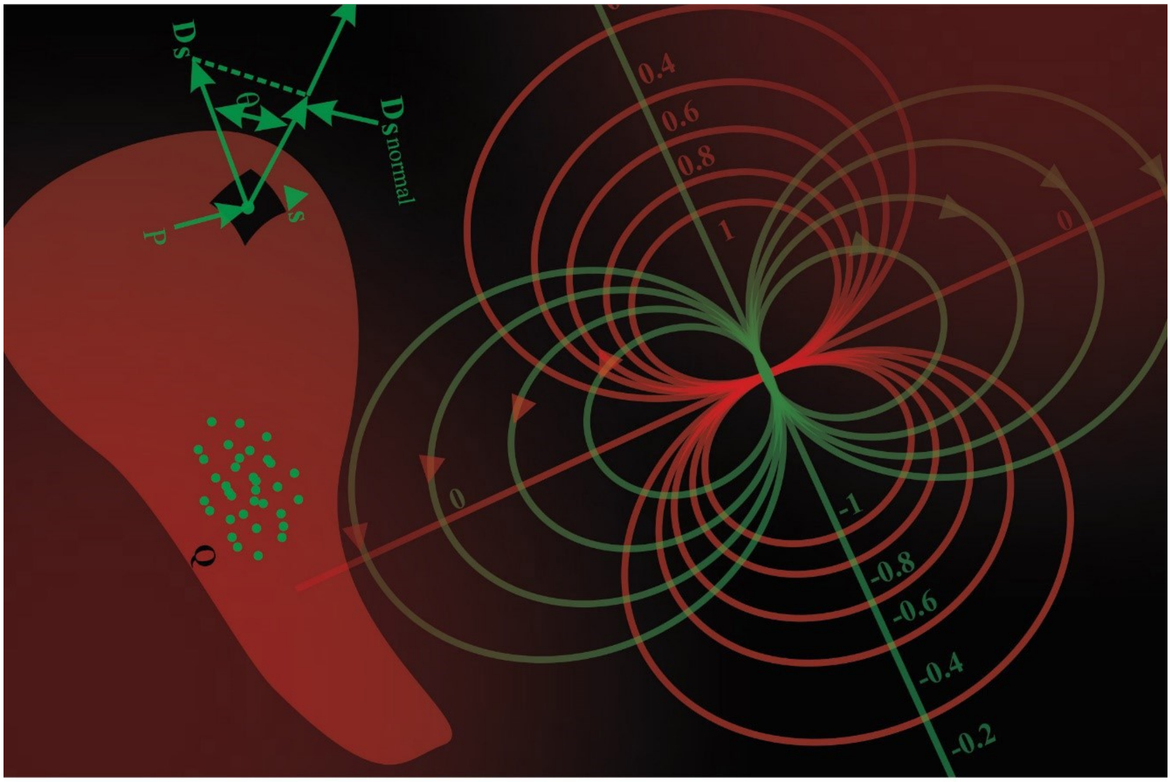

ELECTROMAGNETIC FIELD THEORY

with Application for Undergraduates

KENNETH UGO UDEZE

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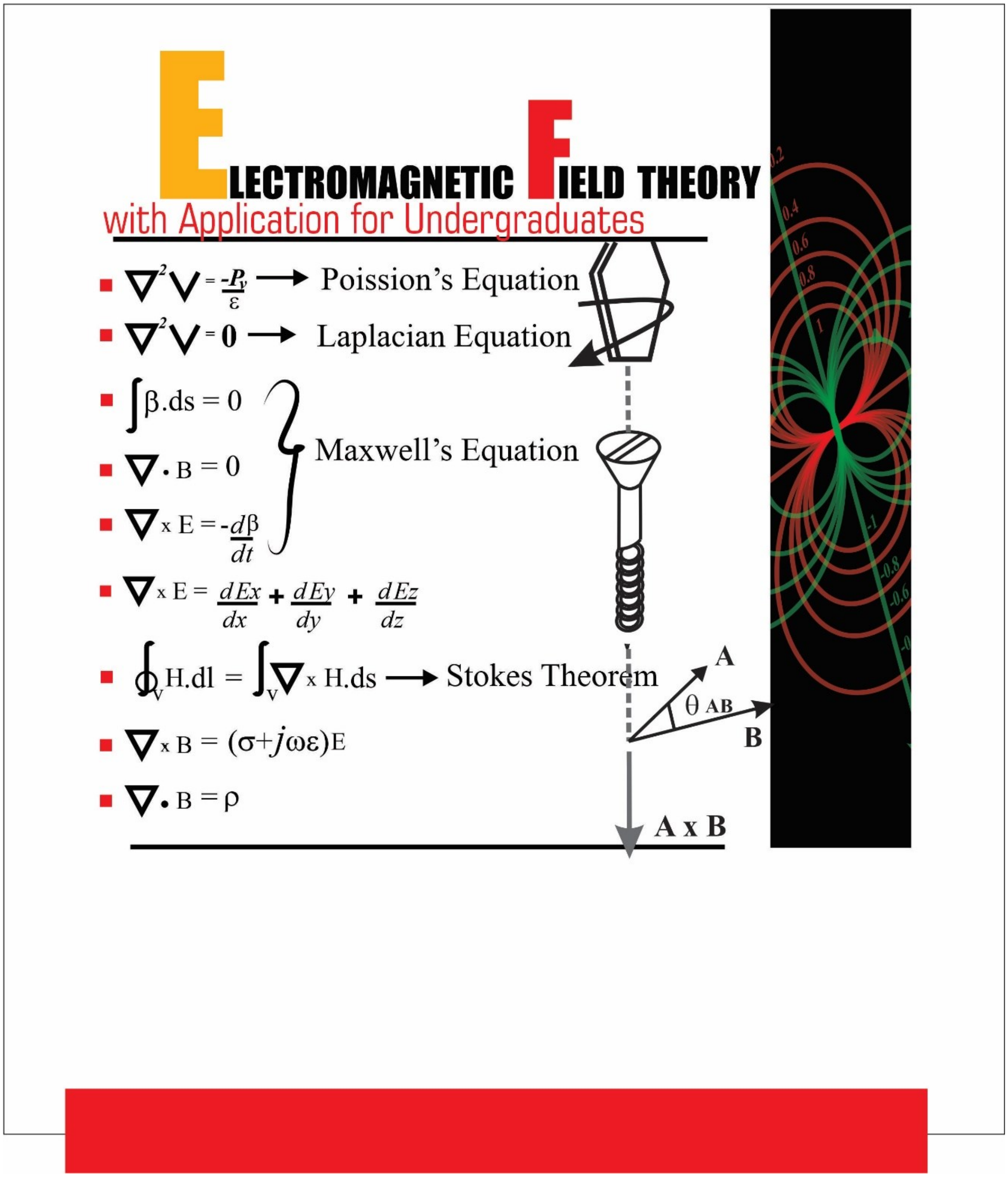
ELECTROMAGNETIC FIELD THEORY
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$\nabla^2 V = \frac{-P_v}{\varepsilon}$ → Poission's Equation
$\nabla^2 V = 0$ → Laplacian Equation
$\int \beta . ds = 0$
$\nabla \cdot B = 0$
Maxwell's Equation
$\nabla \times E = -\frac{d\beta}{dt}$
$\nabla \times E = \frac{dEx}{dx} + \frac{dEy}{dy} + \frac{dEz}{dz}$
$\oint_V H.dl = \int_V \nabla \times H.ds$ → Stokes Theorem
$\nabla \times B = (\sigma + j\omega\varepsilon)E$
$\nabla \cdot B = \rho$
A
θ AB
B
A x B

ELECTROMAGNETIC FIELD THEORY

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By

KENNETH UGO UDEZE

Electromagnetic Field Theory (with Application for Undergraduates)

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PREFACE

This book originates from notes used in teaching Electromagnetic Field Theory course at the final year HND level in Electrical/Electronic Engineering Department, Federal Polytechnic, Oko, Anambra State, Nigeria. Along with other materials gathered by the author during his degree and post-degree years of academic pursuit, and over fifteen years of teaching experience in accordance with course curriculum guidelines from the National Board for Technical Education (NBTE), this text, "ELECTROMAGNETIC FIELD THEORY with Application for Undergraduate Students", was written.

The content of each chapter was designed to accommodate Higher National Diploma (HND) and Bachelor of Science/Engineering (B.Sc./B.Eng.) undergraduate students as the materials presented were made comprehensive enough to cover both classes of programs at their final year and mid-course levels respectively.

Chapter 1 covers the introduction to electric and magnetic fields, vectors and scalar quantities, gradients and curl of vectors. Chapter 2 covers introduction to divergent and Stokes's theorems, Maxwell's equations and Gauss's law with their applications and chapter 3 talks about Electrostatics.

Chapter 4 and 5 cover capacitance of capacitor and electromagnetic induction. electromagnetic equations, Ampere's circuital laws, Faraday's law, Gauss's law and more and in chapter 6, while electromagnetic waves analysis and wave propagation in a lossless medium are in chapters 7 and 8 respectively.

Also, at the end of the chapters are enough review problems designed to help student exercise their level of comprehension of the treated matters, and by so doing, internalizes the underlying principles of lessons taught.

ABOUT THE AUTHOR

Udeze Kenneth Ugo hails from Onicha Ugbo in Aniocha North Local Government Area in Delta State, Nigeria. He attended his primary school at Aniemeke Primary School Onicha Ugbo, Delta State, Nigeria and attended his secondary education at Model Secondary School Maitama, Abuja, FCT, Nigeria where he obtained his Senior School Certificate in 2003.

Between 2005 and 2010, he obtained his National Diploma (ND) and Higher National Diploma (HND) in Electrical and Electronics Engineering (Telecommunication and Electronics Options) with a CGPA of 3.68/4 i.e., Distinction Honors from Federal Polytechnic Oko, Anambra State, Nigeria. He also obtained his first degree in Electrical and Electronics Engineering (Power, Telecommunication and Electronics Options), in 2013 from the prestigious University of Ibadan, Oyo State, Nigeria with a CGPA of 5.8/7 i.e., a Second-Class Upper Division (2.1).

After his one year mandatory National Youth Service Corp (NYSC) in Electrical and Electronics Engineering Department in Federal Polytechnic Oko, Anambra State, Nigeria in 2015, he proceeded to obtain his Masters degree in Offshore Engineering in 2016, majored in Offshore Design and installations, Subsea umbilical cables designed, Installation and maintenance of offshore facilities, Submarine power cable design and maintenance, Subsea instrumentation and control system (E&I) from Offshore Technology Institute, School of Advance Engineering, University of Port-Harcourt, Rivers State, Nigeria. Then a second Masters degree in Electrical and Electronics Engineering and majored in Power System Engineering, from University of Lagos, Lagos State, Nigeria in 2023. He graduated with a CGPA of 4.7/5 i.e., Distinction Honors.

He is currently a staff of Federal Polytechnic Oko, Anambra State, Nigeria attached to Electrical and Electronics Engineering Department. He teaches Mathematics and Electrical Engineering courses.

He is presently prospecting for PhD admission overseas for researches in Renewable Energy.

CHAPTER 1

ELECTRIC/MAGNETIC FIELD BASIC THEORY

1.0 Circuit and Field Vectors

The circuit theory studied in electrical electronics engineering is used to predict with accuracy the "performance" of electrical network with regards to the followings:

- The voltages and currents
- The simplicity (as can be applied to any electrical network)
- The usefulness (helps in evaluating performance parameters of any network).

