{\sqrt{1-(0.5)^2}}{2 \times 0.5}\right)$$

$$= \frac{\lambda}{2\pi} \times 0.227\pi = 3.4 \text{ cm}$$

## 9.14 Further Examples

**1.** A transmission line with the characteristic impedance of $250\ \Omega$ is terminated in a load of $100\ \Omega$. If the load is dissipating a continuous sinusoidal power of 50 watts, calculate:

(i) the reflection coefficient
(ii) Voltage standing wave ratio
(iii) reflected voltage $|V_r|$

Solution:

(i) $|\rho| = \left|\frac{100-250}{100+250}\right| = 0.43$

(ii) $VSWR\ (S) = \frac{(|\rho|+1)}{(1-|\rho|)} = \frac{1.43}{0.57} = 2.50$

(iii) $50 = \frac{(V_{max})(V_{min})}{Z_0}$

$$50 = \frac{(V_i + V_r)(V_i - V_r)}{250}$$

$$V_i^{\,2} - V_r^{\,2} = 12500$$

$$V_r = \sqrt{V_i^{\,2} - 12{,}500} = 0.43V_i$$

$$V_i^{\,2} = (0.43V_i)^2 + 12500$$

$$V_i = \pm\sqrt{\frac{12500}{(1-0.43^2)}}$$

$$= 123.84\ \mathrm{V}$$

**2.** A lossless transmission line with $Z_0 = 60\ \Omega$ is 40 m long and operates at 3 MHz, the line is terminated with a load of $Z_L = 120 + j60\ \Omega$. Given that $u = 0.8c$ on the line, determine analytically. $c = 3 \times 10^8\ \mathrm{m/s}$:

(i) Load admittance
(ii) Voltage reflection coefficient (magnitude & phase)
(iii) VSWR
(iv) $Z_{in}$
(v) $Z_{max}$

(vi) $Z_{min}$

Solution:

(i) $$Y_L = \frac{1}{Z_L} = \frac{1}{(120 + j60)} = \frac{(120 - j60)}{(14400 + 3600)} = 0.0067 - j0.00333\ \Omega$$

(ii) $$\rho = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(120 + j60 - 80)}{(120 + j60 + 80)} = \frac{(40 + j60)}{(200 + j60)} = \frac{(2 + j3)}{(10 + j3)}$$

$$= \frac{(2 + j3)(10 - j3)}{(100 + 9)} = \frac{(20 + 9 + j30 - j6)}{109} = \sqrt{29^2 + 24^2}\angle \tan^{-1}\left(\frac{24}{29}\right)$$

$$\rho = 0.34\angle 39.64^o$$

(iii) $$VSWR\ (S) = \frac{(1 + 0.34)}{(1 - 0.34)} = \frac{1.34}{0.66} = 2.03$$

(iv) $$\lambda = \frac{u}{f} = \frac{(0.8)(3 \times 10^8)}{3} \times 10^{-6} = 80\ \text{m} \implies \beta l = \frac{2\pi}{\lambda}\left(\frac{40\lambda}{80}\right) = \pi$$

$$Z_{in} = Z_0\left[\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}\right] = 80\left[\frac{120 + j60 + j80 \tan \pi}{80 + j(120 + j60)\tan \pi}\right] = 80\left[\frac{120 + j60}{80}\right]$$

$$Z_{in} = 120 + j60\ \Omega$$

(v) $$Z_{max} = Z_0\, S = 80(2.03) = 162.4\ \Omega$$

(vi) $$Z_{min} = \frac{Z_0}{S} = \frac{80}{2.03} = 39.41\ \Omega$$

**3.** A distortionless line ($RC = GL$) has $Z_0 = 80\ \Omega$, $\alpha = 25\ \text{mNP/m}$, $u = 0.5$, where c is the speed of the light in a vacuum. Determine

(i) R
(ii) L
(iii) G
(iv) C
(v) $\lambda$ at 100 MHz, $(c = 3 \times 10^8\ \text{m/s})$