Meanwhile, in microwave or R.F transmission, we generally deal with transmitting power and voltage or current, as it is very difficult to open or short circuit at high frequency. Power is actually expressed in terms of integrated effects of voltages ($\vec{E}, electric\ field$) and current ($\vec{H},\ magnetic\ field$)

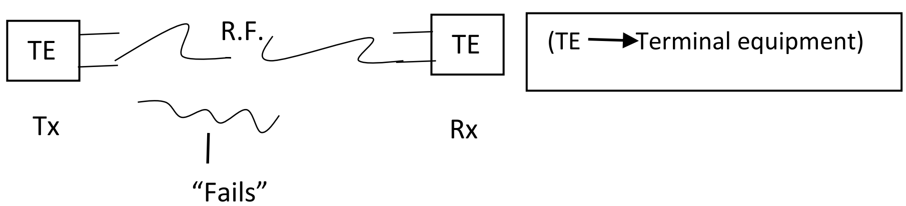


Figure. 1.1 Radio Frequency link

The Fig. 1.1 shows that the ratio V/I cannot be evaluated on any high frequency link, instead $\vec{E}/\ \vec{H}$ are evaluated for calculation of power transmitted.
Also, Electromagnetic field theory. Electromagnetic field theory deals with field vectors $\vec{E}\ and\ \vec{H}$. Voltage and currents are integrated effects of electric and magnetic fields.
Again, Electromagnetics: Electromagnetics is study of effects of electrical charges at rest and in motion. Moving charges produce a current, which gives rise to magnetic field.
The varying electric and magnetic fields are coupled, producing electromagnetic field.
More difficult. Because of large number of variables involved, calculations are difficult in electromagnetism.

It should be noted that when current is constant around a circuit, voltages and currents are functions of one variables "time"
In *uniform transmission-line theory*, the "distance" along the line is an added variable. In this, we define R, L, C, G in terms of length.
Finally, the Four Fundamental Vector Field Quantities in Electromagnetics are:

$\vec{E}$= Electric field intensity

$\vec{D}$ = Electric flux density

$\vec{B}$ = Magnetic flux density

$\vec{E}$ = Magnetic field density

Used in study of electric and magnetic fields in materials media, as $\vec{D} = \varepsilon\vec{E} and\ \vec{B} = \mu\vec{H}\ \ where\ \epsilon\ and\ \mu$, are permittivity and permeability of medium respectively.

1.1 Scalars and Vectors Review

Elementary physics taught is that scalars have only magnitude (size) whereas vector have both magnitude and direction. That means that the former cannot have a negative value (at least when not technically speaking), examples of which include mass, distance, speed commonly encountered in mechanics, and energy etc. that is met commonly in thermodynamics. However, mass with direction becomes force (or weight when caused by the acceleration due to earth's gravity), distance with direction be one's displacement while directed speed is velocity. Energy with direction is used to perform work (both have the unit of joules); these results are all examples of vector quantities; and these unlike scalars, can indeed take up negative values. Scalars can be seen as one-dimensional quantities along x axis that start from zero and only move "eastward" (to the right on x-y coordinate system). Vectors, on the other hand, can exist in one -, two- and three dimensions and can take up negative values. The above is without prejudice to the fact that, from a pure mathematical point of view, scalars can indeed also be presented by negative quantities as long as they are single real numbers. Here, also, vectors are restricted to two- and three-dimensional spaces, thereby leaning out the trivial one dimension.

This course deals with SCALAR and VECTOR FIELDS, and by field is meant summarily a function that connects a point (usually taken at the origin) to a general point in space.

1.1.0 Vector Algebra

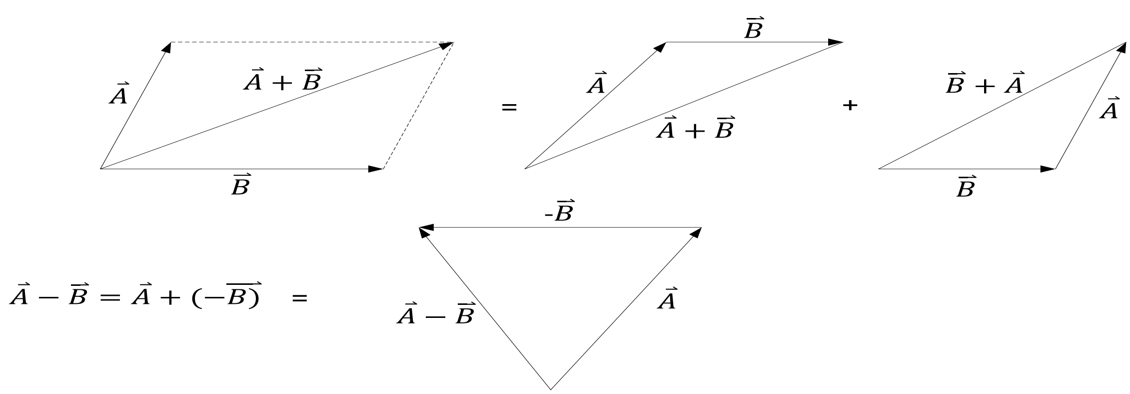
By definition, a typical vectors location is taken to be the origin (tail) of a representative arrow. In textbooks the bold face type is used to indicate a vector quantity, and the italic type is for a scalar. In these notes, however, vectors will be indicated by letters with an arrow at the top, while scalars will be letters without an arrow, or equally, arrowed letters straddled by two upright lines: A = $|\vec{A}|$.

Addition or subtraction of vectors in two (or three) dimensions follow the parallelogram law (rule), and is associative and well as commutative and distributive:

$$\vec{A} + \vec{B} + \vec{C} = \vec{A} + \vec{B} + \vec{C} = \vec{A} + \vec{B} + \vec{C}, \quad 1.1$$

$$(a + b)\,(\vec{A} + \vec{B} + \vec{C}) = a(\vec{A} + \vec{B} + \vec{C}) + b(\vec{A} + \vec{B} + \vec{C}) \quad 1.1.1$$

$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$, that is to say, the positive $\vec{A}$ is added to the negative of $\vec{B}$. Two vectors are identical off (if and only if) they have the same magnitude and direction. By this is meant that the actual (tail) locations of the vector to be compared are, mathematically speaking, inconsequential.

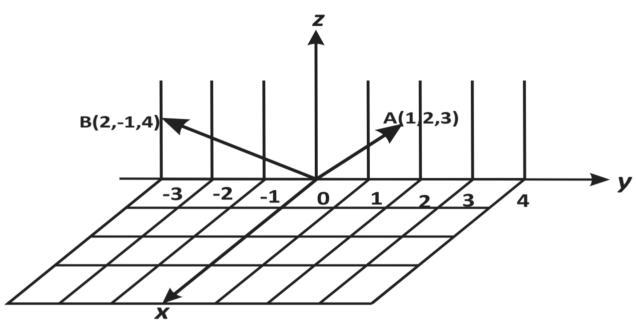


For vectors, there are three different types of coordinate systems, namely, (1) the rectangular (2) the circular cylindrical and (3) spherical coordinate systems and each one can be converted to the others equivalent.

1.2 The Rectangular Coordinates

Here the right-hand-rule is employed or the right-handed screw with positive y axis pointing to the right ("east") of the board, the positive x axis pointing to the "North" and positive x axis pointing out of the board. Another view is of the right hand with the thumb, the forefinger and the middle finger representing, respectively, x, y and z axis. So, negative x will point to the "inside" of the board, negative y "westward", and negative z "southward".

Example 1.0:



For the above xyz plane, A is the point at which the planes $x = 1, y = 2 \; and \; z = 3$ intersect (at each location the plane is parallel to the other two coordinates), and B is where the planes $x = 2, y = -1 \; and \; z = 4$ intersect.

$$\vec{r}_A = x + 2y + 3z$$

$$\vec{r}_B = 2x - y + 4z$$

$$\vec{R}_{AB} = \vec{r}_B - \vec{r}_A = (2-1)\,\hat{a}_x + (-1-2)\,\hat{a}_y + (4-3)\hat{a}_z$$

$$= \hat{a}_x - 3\hat{a}_y + \hat{a}_z$$

with $\hat{a}_x, \; \hat{a}_y and \; \hat{a}_y$ being unit vectors in the directions of x, y and z axis respectively. The scalar values of $\vec{r}_A, \vec{r}_{\boldsymbol{B}}$ (also known as their modulus) Are:

$$|\vec{r}_A| = r_A = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$|\vec{r}_B| = r_B = \sqrt{2^2 + 1^2 + 4^2} = \sqrt{21}$$

Also, $|\vec{R}_{AB}| = |\hat{a}_x - 3\hat{a}_y + \hat{a}_z| = \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11}$

$$\vec{r}_A = \frac{A}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = \frac{\hat{a}_x + 2\hat{a}_y + 3\hat{a}_z}{\sqrt{1^2 + 2^2 + 3^2}}$$

$$= \frac{\hat{a}_x}{\sqrt{14}} + \frac{2\hat{a}_y}{\sqrt{14}} + \frac{3\hat{a}_z}{\sqrt{14}}$$

is the unit vector in the direction of $\vec{A}$ [actually $\vec{r}_A$ which is here not used, to avoid double subscript ($\vec{r}_{a_A}$)].

Check that the modulus of $\vec{a}_A$ $(or\ \ \vec{a}_B)$ gives unity:

$$\sqrt{\left(\frac{1}{14}\right)^2 + \left(\frac{2}{14}\right)^2 + \left(\frac{3}{14}\right)^2} = \sqrt{\frac{1}{14} + \frac{4}{14} + \frac{9}{14}} = \sqrt{\frac{14}{14}} = 1$$

1.3 Vector Field: The Dot Product

By definition, the dot product is the product of the magnitudes of given two vectors and the cosine of the smaller angle **a** between them. Because this results in a scalar, one can therefore only have a dot product of two vectors, since there's no dot product of a scalar and a vector!

$$\vec{A} \cdot \vec{B} \ (''A\ dot\ B'') = |A||B| \cos\theta_{AB} \qquad 1.2$$

$$= |B||A| \cos\theta_{BA} = \vec{B} \cdot \vec{A} \qquad 1.2.1$$

and so, obeys the commutative law. Dotting a vector by itself simply produces the magnitude of the vector squares.

$$\vec{A}.\vec{A} = |\vec{A}||\vec{A}| \cos\theta = |\vec{A}|^2 = A^2$$

Resolving the dotted result of two vectors along their component axis;

$$\vec{A} \cdot \vec{B} = (A_x\,\hat{a}_x + A_y\hat{a}_y + A_z\,\hat{a}_z) \cdot (B_x\,\hat{a}_x + B_y\,\hat{a}_y + B_z\,\hat{a}_z) \qquad 1.3$$

$$= A_xB_x\,\hat{a}_x \cdot \hat{a}_x + A_xB_y\,\hat{a}_x \cdot \hat{a}_y + A_xB_z\,\hat{a}_x \cdot \hat{a}_z + A_yB_x\hat{a}_y \cdot \hat{a}_x + A_yB_y\,\hat{a}_y \cdot \hat{a}_y + A_yB_z\,\hat{a}_y$$

$$\cdot\,\hat{a}_z + A_zB_x\,\hat{a}_z \cdot \hat{a}_x + A_zB_y\,\hat{a}_z \cdot \hat{a}_y + A_zB_z\,\hat{a}_z \cdot \hat{a}_z$$

Observe that $A_n\,B_m$'s are all scalars and two dotted unlike unit vectors disappear since a_x, a_y, a_z are mutually perpendicular (cos 90^0=0), leaving: [$A \cdot B\ =\ A_xB_x\ + A_yB_y + A_zB_z$,

because $\hat{a}_x \cdot\ \hat{a}_x = 1^2, =\ \hat{a}_y \cdot \hat{a}_y = \hat{a}_z \cdot \hat{a}_z = 1$

dotting a vector $\vec{A}$ with a unit vector in any direction results in mechanically advantageous result of merely determining the component of that vector along the axis on which the unit vector lies:

$$\vec{A}.\hat{a} = |\vec{A}||\hat{a}|\cos\theta_{Aa} = A\cos\theta_{Aa}$$

Example 1.1: Given the vector $\vec{F} = za_x + 3.5a_y - 2xa$ and the point $P\ (3, 2, 4)$ determined (1) $\vec{F}\ at\ P$.