Solution:

$$RC = GL \implies G = \frac{RC}{L}$$

$$\Rightarrow \gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{RG}\sqrt{\left(1 + \frac{j\omega L}{R}\right)\left(+\frac{j\omega C}{G}\right)}$$

$$\gamma = \sqrt{RG\left(1 + \frac{j\omega L}{R}\right)\left(1 + \frac{j\omega L}{R}\right)} = \sqrt{RG}\left(1 + \frac{j\omega L}{R}\right) = \alpha + j\beta$$

$$\alpha = \sqrt{RG}$$

$$\beta = \sqrt{RG}\ \left(\frac{\omega L}{R}\right) = \sqrt{RG}\left(\frac{\omega C}{G}\right) = \omega C\sqrt{\frac{R}{G}} = \omega\left(\sqrt{\frac{L}{C}}\right) = \omega\sqrt{LC}$$

$$Z_0 = \sqrt{\frac{R}{G}}$$

$$\alpha Z_0 = \left(\sqrt{RG}\right)\left(\sqrt{\frac{R}{G}}\right) = R$$

$$R = (25 \times 10^{-3})\ (80) = 2\ \Omega$$

$$u = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}} \Rightarrow \frac{Z_0}{u} = \frac{\left(\sqrt{\frac{R}{G}}\right)}{u} = \left(\sqrt{\frac{L}{C}}\right)\sqrt{LC} = L$$

$$L = \frac{80}{(0.5)(3 \times 10^8)}$$

$$G = \frac{L^2}{R} = \frac{(25 \times 10^{-3})^2}{2} = 625 \times \frac{10^{-6}}{2} = 312.54\ \text{V/m}$$

$$C = \frac{GL}{R} = \frac{(312.5 \times 10^{-6})(533.33 \times 10^{-9})}{2} = 83.33\ \text{pF}$$

$$\lambda = \frac{c}{f} = \frac{0.5 \times 3 \times 10^8}{100} \times 10^{-6} = 1.5\ \text{m}$$

**9.15 Exercise**

**1.** (i) In not more than 15 words, define (explain) what is meant by transmission line.
(ii) Sketch and completely label 2 types of Transmission line
(iii) Name and explain the parameters involved in a typical transmission line.

**2.** (i) Define reflection coefficient.
(ii) Under what load conditions will there be total reflection from the load.
(iii) For lines of zero loss for a quarter wave transformer, determine the expression for the characteristic impedance in forms of the input and load impedance.
(iv) In what way does the quarter wavelength section of a transmission line act as impedance transformer?

**3.** (i) What is a stub, and how is it applied to the transmission line?
(ii) Derive the expression for reflection coefficient in terms of load and characteristic impedances.

**4.** What does (i) $\mathrm{VSWR} = 1$ (ii) $\mathrm{VSWR} = \infty$, signify with reference to matching of the transmission line to the load?

**5.** A transmission line with the characteristic impedance of $250\ \Omega$ is terminated in a load of $100\ \Omega$. If the load is dissipating a continuous sinusoidal power of 50 watts, calculate:
(i) The reflection coefficient
(ii) Voltage standing wave ratio
(iii) Reflected voltage $|V_r|$

**6.** Two voltage waves having equal frequencies and amplitudes propagate in opposite directions in a lossless transmission line.
(i) Determine the total voltage as a function of distance and time.
(ii) What kind of wave results (relating its behaviour with respect to position and time)?
(iii) Where do the zeros in the amplitude (i.e., null position) occur?

**7.** A lossless transmission line of 100 cm and operates at a frequency of 300 MHz, the line parameters are $L = 0.5\ \mu\mathrm{H/m}$ and $C = 200\ \mathrm{pF/m}$. determine:
(a) The characteristic impedance
(b) The phase constant
(c) The phase velocity.

**8.** (i) Define the characteristic impedance of a typical transmission line
(ii) In what other way can it be viewed

**9.** An airline has a characteristic impedance of $60\ \Omega$ and a phase constant of 2 rad/m at 80 MHz, calculate the inductance/meter and the capacitance/meter of the line. $(R = 0 = G, \alpha = 0)$

**10.** What is meant by a distortionless line?

**11.** A distortionless line $(RC = GL)$ has $Z_0 = 160\ \Omega$, $\alpha = 50\ m\text{Np/m}, u = 0.8$, where c is the speed of the light in a vacuum. Determine R, L, G, C and $\lambda$ at 100 MHz, $(c = 3 \times 10^8\ \text{m/s})$

**12.** (i) Show that at high frequencies:

$$(R \ll \omega L, G \ll \omega L),\ \gamma = \left(\frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}}\right) + j\omega\sqrt{LC}$$