Solution:

(1) To determine $\vec{F}\ at\ P$,

$$\vec{F}(\vec{r}_p) = 4a_x + 3.5a_y - (2)(3)a_z = 4a_x + 3.5a_y - 6a_z$$

(2) The scalar component of $\vec{F}\ at\ P$ in the direction of $a_N = (2\hat{a}_x - \hat{a}_y + 2\hat{a}_z)/3$:

$$(\vec{F}.\vec{a}_N) = (4\hat{a}_x + 3.5\hat{a}_y - 6\hat{a}_z).\frac{2\hat{a}_x - \hat{a}_y + 2\hat{a}_z}{3}$$

$$= \frac{8 - 3.5 - 12}{3} = \frac{7.5}{3} = -2.5$$

(3) the vector component of $\vec{F}$ in the direction of $\vec{a}_N$.

$$(\vec{F}.\hat{a}_N)a_N = -2.5 \times \frac{(2\hat{a}_x - \hat{a}_y + 2\hat{a}_z)}{3}$$

$$= -1.667\hat{a}_x + 0.833\hat{a}_y - 1.667\hat{a}_z$$

(4) The angle θ_{Fa} between $\vec{F}(\vec{r}_p)$ and $\hat{a}_N$:

$$-2.5 = \vec{F}.\hat{a}_N = |\vec{F}|\cos\theta_{Fa} \Longrightarrow \cos\theta_{Fa} = -\frac{2.5}{|\vec{F}|}$$

$$\cos\theta_{Fa} = -\frac{2.5}{\sqrt{6 + 12.25 + 36}} = -\frac{2.5}{8.02}$$

$$\theta_{Fa} = \cos^{-1}(-\frac{2.5}{8.02}) = 71.83°$$

You can try this: Consider the vector field $\vec{C}$ = ya_x – 2.5xa_y +3.5a_z and the point P (4, 5, 2). We wish to find: C at P; the scalar component of $\vec{C}$ at P in the direction of a_N = $\frac{1}{3}$ (2ax + ay – 2az); the vector component of $\vec{C}$ at Q in direction of a_N; and finally, the angle θ_{CR} between C (r_C) and a_N.

Notice that in this proceeding example, the component of the vector $\vec{F}$ can be dependent on the magnitude of the scalar component of a vector located at P, in any direction.

1.4 The Vector Product

By definition, the vector (or cross) product of $\vec{A}$ and $\vec{B}$ is:

$\vec{A} \times \vec{B}$(A cross B) $= \hat{a}_N|\vec{A}||\vec{B}|\sin\theta_{AB},\ \ with\ \ \theta_{AB}$ being the smaller angle between $\vec{A}\ and\ \vec{B}$.

Therefore, $180^o > \theta_{AB} > 0^o$; and $\hat{a}_N$ is the unit vector normal to the plane containing vectors $\vec{A}\ and\ \vec{B}$ with its direction obeying the right-handed screw rule when $\vec{A}$ is turned into $\vec{B}$.

$$\vec{A} \times \vec{B} = (A_x\hat{a}_{xy} + A_y\hat{a}_y + A_z\hat{a}_z) \times (B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z) \qquad 1.4$$

$$= A_xB_x\hat{a}_x \times \hat{a}_x + A_xB_y\hat{a}_x \times \hat{a}_y + A_xB_z\hat{a}_x \times a_z$$

$$+A_yB_x\hat{a}_y \times \hat{a}_x + A_yB_y\hat{a}_y \times \hat{a}_y + A_yB_z\hat{a}_x \times \hat{a}_z$$

$$+A_zB_x\hat{a}_z \times \hat{a}_x + A_zB_y\hat{a}_z \times \hat{a}_y + A_zB_z\hat{a}_z \times \hat{a}_z$$

This time around $\hat{a}_x \times \hat{a}_x = \hat{a}_y \times \hat{a}_y = \hat{a}_z \times \hat{a}_z = 0$, because $\sin 0 = 0$, when a vector is crossed onto itself!

Combined,

$$A \times B = \left(A_yB_z - A_zB_y\right)\hat{a}_x + (A_zB_x - A_xB_z)\hat{a}_y + \left(A_xB_y - A_yB_x\right)\hat{a}_z$$

Though this looks quite complicated, the cyclic nature of the terms lends itself to easy remembrance, once the first grouping is put down. Notice that y into z and x into y implies $\left(A_yB_z \ldots \ldots\right)\hat{a}_x$, and the negative second grouping is put down by merely exchanging the subscripts y, z attached to A, B respectively: $A_yB_z \Longrightarrow A_zB_y$
xyz; yzx; zxy cyclically for the second and third grouped terms.

However, in a more compact easily remembered form we write the determinant:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad 1.4.1$$

Example 1.2: Given $\vec{A} = 3\hat{a}_x - 2\hat{a}_y + a_z$, $\vec{B} = -2\hat{a}_x - \hat{a}_y + 4\hat{a}_z$, then

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 3 & -2 & 1 \\ -2 & -1 & 4 \end{vmatrix}$$

$$= [(-2)(4) - (-1)(1)]\hat{a}_x + [(1)(-2) - (3)(4)]\hat{a}_y + [(3)(-1) - (-2)(-2)]\hat{a}_z$$

$$= (-8 + 1)\hat{a}_x + (-2 - 12)a_y + (-3 - 4)a_z$$

$$= -7\hat{a}_x - 14\hat{a}_y - 7\hat{a}_z$$

1.5 Unit Vector

A unit vector ($\hat{A}$) has the direction of main vector ($\vec{A}$) but is of unit magnitude. It is the ratio of vector itself by its magnitude.

$$\hat{A} = \frac{\vec{A}}{|A|} \qquad 1.5$$

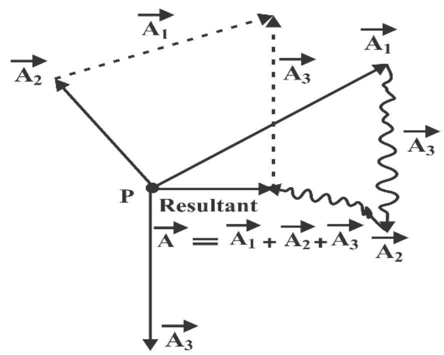


Fig 1.2 Vector Addition

$|A| = \sqrt{A_x^2 + A_y^2 + A_z^2}$, where A_x, A_y, A_z are components of vector $\vec{A}$ along x, y, z directions.

Example 1.3 given point **M** (-1,2,1), **N** (4, -4,0) and **P** (-1, -3, -4), find:

(a) $\mathbf{R}_{MN}$

(b) $\mathbf{R}_{MN} + \mathbf{R}_{MP}$

(c) $|r_M|$

(d) $\hat{a}_{MP}$

(e) $2|r_P| - 3|r_M|$

Solution

(a) $[4 - (-1)]a_x + [-4 - 2]a_y + [0-1]a_z = 5a_x - 6a_y - a_z$
(b) $(5a_x - 6a_y - a_z) + (-a_x - 5a_y - 5a_z)) = 4a_x - 11a_y - 6a_z$

(c) $\sqrt{1+4+1} = \sqrt{6} = 2.45$

(d) $\left(\frac{a_x - 5a_y - 5a_z}{\sqrt{25+25+1}}\right)$ = -0.14a_x – 0.7a_y – 0.7a_z

(e) $[2\sqrt{(4+9+16)}$-3$(\sqrt{1+4+1})] = 3.42$

Example 1.4: A vector field $\vec{r}$ is expressed in rectangular coordinates as:

$\vec{r}$ ={125/[(x-1)^2+(y-2)^2+(z+1)^2]} {x-1) a_x +(y-2) a_y + (z+1) a_z}

(a) Evaluate $\vec{r}$ at A(3 5, 4)

(b) Determine a unit vector that gives the direction of $\vec{r}$ $at\ A(2{,}4{,}3)$

(c) Specify unit vector extending from origin towards point M (2, -2, 1)

Solution

(a) 6.579a_x + 9.869a_y + 16.447a_z (Put x=3, y=5, z=4)

(b) 0.218a_x + 0.436a_y +0.873a_z $\vec{r}$ at A(2,4,3)

(c) (d) $\hat{a}_M = \frac{M}{|M|} = \frac{2a_x - 2a_y - a_z}{\sqrt{(2)^2+(-2)^2+(-1)^2}}$

= **0.667a_x – 0.667a_y – 0.333a_z Ans**

Example 1.5: Add the following vectors

(a) $\vec{A} = 16\hat{x} + 3\hat{y}\ and\ \vec{B} = 3\hat{x} - 7\hat{y}$

(b) $\vec{A} = -8\hat{x} + 12\hat{y}; \vec{B} = -5\hat{y} + 15\hat{x}; \vec{C} = -2\hat{x} + 47\hat{y}$

Solution: (a) $\vec{A} + \vec{B} = (16+4)\ \hat{x} + (3-7)\hat{y}$

$\vec{A} + \vec{B} = 20\ \hat{x} - 4\hat{y}$ **Ans**

(b) $\vec{A} + \vec{B} + \vec{C} = (-8+15-2)\ \hat{x} + (12-5+4)\hat{y}$

$\vec{A} + \vec{B} + \vec{C} = 5\hat{x} + 54\hat{y}$ **Ans**

1.6 Properties of Vector Product

1. Associative Law $(\vec{A}\ x\ \vec{B})x\vec{C} \neq \vec{A} \times (\vec{B} \times \vec{C})$ 1.6

2. Distributive law $\vec{A}\,x\,(\vec{B}+\vec{C}) = \vec{A}\,x\vec{B} + \vec{A}\,x\vec{C}$ 1.6.1

3. Cross product AB sin θ is the area of parallelogram PQRS of which vectors $\vec{A}$ *and* $\vec{B}$ are two adjacent sides, see in Fig. 1.3

The cross-product operation is useful for obtaining the unit vector normal to two given vectors at a point, i.e.

$$1_N = \frac{\vec{A}\,x\,\vec{B}}{ABsin\theta}$$

For unit vectors along $\hat{x}, \hat{y}, \hat{z}$, we have

$i_x\ x\ \ i_x =$	$0;$
$i_y\ x\ \ i_x =$	$-i_z$
$i_z\ x\ \ i_x =$	i_y

$i_x\ x\ \ i_y =$	$i_z;$
$i_y\ x\ \ i_y =$	$0;$
$i_z\ x\ \ i_y =$	$-i_x$

$i_x\ x\ \ i_z =$	$i_y;$
$i_y\ x\ \ i_z =$	i_x
$i_z\ x\ \ i_z =$	0

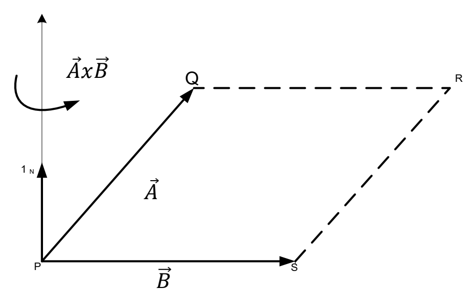


Figure 1.3 (a) Product of two vectors related to Area of Parallelogram

Thus, cross product of two identical unit Vectors is the null vector 0. If we arrange the unit vectors in the manner i_x i_y $i_z i_x$ i_y, then going to right the cross product of any two successive unit vectors is the following unit vector, whereas going to left the cross product of any two, successive unit vectors is negative of the following unit vector. Also, cross product is not commutative since.

$$\vec{B}\times\vec{A} = |B||A|\sin\theta\,(-i_N) = -ABsin\theta i_N = -\vec{A}\times\vec{B}$$

Component of a vector in a vector in a particular direction is given by dot multiplication of the vector and unit vector in that direction as in Fig 1.3(b)

$$\vec{C} = (\vec{C}.1_A)\,1_A + (\vec{C}.1_B)\,1_B \qquad 1.7$$

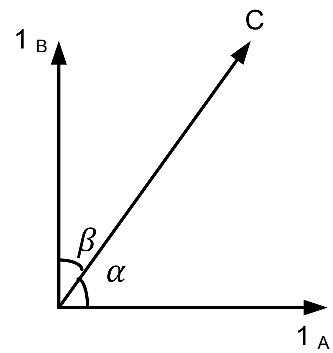


Figure 1.3 (b) Component of vector $\vec{C}$ in 2 directions

1.7 Scalar Triple Product

[$\vec{A}\cdot(\vec{B}\times\vec{C})$] Scalar triple product involves three vectors in a dot product operation and a cross product operation.

It gives the value of volume of parallel-piped having the three vectors as three of its contiguous edges.

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_3 & C_3 \end{vmatrix} \qquad 1.8$$

Example 1.7 The three vertices of a triangle are located at P(6, -1, 2), Q(-2, -3, -4) and N(-3, 1, 5). Find: (a) R_{PQ}; (b) R_{PN}; (c) the angle.

Solution:

(a) $-8a_x - 2a_y - 6a_z$ [Hint: $R_Q - R_P = (-2-6)a_x + (-3+1)a_y + (-4-2)a_z$)]
(b) $-9a_x + 2a_y + 3a_z$ [Hint: $R_N - R_P = (-3-6)a_x + (1+1)a_y + (5-2)a_z$

(c) 59.6^0 [Hint: $\theta = \cos^{-1}\left(\frac{(R_Q - R_P).(R_N - R_P)}{|R_Q - R_P|\,|R_N - R_P|} = \frac{50}{\sqrt{(64+4+36)(81+4+9)}}\right)$

Example 1.8 If $\vec{P} = 2a_x - 3a_y + a_z$ *and* $\vec{Q} = -4a_x - 2a_y + 5a_z$, we have to find $\vec{P}x\vec{Q}$.

Solution: $\vec{P}\times\vec{Q} = \begin{vmatrix} a_x & a_y & a_z \\ 2 & -3 & 1 \\ -4 & -2 & 5 \end{vmatrix}$

$$= [(-3)(5) - (1)(-2)]a_x - [(2)(5) - (1)(-4)]_{ay} + [(21)(-2) - (-3)(-4)]a_z$$

$\vec{P} \times \vec{Q}$= **$-13a_x - 14a_y - 16a_z$ Ans**

Example 1.9: The three vertices of a triangle are located at A(6, -1, 2), B (-2, 3, -4), and C(-3, 1, 5). Find: R_{AB} x R_{AC} (b) the area of triangle (c) a unit vector perpendicular to plane in which triangle is located. Solution

(a) $24\hat{a}_x + 78\hat{a}_y + 20\hat{a}_z$
(b) 42
(c) $0.286\hat{a}_x + 0.928\hat{a}_y + 0.238\hat{a}_z$.

Example 1.10: Refer to **Example 1.5** and find angle of the resultant vector with respect to x-axis.

Solution: let us say $\vec{C} = \vec{A} + \vec{B} = \mathbf{20\hat{x} - 4\hat{y}}$

∴ magnitude of $\vec{C} = |C| = \sqrt{(20)^2 + (4)^2} = 20.396$

Angle of 'C' w.r.t. x-axis

Solution: lets us say $\vec{C} = \vec{A} + \vec{B} = \mathbf{20\hat{x} - 4\hat{y}}$

∴ magnitude of $\vec{C} = |C| = \sqrt{(20)^2 + (4)^2} = 20.396$

Angle of "C" w.r.t. x-axis

$$\propto = cos^{-1}\left(\frac{C_x}{|C|}\right) = cos^{-1}\left(\frac{20}{20.396}\right)$$

$$\propto = 11.31^0 \textbf{ Ans.}$$

Example 1.11: Given $\vec{A} = 6\hat{x} + 2\hat{y} + 10\hat{z}$, $\vec{B} = 3\hat{x} + 4\hat{y} + 7\hat{z}$. Find (a) the area of parallelogram of which $\vec{A} x\ \vec{B}$ are adjacent sides, (b) unit vectors normal to plane containing $\vec{A}\ and\ \vec{B}$

Solution: (a) area of parallelogram of which $\vec{A}\ and\ \vec{B}$ are adjacent sides

$$|\vec{A} \times \vec{B}| = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \mathbf{6} & \mathbf{2} & \mathbf{10} \\ \mathbf{3} & \mathbf{4} & \mathbf{7} \end{vmatrix}$$

$$= \hat{\boldsymbol{x}}(14 - 40) - \hat{y}(42 - 30) + \hat{z}(24 - 6)$$

$$= \mathbf{-26\hat{x} - 12\hat{y} + 18\hat{z}}$$

$$\text{Area of } ||gm = \sqrt{26^2 + 12^2 + 18^2}$$

$$Area = 33.82 \textbf{ Ans}$$

(b) $\quad i_n = \frac{\vec{A} x\ \vec{B}}{|\vec{A} x\ \vec{B}|}$

$$\Rightarrow from\ (a)\ |\ \vec{A}\boldsymbol{x}\ \vec{B}\ | = 33.82$$

$$\vec{A}\boldsymbol{x}\ \vec{B} = -26\hat{x} - 12\hat{y} + 18\hat{z}$$

$$\therefore\ \boldsymbol{i_n} = (0.029)(-26\hat{x} - 12\hat{y} + 18\hat{z})$$

$$= -0.77\hat{x} - 0.35\hat{y} + 0.53\hat{z}$$

$$\boldsymbol{i_n} = -0.77\hat{x} - 0.35\hat{y} + 0.53\hat{z} \textbf{ Ans}$$

Example 1.12: Evaluate:

(a) $A \times (\vec{B} \times \vec{C})$
(b) $(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D})$
(c) $(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D})$

Solution: (a) let us first evaluate $(\vec{B} \times \vec{C})$ assume it to be $\vec{D}$

$$\therefore (\vec{B}\ x\ \vec{C}) \times \vec{D} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$\hat{x}(B_yC_z - B_zC_y) - \hat{y}(B_xC_z - C_xB_z) + \hat{z}(B_xC_y - B_yC_x)$$

$$\Rightarrow \qquad \vec{A}x\ \vec{B}x\ \vec{C} = \vec{A}x\vec{D} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ D_x & D_y & D_z \end{vmatrix}$$

There $D_x = B_yC_z - B_zC_y$

$$D_y = C_xB_z - C_zB_x$$

$$D_z = B_xC_y - B_yC_x$$

$$\therefore \qquad \vec{A}x(\vec{B}x\vec{C}) = \sum \hat{x}\ (A_y\big(B_xC_y - B_zC_y\big) - A_Z(C_xB_z - C_zB_x)$$

$$= \sum \hat{x}\ \big(\ A_y\ B_xC_y - B_zC_y\big) - A_Z(C_xB_z - C_zB_x)$$

$$= \sum \hat{x}\ \big(B_x(A_yC_y + A_zC_z\big) - C_x\big(A_yB_y - A_zB_z\big)$$

$$= (\vec{A} \cdot \vec{C})\ B - (\vec{A} \cdot \vec{B})\ \vec{C}$$

$$\Rightarrow \vec{A}x\ (\vec{B}x\vec{C}) = (\vec{A} \cdot \vec{C})\ B - (\vec{A} \cdot \vec{B})\ \vec{C}$$

(b) let us assume $\vec{C}x\vec{D} = \vec{E}$

$$\therefore \qquad \big(\vec{A}\ x\vec{B}\big).\big(\vec{E}\big) = \vec{A}\big(\vec{B}x\ \vec{E}\big)$$

[intersecting dot and cross products]

$$= \vec{A} \cdot (\vec{B}x(\vec{C}x\vec{D}))$$

$$= \vec{A} \cdot ((\vec{B} \cdot \vec{D})\ \vec{C} - (\vec{B} \cdot \vec{C})\vec{D})$$

$$= (\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})\ (\vec{A} \cdot \vec{C})$$

$$\therefore \qquad (\vec{A}\,x\vec{B}).(\vec{C}x\vec{D}\,)\,(\vec{A}\cdot\vec{C}) = (\vec{A}\cdot\vec{C})\;\;(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})\,(\vec{B}\cdot\vec{C})\;\;Ans$$

(c) let us assume $\vec{C}x\vec{D} = \vec{E}$

$$\therefore \qquad (\vec{A}\,x\vec{B})\;x\,\vec{E} = (\vec{A}\cdot\vec{E})\,\vec{B} - (\vec{B}\cdot\vec{E})\vec{A}\;\;[from\;(a)]$$

$$= \;\; (\vec{A}\cdot(\vec{C}x\vec{D})\,) - (\vec{B}\cdot(\vec{C}x\vec{D})\,)\,\vec{A}$$

$$(\vec{A}\,x\vec{B})\;x(\vec{C}x\vec{D})\;\;([\overrightarrow{A}\cdot(\vec{C}x\vec{D})\,]\;\;\vec{B}) - ([\vec{B}\cdot(\vec{C}x\vec{D})\,]\,\vec{A})$$

1.8 Physical Interpretation of Gradient

Maximum space rate of change of a physical function is called gradient of that function, e.g if scalar function V represent temperature, then $\nabla V = grad\;V$ is temperature gradient or rate of change of temperature with distance.

∇V is a "***vector quantity"***, its direction being that in which the temperature changes most rapidly.

$$\nabla V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z} \qquad 1.9$$

With reference to fig 1.4 (a), we have defined an equipotential surface V(r) wherein

$\frac{\partial V}{\partial l}$ is derived of scalar V at point P in direction ∂l (change in length)

$\frac{\partial V}{\partial n}$ is derivation in direction of normal

$\frac{\partial V}{\partial n} > \frac{\partial V}{\partial l}$ ($\therefore PQ\;is\;minimum\;distance\;between\;V_0 and\;V_1$

(as shortest distance between any two lines is perpendicular between them)

$$\Rightarrow \qquad \frac{\partial V}{\partial l} = \frac{\partial V}{\partial l\cos\theta}$$

$$\Rightarrow \qquad \frac{\partial V}{\partial l} = \frac{\partial V}{\partial n}\cos\theta \qquad 1.10$$

$$\left(\frac{\partial V}{\partial n} 1_N \Rightarrow gradient\ of\ scalar\ V\right)$$

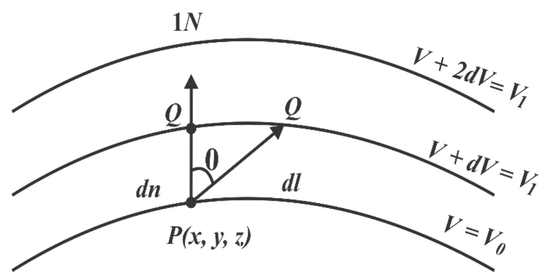


Figure 1.4 Equipotential Surface V_r = V(x , y , z)

The gradient lines are orthogonal (perpendicular) to the equipotential lines (level surface).