(ii) Obtain a similar formula for $Z_0$

**13.** (i) Define reflection coefficient

(ii) Under what load conditions will there be total reflection from the load

**14.** Derive the expression for reflection coefficient in terms of load and characteristic impedances.

**15.** (i) Define the characteristic impedance of a typical transmission line

(ii) In what other way can it be viewed

**16.** An airline has a characteristic impedance of 80 Ω and a phase constant of 3.5 rad/m at 100 MHz, calculate the inductance/meter and the capacitance/meter of the line. $(R = 0 = G, \alpha = 0)$

**17.** What is meant by a distortionless line?

**18.** A distortionless line $(RC = GL)$ has $Z_0 = 80\ \Omega, \alpha = 25\ \text{mNp/m}, u = 0.5c$, where c is the speed of the light in a vacuum. Determine R, L, G, C and $\lambda$ at 100 MHz, $(c = 3 \times 10^8\ \text{m/s})$.

**19.** An airline has a characteristic impedance of $200\ \Omega$ and a phase constant of 4 rad/m at 180 MHz Calculate the inductance/meter and the capacitance/meter of the line. $(R = 0 = G, \alpha = 0)$

**20.** A distortionless line $(RC = GL)$ has $Z_0 = 120\Omega$, $\alpha = 50\ \text{mNp/m}, u = 0.75c$, where c is the speed of the light in a vacuum. Determine at 160 MHz, $(c = 3 \times 10^8\ \text{m/s})$.

(i) R

(ii) L

(iii) G

(iv) C and $\lambda$

**21.** A lossless transmission line is 100 cm and operates at a frequency of 400 MHz the line parameters are $L = 0.75\ \mu\text{H/m}$ and $C = 300\ \text{pF/m}$. determine.

(a) The characteristic impedance.

(b) The phase constant.

(c) The phase velocity.

**22.** A load of $25 + j50\ \Omega$ terminates a $50\ \Omega$ line, given that the line is 60cm long and the signal wavelength 2m, $c = 3 \times 10^8\ \text{m/s}$. Determine analytically:

(i) The load admittance.

(ii) The reflection coefficient (amplitude and phase).

(iii) Voltage Standing Wave Ratio.

(iv) Input impedance.

**23.** A lossless transmission line of characteristic impedance $150\ \Omega$ is terminated in a load of $350 + j200\ \Omega$, given that the length of the line is 80 cm and the signal wavelength is 50 cm, $c = 3 \times 10^8\ \text{m/s}$ determine analytically the:

(i) Load admittance.

(ii) Reflection coefficient VSWR.

(iii) Distance between the load and the nearest voltage minimum to it normalized input impedance.

**24.** A lossless transmission line with $Z_0 = 60\ \Omega$ is 80 m long and operates at 6 MHz, the line is terminated with a load of $Z_L = 120 + j60\ \Omega$. Given that $u = 0.5c$ on the line, determine analytically. $c = 3 \times 10^8\ \text{m/s}$:

(vii) Load admittance.

(viii) Voltage reflection coefficient (magnitude & phase).

(ix) VSWR.

(x) $Z_{in}$

**25.** In a lossless transmission line, the velocity of propagation is $3.5 \times 10^8\ \text{m/s}$. capacitance of the line is 40 pF/m. determine:

(i) Inductance per meter of the line.

(ii) Phase constant at 100 MHz

(iii) The characteristic impedance.

**26.** A lossless transmission line is 100 cm and operates at a frequency of 300 MHz the line parameters are $L = 0.5\ \mu\text{H/m}$ and $C = 200\ \text{pF/m}$. determine:

(a) The characteristic impedance.

(b) The phase constant.

(c) The phase velocity.