Note: $\boldsymbol{\nabla V . dl = 0}$ $on\ equipotential$ surface (as potential remains same throughout)

Example: 1.13 Given the potential field, V=2x^{2}y – 5z, and point P(4, 3, 6), we wish to find

(a) The potential V (i.e E=- $\boldsymbol{\nabla V}$)
(b) The electricity field intensity

Solution: (a) The potential at P(-4, 5, 6) is

$$V_p = 2(-4)^2\,(3) - 5(6) = \ 66\text{ V } \boldsymbol{Ans}$$

(b) we may use gradient operation to obtain electric field intensity

$$E = -\nabla V = -4xy\ a_x - 2x^2\ a_y + 5a_z\ V/m$$

The value of $\vec{E}$ at point P is

$$\overrightarrow{E_p} = 48a_x\ -32a_y + 5a_z\text{ V/m}$$

And $$\left|E_p\right| = \sqrt{(48)^2 + -32)^2 + (5)^2} \quad = 57.9\text{ V/m } \boldsymbol{Ans}$$

Example 1.14: Find gradient of a scalar function of position v where v(x, y, z)= x^2 y +e^z. calculate the magnitude of gradient at point P (1,5, -2).

Solution: $v\,(x, y, z) \;=\; x^2 y + e^z$

Gradient: $$v = \boldsymbol{\nabla V} = \frac{\partial v}{\partial x}\hat{a}_x + \frac{\partial v}{\partial y}\hat{a}_y + \frac{\partial v}{\partial z}\hat{a}_z$$

$$= (2xy)\hat{a}_x + (x^2)\,\hat{a}_y + (e^z)\hat{a}_z$$

At P (1, 5, 2) $$\nabla V = 10\hat{a}_x + \hat{a}_y + e^{-2}\,\hat{a}_z = 10\hat{a}_x + \;\hat{a}_y + \tfrac{1}{e^2}\hat{a}_z$$

$$= 10\hat{a}_x + \hat{a}_y + 0.1353\hat{a}_z$$

$$|\nabla v| = \sqrt{(10)^2 + (1)^2 + (0.1353)^2} = 10.051$$

- "∇V" at any point is perpendicular to constant V surface that passes through that point.
- In electrostatics, $\vec{\nabla} x\, \vec{E} = 0$, as curl of any vector is zero, then vector is represented by gradient of another scalar i.e, $\vec{E} = -\nabla v$ where "V" is scalar potential.

1.9 Physical Interpretation of Divergence

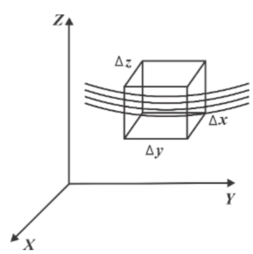


Figure 1.5 Divergence

Net outward flow per unit volume is called divergence (for a compressible fluid).

Derivation: the rectangular parallel-piped is assumed, $\nabla x, \nabla y, \nabla z$ is an infinitesimal volume element within the fluid (e.g., water, or steam) as in Fig. 1.5.

If ρ_m is mass density of fluid, flow into the volume through L.H.S face is $\rho_m\, v_y\, \Delta_x . \Delta_z \Rightarrow v_y$ is the average of y component of fluid velocity through left hand face (as density =mass/volume and velocity=distance/time, $\Delta_x . \Delta_z = area$).

Corresponding velocity through the R.H.S face will be

$$v_y + \left(\frac{\partial v_y}{\partial y}\right)\Delta y \qquad 1.11$$

Flow through this face is

$$\rho_m v_y + \left[\frac{\partial(\rho_m v_y}{\partial y}\ \Delta y\right]\ \Delta x\ \Delta z \qquad 1.12$$

The net outward flow in y direction is

$$\frac{\partial(\rho_m v_{y)}}{\partial y}\ \Delta x\ \Delta y\ \Delta z\ (i.e, R.H.S\ flow - L.H.S\ flow) \qquad 1.13$$

Similarly, the net outward flow in z dir^n $\frac{\partial(\rho_m v_{y)}}{\partial y}\ \Delta x\ \Delta y\ \Delta z$ 1.14

X dir^n $\frac{\partial(\rho_m v_{y)}}{\partial y}\ \Delta x\ \Delta y\ \Delta z$ 1.15

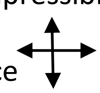
Total net outward flow is $\left[\frac{\partial(\rho_m v_{y)}}{\partial y} + \frac{\partial(\rho_m v_{y)}}{\partial y} + \frac{\partial(\rho_m v_{y)}}{\partial z}\right]\ \Delta x\ \Delta y\ \Delta z$ 1.16

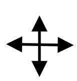
Where $\Delta x\ \Delta y\ \Delta z$ represent volume of parallel-piped.

The next outward flow per unit volume is

$$\frac{\partial(\rho_m v_{x)}}{\partial x} + \frac{\partial(\rho_m v_{y)}}{\partial y} + \frac{\partial(\rho_m v_{z)}}{\partial z} = \partial iv(\rho_m v) = \nabla.(\rho_m x \qquad 1.17$$

For incompressible fluid, divergence is always zero. (**Solenoidal)**

Divergence  +ve for valve on steam boiler opened

 -ve for evaluated light bulb broken

Example 1.15 Find div $\vec{D}\ at\ origin\ if\ \ \vec{D} = e^{-x}\sin y\ \hat{a}_y + 2z\hat{a}_z$

Solution: Divergence is given by

$$div\ \vec{D} = \frac{dD_x}{dx} + \frac{dD_y}{dy} + \frac{dD_z}{dz} = -e^{-x}\sin y + e^{-x}\cos y + 2$$
$$= 2\ (at\ origin\ \sin 0 = 0).$$

Also, the value is the constant 2, regardless of location.

Example 1.16: Find the value of constant c for which vector

$\vec{A} = (x+3y)\hat{a}_x + (y-2z)\hat{a}_y + (x+C_z)\hat{a}_z$ is solenoidal.

Solution: for a solenoidal vector field $\vec{A}$

Div A=0

$\nabla . A = 0$

$\Rightarrow \quad \left[\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z}\right]\left[(x+3y)\hat{a}_x + (y-2z)\hat{a}_y + (x+cz)\hat{a}_z\right] = 0$

$\Rightarrow \quad \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+cz) = 0$

$\Rightarrow \quad c = -2$

- Divergence of scalar has no significance, as dot product cannot be applied to scalar quantities.
- Divergence is positive when source spreads out every, as then the net flow of flex is in outward direction.
- Divergence is negative when sink intakes the power given by source, as then system has inward flown flux.
- Divergence is zero, if system is neither a source or a sink, as then there is zero net outward flow of flux.

1.10 Physical Interpretation of Curl

Rate of change of vector field is called curl, “circular rotation”.

Assumed here is stream of water over which leaf a float, therefore, if velocity at surface is entirely in y direction, ⇒ translational motion.

If there are eddies or vertices in stream flow ⇒ Rotational +Translational

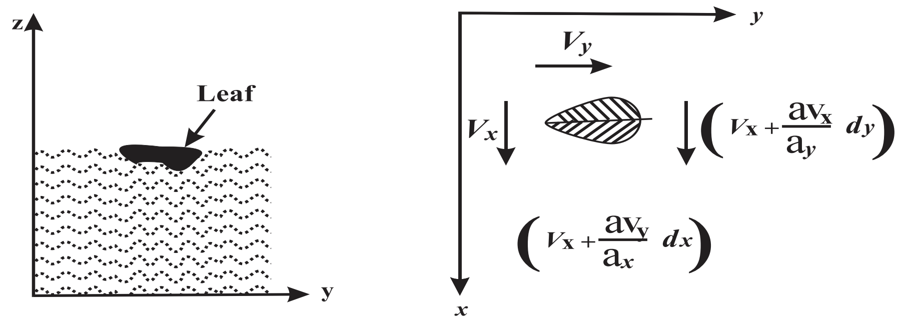


Figure 1.6 Stream on surface of which floats a leaf

The rate of rotation/angular velocity at any point is measure of curl of velocity of water at that point.

$(\Delta\ x\ v)_z$ Rotation/curl of v in z direction.

+ve value denotes rotation from x to y. i.e counterclockwise dir” with reference to Fig. 1.6;

A positive value for $\left(\frac{\partial v_y}{\partial x}\right)$ → rotate leaf in **CW** dir”

A positive value for $\left(\frac{\partial v_x}{\partial y}\right)$ → rotate leaf in **CW** dir”

The rate of rotation about z axis is proportional to difference between these two quantities

$$(\Delta\ x\ v)_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \qquad 1.18$$

Similarly $(\Delta\ x\ v)_z = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}$ 1.19

And $(\Delta\ x\ 0)_z = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}$ 1.20

⇒ $$\nabla x v = \left(\frac{\partial y_z}{\partial_y} - \frac{\partial v_y}{\partial_z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z} \qquad 1.21$$

Not necessary to have circular motion to have CURL,

$$\boldsymbol{if\ v_x = 0,\qquad v_y\ varies\ in\ x\ dir", (\nabla x v)_z = \frac{\partial v_y}{\partial x} \Rightarrow CURL}$$

Other way. The curl of vector $\vec{B}$ is a vector whose magnitude is maximum net circular of $\vec{B}$ per unit area as the area tends to zero and whose direction is normal direction of area when the area is oriented to make the net circulation maximum.

$$\nabla \times \vec{B} = \lim_{\nabla s \to 0} \oint_{AC} \frac{\overrightarrow{B.dl}}{\nabla s}\, 1_n \qquad 1.22$$

If curl of any vector is zero then it is called irrotational.

- If the divergence of vector is zero, then that vector quantity can be expressed as curl of another, $\nabla.\vec{B} = 0$ then $\vec{B} = \nabla x \vec{A}$ ($\vec{A}$ is magnitude vector potential)

Example 1.17 If $\vec{A} = xz^3\hat{a}_x - 2x^2yz\,\hat{a}_y + 2yz^4\,\hat{a}_z$, find $\vec{\nabla}x\,\vec{A}$ at point (1, -1, 1).

Solution $\vec{\nabla}x\,\vec{A}$= $\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$

$$= \hat{a}_x\left[\frac{d}{dy}(2yz^4) - \frac{d}{dz}(-2x^2yz)\right] + \hat{a}_y\left[\frac{d}{dz}(xz^3) - \frac{d}{dx}(2zy^4)\right] + \hat{a}_z\left[\frac{d}{dx}(-2x^2yz) - \frac{d}{dz}(x^3z)\right]$$

$$= \hat{a}_x(2z^4 + 2x^2y) + \hat{a}_y(3xz^2) + \hat{a}_z(-4xyz) at\ point\ (1, -1, 1)$$

$$\vec{\nabla}x\,\vec{A} = \hat{a}_x(2-2) + \hat{a}_y(3) + \hat{a}_z(4) = 3\hat{a}_y + 4\hat{a}_z$$

Example 1.18: Show that the field $\vec{F} = \left(\frac{150}{r^2}\right)\,\hat{a}_r + 10\hat{a}_\emptyset$ (cylindrical coordinate) is irrotational and non-solenoidal.

Solution: For field to be irrotational $\vec{\nabla}x\,\vec{F} = 0$

Using formula for curl $\vec{F}\ \ in\ cylindrical\ coordinate$

We get $\vec{\nabla}x\ \vec{F}= \begin{vmatrix} \frac{\hat{a}_r}{r} & a\emptyset & \frac{\hat{a}_z}{r} \\ \frac{d}{dr} & \frac{d}{d\emptyset} & \frac{d}{dz} \\ \frac{150}{r^2} & 10r & 0 \end{vmatrix}$ (formula for curl in cartesian co-ordinates)

$$= \frac{\hat{a}_r}{r}\left(0-\frac{d}{dz}(10r)\right)+\hat{a}_\emptyset\left[\frac{d}{dz}\left(\frac{150}{r^2}\right)\right]+\frac{\hat{a}_z}{r}\left[\frac{d}{dr}(10r)\right]-\frac{d}{d\emptyset}\left(\frac{150}{r^2}\right)$$

$$= 0 \quad proved$$

For field to be solenoidal $\nabla.\vec{F}= 0$

$$\Rightarrow \qquad \nabla.\vec{F}=\frac{1}{r}\,\frac{d}{dr}\left(r\frac{150}{r^2}\right)+\frac{1}{r}\,\frac{d}{d\emptyset}(10)=\frac{150}{r^3}\neq 0$$

So, non-solenoidal.

Example 1.19: If $\vec{r}= x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ show that

i) $\quad \vec{\nabla}\cdot\vec{r} = 3$

Solution: (i) $\vec{\nabla}\cdot\vec{r} = \left[\hat{a}_x\frac{d}{dx}+\hat{a}_y\frac{d}{dy}+\hat{a}_z\frac{d}{dz}\right]\cdot\left[\hat{a}_x x+\ \hat{a}_y y+\hat{a}_z z\right]$

$$=\frac{d(x)}{dx}+\frac{d(y)}{dy}+\frac{d(z)}{dz}$$

$$\vec{\nabla}\cdot\vec{r} = 1+1+1 = 3\ \boldsymbol{proved}$$

ii) $\quad \vec{\nabla}\cdot\vec{r} = \left[\hat{a}_x\frac{d}{dx}+\hat{a}_y\frac{d}{dy}+\hat{a}_z\frac{d}{dz}\right].\left[\hat{a}_x x+\ \hat{a}_y y+\hat{a}_z z\right]$

$$\vec{\nabla}\cdot\vec{r} = \hat{a}_x\left[\frac{dz}{dy}-\frac{dy}{dz}\right]+\hat{a}_y\left[\frac{dx}{dz}-\frac{dz}{dx}\right]+\hat{a}_z\left[\frac{dy}{dx}-\frac{dx}{dy}\right]= 0 \qquad Proved$$

Example 1.20: For a vector field $\vec{A}$ in cylindrical coordinates

$\vec{A}$(r,$\theta, z) = r^3\sin\theta\,\hat{r} + rcos^2\theta\,\hat{\emptyset} + z\,tan\,\theta\hat{z}$ determine (i)$\nabla.\vec{A}$ $and\ \nabla\times\vec{A}$

Solution: (i) $\frac{1}{r}\frac{d}{dr}(\,Ar) + \frac{1}{r}\frac{dAz}{d\theta} + \frac{dAz}{dz} = \nabla.\vec{A}$

Where $A_r = r^2 sin\theta$

$A_\theta = r\, cos^2\theta$

$A_z = z\, tan\theta$

$\Rightarrow$ $\nabla.\vec{A} = \frac{1}{r}\frac{d}{dr}(r^3\, sin\theta) + \frac{1}{r}\frac{drcos^2\theta}{d\theta} + \frac{d}{dz}(z\, tan\theta)$

$= \frac{1}{r}\, 3r^2 sin\theta + \frac{d}{d\theta}\,(cos^2\theta) + tan\theta$

$= 3r\, sin\theta + 2cos\theta(-sin\theta) + tan\theta$

$\nabla.\vec{A} = 3r\, sin\theta - sin2\theta + tan\theta$

(ii) $\nabla \times \vec{A} = \begin{vmatrix} \frac{\hat{r}}{r} & \theta & \frac{\hat{z}}{r} \\ \frac{d}{dr} & \frac{d}{d\theta} & \frac{d}{dz} \\ A_r & rA_0 & A_z \end{vmatrix}$

$$\boldsymbol{\nabla}.\vec{A} = \frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}}\,[\frac{\boldsymbol{d}}{\boldsymbol{d\theta}}(ztan\theta) - \frac{d}{dz}(r^2cos^2\theta)$$

$$-\theta\,\left[\frac{d}{dr}(z\, tan\emptyset) - \frac{d}{dz}(r^2 \sin\theta)\,\right]$$

$$+\,\frac{\hat{z}}{r}\left[\frac{d}{dr}(r^2cos^2) - \frac{d}{d\theta}(r^2 \sin\theta)\,\right]$$

$$= \left[\frac{\hat{r}}{r}(z\, sec^2\emptyset) - \hat{0}[0] + \frac{\hat{z}}{r}(2rcos^2\theta - y^2\, cos\theta\right]$$

$$\nabla.\vec{A} = \frac{1}{r}[zsec^2\theta\hat{r} + (2r\, cos^2\theta - r^2\, \cos\theta)\hat{z}]$$

Two null identities

(a) $\nabla.\nabla V = 0$

The curl of gradient of any scalar field is identically zero.

Proof: from strokes theorem we can write.

$$\int (\nabla \times \nabla V) \cdot \overrightarrow{ds} = \oint (\nabla V) \cdot \overrightarrow{dl}$$

i.e surface integral of $\nabla x \nabla V$ over any surface gives line integral of ∇V around closed path bounding that surface.

Also $\oint (\nabla \times \nabla V) \cdot \overrightarrow{dl} = \oint (\mathrm{d}V)$ (which is zero)

$\Rightarrow \qquad \oint (\nabla \times \nabla V) \cdot \overrightarrow{ds} = 0$

Or $\qquad \nabla \times \nabla V = 0$

Converse statement is: if a vector field is curl, free then it can be expressed as the gradient of scalar field.

Suppose $\vec{E}$ is vector field then if $\nabla x\, \vec{E} = 0$ we can say that $\vec{E} = -\nabla V$.

$An\ irrotational\ (or\ conservative) vector\ \ field\ can\ always\ be\ expresed\ as\ gradient\ of\ scalar\ field$

B $\qquad \nabla \times (\nabla \times \vec{A}) = 0$

The divergence of curl of any vector field is zero

Proof: from divergence theorem

$$\oint_v \nabla.(\nabla \times \vec{A})\, dv = \oint_s (\nabla x\, \vec{A}) \cdot \overrightarrow{ds}$$

Here we have taken arbitrary volume V and split it in two open surface S_1 and S_2 connected by a common boundary that has been drawn twice as C_1 and C_2

$$\therefore \quad \oint_s (\nabla \times \vec{A}) \cdot \overrightarrow{ds} = \oint_s \left(\nabla x \vec{A}\right). \hat{n}_2 ds$$

$$= \oint_{C_1} \vec{A} \cdot \overrightarrow{dl} + \oint_{C_2} \vec{A} \cdot \overrightarrow{dl}$$

Here, the two-line integrals on R.H.S of above equation follow same path but in opposite directions thus sum is zero, hence we can say that

$$\oint_s (\nabla \times \vec{A}) \cdot \overrightarrow{ds} = 0$$

Or $\nabla \times (\nabla \times \vec{A}) = 0$

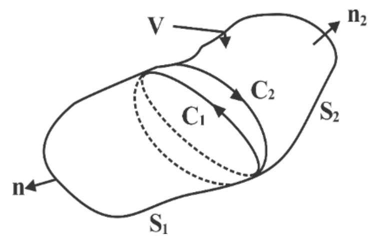


Figure 1.7 Proof of vector B

Converse statement is: if a vector field is divergenceless, then it can be expressed as curl of another vector field is if B is vector field an $\nabla . B = 0$,

hence.

$$\vec{B} = \nabla \times \vec{A}$$

1.11 Divergence Theorem

"The volume integral to the divergence of vector field $\vec{A}$, taken over any volume V is equal to surface integral of $\vec{A}$ taken over the closed surface (total outward flux of vector through surface) that bounds the volume V" is definition of Divergence Theorem

$$\oint_v (\nabla \cdot \vec{A}) \cdot dV = \oint_s \vec{A} \cdot \overrightarrow{ds} \qquad 1.23$$

Example 1.21: Show that (a) $\oint_s \vec{F} \cdot \overrightarrow{ds} = 6V$ where s is a closed surface, enclosing a volume V and $\vec{F} = 2x\hat{a}_x + 3y\hat{a}_y + z\hat{a}_z$.

(b) Use divergence theorem to evaluate $\int_S \vec{F} \cdot \overrightarrow{ds}$ when $\vec{F} = x^3 a_x + y^3 a_y + z^3 a_z$ and S is surface of sphere $x^2 + y^2 + z^2$.

Solution: from divergence theorem

(a) $\int_s \vec{F} \cdot \overrightarrow{ds} \int_v \vec{\nabla} \cdot \vec{F} dV = \int_v \left(\frac{d}{dx}(2x) + \frac{d}{dy}(3y) + \frac{d}{dz}(z) \right) dV$

$$\int_v (2+3+1) dV = 6 \int_v dV$$

$$= 6 \text{ V}$$

(b) Divergence theorem $\int_s \vec{F} \cdot \overrightarrow{ds} = \int_v \vec{\nabla} \cdot \vec{F} dV$

$$\text{R.H.S} = \frac{d(x^3)}{dx} + \frac{d(y^3)}{dy} + \frac{d(z^3)}{dz} = \int_v (3x^2 + 3y^2 + 3z^2)\, dV$$

$$= 3 \int_v (x^2 + y^2 + z^2) dV = 3a^2 \int_v \overrightarrow{dV}$$

$= 3a^2$ x Volume of sphere

$$= 3a^2 \times \frac{4}{3} \pi a^3 = 4\pi a^5$$

EXAMPLE 1.22: Given that $\vec{E}(y, \theta, s) = \frac{1}{r^4} sin^2\theta\, \hat{r}$, evaluate (i) $\int_s \vec{E} \cdot \overrightarrow{ds}$(ii) $\iiint (\nabla . \vec{E})\, dv$ over the region between spherical surfaces $r = 2$ and $r = 4$. What is the inference drawn from results obtained in (i) and (ii)?