**Name………………………………………………………………………….**

**REG NUMBER…….……………………………………………………….**

## Semi-Logarithmic Graph Paper

Horizontal axis: Logarithmic, 4 cycles
Vertical axis: Linear, 5 mm

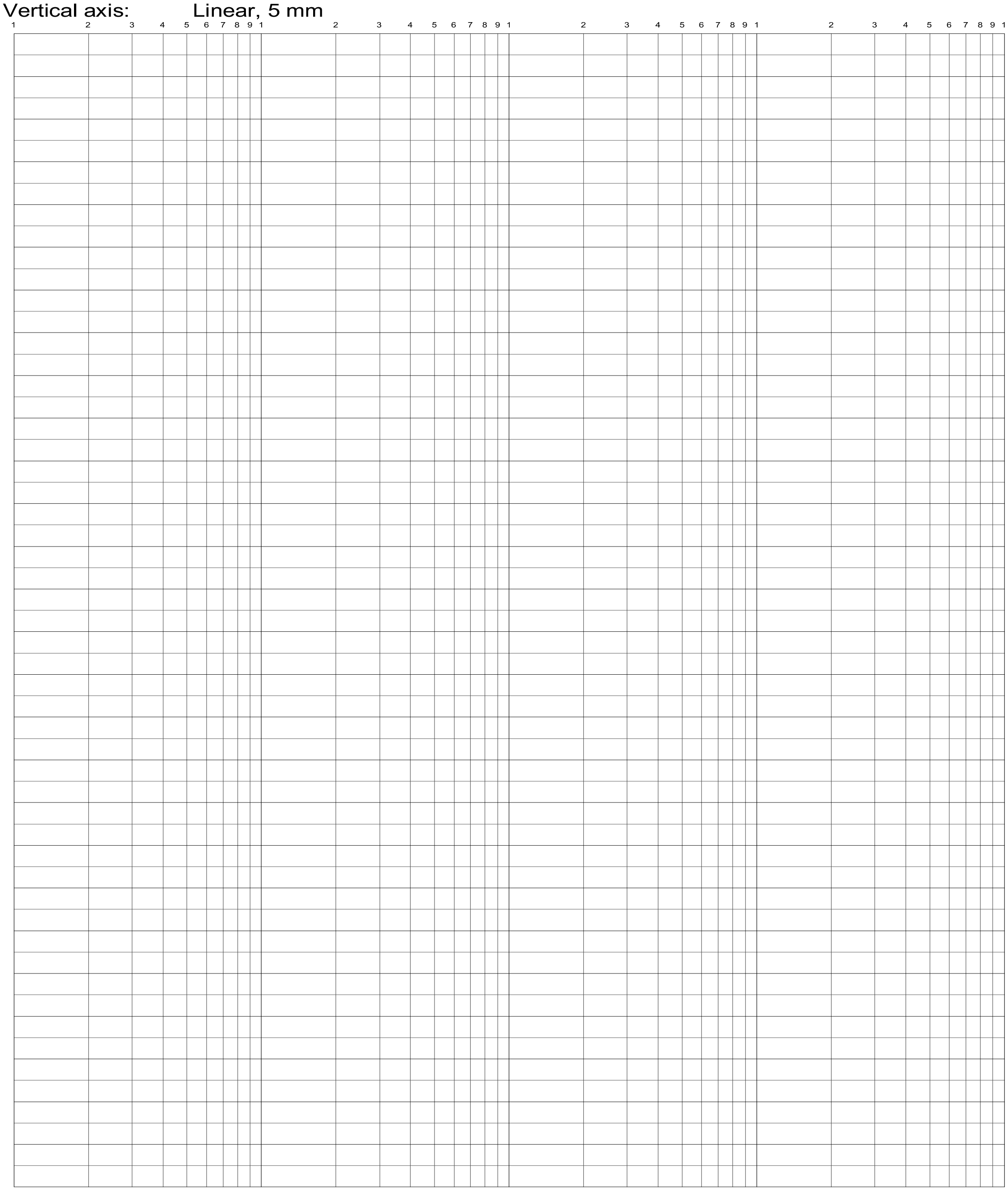



Figure Q

# BIBLIOGRAPHY

1. Charles K. Alexander and Mathew N.O Sadiku, (2002) Fundamentals of Electric Circuits (2ndedition). McGraw-Hill, New York.
2. Dass, H. K, (2008). Advanced Engineering Mathematics (9th edition). S. Chand& Company, Ram Nagar, New Delhi.
3. Parul Dawar, (2009). Electromagnetic Field Theory (1st edition). Kataria S.K & sons, New Delhi
4. Syed A. Nasar, (1988). 3000 Solve Problems in Electric Circuits. McGraw-Hill, New York.
5. William, H.H, Jack, E.K, Steven, M. D, (2007). Engineering Circuit Analysis (7th edition). McGraw-Hill, New York.

# INDEX