Solution: (i) $\int_s \vec{E} \cdot \overrightarrow{ds}$ (you need to refer spherical coordinate system text).

$$\Rightarrow \quad \int_s \vec{E} \cdot \overrightarrow{ds} = \int_{\theta=0}^{\pi} \int_{\emptyset=0}^{2\pi} \frac{1}{4} . \sin^2\theta\, r \sin\theta\, d\emptyset\, r d\theta$$

$$= \frac{1}{r^2} \int_{\emptyset-=0}^{2\pi} sin^2\theta\, d\theta \int_{\theta=0}^{\pi} \sin\theta\, d\theta = \frac{1}{r^2} \left[\frac{1 - \cos 2\theta}{2} \right]_0^{2\pi} [-cos\theta]_0^{\pi}$$

$$= \frac{1}{r^2} \times \left[\left(\frac{1-1}{2} \right) - \left(\frac{1-1}{2} \right) \right] [-(-1-1)]$$

$$\int_s \vec{E} \cdot \overrightarrow{ds} = 0$$

(ii) $\nabla \cdot \vec{E} = \frac{1}{r^2}\frac{d}{dr}\ (r^2\, Er) = \frac{1}{r^2}\frac{d}{dr}\left(r^2.\frac{1}{r^4}\,sin^2\theta\right)$

$$= \frac{1}{r^2}\frac{d}{dr}(sin^2\theta . r^{-2}) = \frac{sin^2\theta}{r^2} \times \left[-\frac{1}{r}\right]$$

$$\nabla \cdot \vec{E} = \frac{-sin^2\theta}{r^3}$$

$$\int_v \nabla \cdot \vec{E} dv = \int_{r=2}^{4} \int_{\emptyset=0}^{2\pi} \frac{-\sin^2\theta\, r}{r^3} sin\theta\ d\theta\ dr\ d\theta$$

$$= \left[\int_2^4 \frac{1}{r} dr\right]\left[\int_0^{2\pi} sin^2\theta\ d\theta\right]\left[\int_0^{\pi} \sin\theta\ d\theta\right] = \ln\left(\frac{4}{2}\right) \times 0 \times 2$$

$$\int_v \nabla \cdot \vec{E} dV = 0$$

We see that result obtained from (i) and (ii) are equal and this is because they satisfy divergence theorem i.e.

$$\int_v \nabla \cdot \vec{E}\ dV = \int_s \vec{E} ds$$

1.12 Stoke's Theorem

The surface integral of curl of a vector field **A** over an open surface equals line integral of vector field over the closed curve bounding the surface area "is definition of Stokes' theorem".

$$\int_s (\vec{\nabla} x\ \vec{A}) \cdot \overrightarrow{ds} = \oint_c \vec{A} \cdot \overrightarrow{dl} \qquad 1.24$$

Example 1.23: Evaluate both sides of stokes theorem for field $H = 6xy\,\hat{a}_x - 3y^2\hat{a}_y$ A/m and rectangular path around region, $2 \le x \le 5, -1 \le y \le 1, z = 0$. Let the positive direction of ds be $\hat{a}_z$.

Solution: Stokes theorem is $\int_C \vec{H}\cdot \vec{dl} = \int_s (\vec{\nabla}\times \vec{H})\,\vec{dS}$

$$R.H.S. \qquad \vec{\nabla}\times\vec{H} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ 6xy & -3y^2 & 0 \end{vmatrix}$$

$$\Longrightarrow \qquad \vec{\nabla}\text{x}\vec{H} = \hat{x}(0) - \hat{y}(0) + \hat{z}\left[\frac{d}{dx}(3y^2) - \frac{d}{dy}(6xy)\right]$$

$$\vec{\nabla}\times\vec{H} = -6x\hat{z}$$

$$R.H.S. = \int_S \vec{\nabla}\, x\vec{H}\,\overrightarrow{dS}$$

$$\int_S -6x\hat{z}.(\hat{z}.dx\,dy) = -\int_S 6x\,dx\,dy\,\hat{z}$$

$$= -\int_{y=-1}^{1}\int_{x=2}^{5} 6x\,dx\,dy\,\hat{z}$$

$$= -6\left[\frac{x^2}{2}\right]_2^5 [dy]_{-1}^{1}\,\hat{z} = -6\times\frac{1}{2}\times(25-4)(2)\hat{z}$$

$$= -6\times 21\,\hat{z}$$

$$-126\,\hat{z}$$

$$L.H.S \qquad \oint_C \vec{H}\cdot\vec{dl} = \oint_C (6xy\,\hat{a}_x - 3y^2\,\hat{a}_y).(dx\,\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z)$$

$$= \oint_C (6xy\,dx - 3y^2dy)$$

For this we need to find out equation of line integral.

$$\oint_C \vec{H}\cdot \vec{dl} = \int_{x=5,y=1}^{2}\int 6xy\, dx + \int_{x=2y=-1}^{5}\int 6xy\, dx$$

$$= \frac{6x^2}{2}(1)\Big|_5^2 + \frac{6x^2}{2}(-1)\Big|_2^5 = -126$$

Example 1.24: Given $\vec{E} = 2r\cos\emptyset\hat{r} + r\widehat{\emptyset}$ in cylindrical coordinates verify Stokes theorem for the contour in Fig. 1.8 which lies in x-y plane completely.

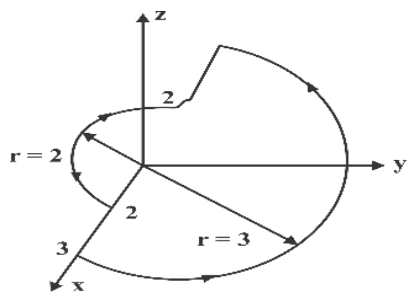


Figure 1.8

Solution: according to Stokes theorem

$$\int_S \nabla \times\vec{E}\cdot ds = \oint \vec{E}\cdot dl$$

R.H.S. we have to evaluate line integral over the given curve as we see that we have 4 defined contours as shown:

$$\oint \vec{E}\cdot dl = \int_{r=2}^{3} 2r\cos\emptyset\, dr + \int_{r=3\ \emptyset=0}^{\pi} r.rd\emptyset + \int_{r=3}^{2} 2r\cos\emptyset\, dr + \int_{r=2\ \emptyset=\pi}^{0} r.rd\emptyset$$

$$= 2\cos\emptyset\left[\frac{r^2}{2}\right]_2^3 + 9[\emptyset]_0^{\pi} + 2\cos\emptyset\left[\frac{r^2}{2}\right]_3^2 + 4[\emptyset]_{\pi}^{0}$$

$$= 9[\pi - 0] + 4[0-\pi] = 9\pi - 4\pi$$

$$\int \vec{E}\cdot\vec{dl} = 5\pi$$

L.H.S $$\nabla \times \vec{E} = \begin{vmatrix} \frac{\hat{r}}{r} & \widehat{\emptyset} & \frac{\hat{z}}{r} \\ \frac{d}{dr} & \frac{d}{d\emptyset} & \frac{d}{dz} \\ Er & rE\emptyset & Ez \end{vmatrix} = \begin{vmatrix} \frac{\hat{r}}{r} & \widehat{\emptyset} & \frac{\hat{z}}{r} \\ \frac{d}{dr} & \frac{d}{d\emptyset} & \frac{d}{dz} \\ 2r\cos\emptyset & r^2 & 0 \end{vmatrix}$$

$$= \frac{\hat{r}}{r}\,[0] - \emptyset[0] + \frac{\hat{z}}{r}\,[2r + 2r\sin\emptyset]$$

$$= \ (2 + 2\sin\emptyset)\,\hat{z}$$

So, $\int_s \nabla \times \vec{E}.ds = \int_s (2 + 2\sin\emptyset)\ (rd\emptyset\ dr)$

$$= \int_S (2r\ ds\ dr + 2r\sin\emptyset\ d\ dr)$$

$$= 2\int_2^3 r\ dr \int_{\emptyset=0}^{\pi} d\emptyset + 2\int_2^3 r\ dr \int_{\emptyset=0}^{\pi} \sin\emptyset\ d\emptyset$$

$$= 2\left[\frac{r^2}{2}\right]_2^3 [\emptyset]_0^{\pi} + 2\left[\frac{r^2}{2}\right]_2^3 [\cos\emptyset]_0^{\pi} = 2\pi\left[\frac{9-4}{2}\right] + 2\ \left[\frac{9-4}{2}\right][0]$$

$$\int_S \nabla x \underset{E}{\rightarrow}.ds = 5\pi$$

Thus, Stokes theorem is verified

1.13 Line Integral of a Vector Field

Consider path OMN between two points O and N and let 'dl' be an element of length at a point on a smooth curve ON drawn in a vector field and $\vec{A}$, a continuous vector point function inclined at an angle 'θ' to 'dl' as shown in Fig 1.9 (a) such that it continuously varies in magnitude as well as direction as we proceed along the curve. Then, the integral.

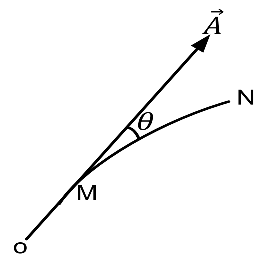


Figure 1.9 (a) Line Integral

$\int_O^N \underset{A}{\rightarrow} . \underset{dl}{\rightarrow} = \int_O^N A\cos\theta\, dl$This is referred as the line integral of vector $\vec{A}$ along curve ON.

Note: if $\oint \vec{A} \cdot \overrightarrow{dl} = 0$, then the field is called a conservative field or lamellar field.

In terms of components of $\vec{A}$ along the three Cartesian coordinate, we have

$$\int_O^N \vec{A} \cdot \overrightarrow{dl} = \int_O^N \left(A_x dx + A_y dy + A_z dz\right) \quad 1.25$$

Where 'dl' represents differential length

But $\left(\overrightarrow{dl} = \overrightarrow{dl}\hat{x} + \overrightarrow{dl}\hat{y} + \overrightarrow{dl}\hat{z}\right)$

Differential length vector: the differential length vector dl is the vector drawn from point p (x, y, z) to a neighboring point $Q(x + dx, y + dy, z + dz)$ obtained by incrementing the coordinates of P by infinitesimal amounts. Thus, it is the vector sum of the three differential length elements, as shown.

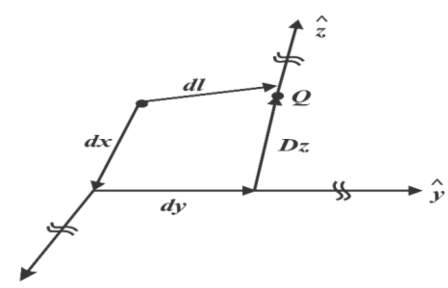


Figure 1.9 (b) Differential Length

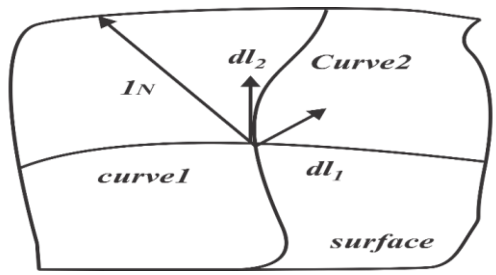


Figure 1.9 (c) Finding unit vector Normal to Surface

$$\overrightarrow{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

The differential lengths dx, dy and dz are, however, not independent of each other since in evaluation of line integrals, the integration is performed along a specific path.

Differential length vectors are useful for finding the unit vector normal to a surface at a point on that surface. This is done by considering two differential length vectors at that point under consideration and tangential to two curves on surface,

$$i_N = \frac{\overrightarrow{dl}_1 \times \overrightarrow{dl}_2}{\overrightarrow{dl}_1 \times \overrightarrow{dl}_2}$$

1.14 Surface Integral of Vector Field

Let $\vec{A}$ be a continuously varying vector point function (or vector) at a point B in a small elements dS_1 of the surface S_1 as shown in Fig. 1.10. (a)

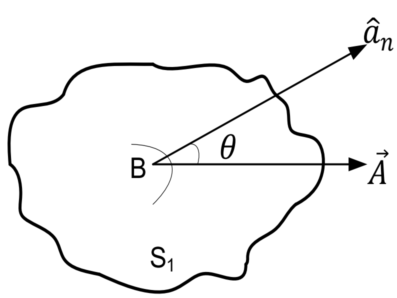


Figure 1.10 (a) Surface Integral

$\vec{A}$ vector is at angle θ with normal to surface, at that point (drawn outwards) if the surface be closed and always towards the same side otherwise). Then the surface integral of vector field $\vec{A}$ is defined as the sum of product of normal components of A and surface elements covering the whole surface.

If $\hat{n}$ is unit vector normal to surface elements dS_1 then normal component of field $\vec{A}$ is $(\vec{A} \cdot a\hat{n})$.

Hence surface integral of vector $\vec{A}$ over the surface is given as

$$\int_{S_1}\int (\vec{A} \cdot \hat{a}_n) dS_1 \;\; or \;\; \oint_{S1} (\vec{A} \cdot \hat{a}_n) dS_1$$

In terms of Cartesian coordinates of $\vec{A}$ we have

$$\int_{S1}\int \vec{A}\, dS_1 = \int_{S1}\int \left(A_x dS_x + A_y dS_y + A_z dS_z\right) \qquad 1.26$$

If $\oint_{S1} \int \left(\vec{A}. dS_1\right) = 0$, then vector field is said to be solenoidal vector field.

Where 'ds' represents differential area

$$\overrightarrow{ds} = \overrightarrow{ds}\hat{x} + \overrightarrow{ds}\hat{y} + \overrightarrow{ds}\hat{z}$$

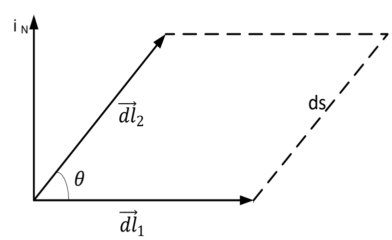


Figure 1.10 (b) Differential Surface Vector Concept

Differential surface vector: two differential length vectors $\overrightarrow{dl}_1$ and $\overrightarrow{dl}_2$ originating at a point define a differential surface whose area 'ds' is that of a parallelogram having $\overrightarrow{dl}_1$ and $\overrightarrow{dl}_2$ as two of its adjacent sides.

$$ds = \; dl_1 dl_2 \sin\alpha = \overrightarrow{dl}_1 \times \overrightarrow{dl}_2$$

$$\pm\, dy\, \hat{y} \times dz\, \hat{z} = \pm dy\, dz\, \hat{x}$$
$$\pm\, dz\, \hat{z} \times dx\, \hat{x} = \pm dz\, dx\, \hat{y}$$
$$\pm\, dx\, \hat{x} \times dy\, \hat{y} = \pm dx\, dy\, \hat{z}$$

Associated with planes x=constant, y =constant, z= constant. Respectively

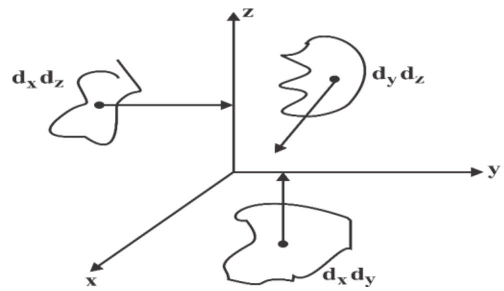


Figure 1.10 (c) Differential Surface Vectors in cartesian coordinate system

1.15 Volume Integral

Let $\vec{A}$ be a vector field and V be the volume enclosed by surface, at a point in small element Dv of the region. Then the integral $\int_V \iint \, or \int \vec{A}.dV$ covering the entire region, is called volume integral of a vector $\vec{A}$ over the surface. The differential or elemental volumes are

$dV = dx\; dy\; dz$ is Cartesian coordinate

$dV = r\; d\theta\; d\emptyset\; dz$ is cylindrical coordinate

$dV = r^2 \sin\theta\; dr\; d\theta\; d\emptyset$ is spherical coordinate

Differential volume: three differential length vectors dl_1, dl_2 and dl_3 originating at a point define a differential volume dV which is that of the parallelepiped having $\vec{dl}_1, \vec{dl}_2$ and $\vec{dl}_3$ as three of its contiguous edges, so dV = area of base of parallelepiped x height of parallelepiped.

$$= |\,\vec{dl}_1 \times \vec{dl}_2||\,\vec{dl}_3 \cdot i_N| = |\,dl_1 \times \vec{dl}_2|\,\vec{dl}_3 \cdot \frac{\vec{dl}_1 \times \vec{dl}_2}{\vec{dl}_1 \times \vec{dl}_2} = |\,\vec{dl}_3 \cdot \vec{dl}_1 \times \vec{dl}_2|$$

$$or\; dV = \vec{dl}_1 \cdot \vec{dl}_2 . \vec{dl}_3$$

1.16 Physical Significance of Gauss's Divergence Theorem

Gauss's Divergence theorem is given by

$$\int_S \vec{A}\cdot\overrightarrow{dS} = \int_V \nabla\cdot\vec{A}\cdot\overrightarrow{dV} \qquad 1.27$$

Let us consider a finite volume of any closed surface S in the region of any vector function $\vec{A}$

The flux diverging from the surface S of volume V is

$$\emptyset = \int_S \vec{A}\cdot\overrightarrow{ds}$$

Now lets us divide the volume v into two parts of volume V_1 and V_2 enclosed by surface S_1 and S_2 respectively.

The Flux emerging out of surface S_1 = $\int_{s_1} \vec{A}\cdot\overrightarrow{ds_1}$

The Flux emerging out of surface S_2 = $\int_{s_2} \vec{A}\cdot\overrightarrow{ds_2}$

The flux emerging from shaded surface will cancel each other (because flux emerging in and out of it for both the volumes V_1 and V_2 are equal and in opposite direction). Thus, the flux

$$\emptyset = \int_S \vec{A}\cdot\overrightarrow{ds} = \int_{s_1} \vec{A}\cdot\overrightarrow{ds_1} + \int_{s_2} \vec{A}\cdot\overrightarrow{ds_2} \qquad 1.28$$

In the same way, if we divide the volume V into large number of parts V_1, V_2, V_3………., V_1…….V_n enclosed by surfaces $_1$, S_2, ………, S_1…… S_n respectively, we must have.

$$\emptyset = \int_S \vec{A}\cdot\overrightarrow{ds} = \int_{S_1} \vec{A}\cdot\overrightarrow{ds_1} + \int_{S_2} \vec{A}\cdot\overrightarrow{ds_2} + \cdots + \int_{S_i} \vec{A}\cdot\overrightarrow{ds_i} + \cdots + \int_{sn} \vec{A}\cdot\overrightarrow{ds_n}$$

$$= \sum_{i=1}^{N} \int_{s_i} \vec{A} \cdot \overrightarrow{ds_i}$$

Dividing and multiplying by V_i we get

$$\int_{S} \vec{A} \cdot \overrightarrow{ds} = \sum_{i=1}^{N} V_i \frac{\int_{s_1} \vec{A} \cdot \overrightarrow{ds}}{V_i} \qquad 1.29$$

If N is sufficiently large, then volume V_i becomes infinitely small i.e…, if N tends to infinity V_i tends to zero and in the limit we may write.

$$\lim_{V_i \to 0} \int_{S_1} \frac{\vec{A} \cdot \overrightarrow{ds}}{V_i} = \nabla \vec{A}$$

And convert the summation into integration writing *dV* for infinitely small volume V_i i.e

$$\sum_{i=1}^{n \to \infty} V_i = \int dV$$

So, Equ. (1.32) can be written as

$$\int_{S} \vec{A} \cdot \overrightarrow{ds} = \int_{V} \nabla \cdot \vec{F} \cdot \overrightarrow{dV}$$

$\boldsymbol{Note}: This\ equation\ is\ called\ Gauss's\ Divergence\ Theorem. \Rightarrow this\ theorem\ is\ used\ to\ convert\ the\ volume\ integral\ of\ divergence\ of\ vector\ field\ into\ surface\ integral\ of\ the\ vectro\ field\ and\ vice\ versa.$

1.17 Physical Significance of Stokes Theorem

Stokes's theorem is given by:

$$\int \vec{A} \cdot \overrightarrow{dl} = \int_{S} \nabla \cdot \vec{A} \cdot \overrightarrow{ds} \qquad 1.30$$

Let us consider a surface S with C as its boundary. Let us calculate the line integral of vector function $\vec{A}$ around the boundary C of surface S.

The line integral of $\vec{A}$ around the boundary C of surface S = $\oint_C \vec{A} \cdot \overrightarrow{dl}$

Now divide the surface S into two parts of surface S_1 and S_2 having boundaries C_1 and C_2 respectively as shown in fig 1.12(b)

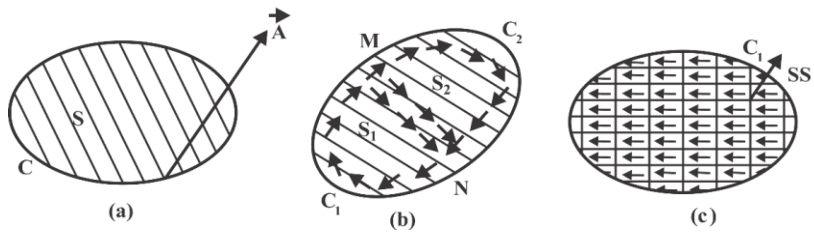


Figure 1.12 Stokes Theorem significance

For boundary S_1: $\oint_{C_1} \vec{A} \cdot \overrightarrow{dl_1}$

For boundary S_2: $\oint_{C_2} \vec{A} \cdot \overrightarrow{dl_2}$

MN is the common boundary of both the surface and flux being equal and opposite, cancel each other when considered together.

The rest of the boundaries C_1 and C_2 are identical to original boundary C.

Thus, obviously

$$\oint_C \vec{A} \cdot \overrightarrow{dl} = \int_{C_1} \vec{A} \cdot \overrightarrow{dl_1} + \oint_{C_2} \vec{A} \cdot \overrightarrow{dl_2} \qquad 1.31$$

Similarly, if we divide the surface S into a large number of parts S_1, S_2,……., S_i, …… S_n having boundaries C_1, C_2, ……., C_i,……. C_n respectively, as shown in fig 1.13 (c), then

$$\oint_C \vec{A} \cdot \overrightarrow{dl} = \oint_{C_1} \vec{A} \cdot \overrightarrow{dl_1} + \oint_{C_2} \vec{A} \cdot \overrightarrow{dl_2} + \cdots + \int_{C_i} \vec{A} \cdot \overrightarrow{dl_\imath} + \cdots + \int_{C_N} \vec{A} \cdot \overrightarrow{dl_N}$$

$$\sum_{i=1}^{N} \int_{C_i} \vec{A} \cdot \overrightarrow{dl_\imath}$$

If N is sufficiently large, then surface area S_i becomes infinitely small, i.e if N tends to infinity, S_i tends to zero and in the limits we may write.

$$\lim_{S_i \to 0} \oint_{C_i} \frac{\vec{A} \cdot \overrightarrow{dl_i}}{S_i} \hat{a}_n = \nabla \times \vec{A}$$

$$or \ \lim_{S_i \to 0} \oint_{C_i} \frac{\vec{A} \cdot \overrightarrow{dl_i}}{S_i} \ \hat{a}_n \cdot \hat{a}_n = curl \ \ \vec{A} \cdot \hat{a}_n$$

$$\lim_{S_i \to 0} \oint_{C_i} \frac{\vec{A} \cdot \overrightarrow{dl_i}}{S_i} = curl \ \ \vec{A} \cdot \hat{a}_n$$

Convert the summation into integration writing $\overrightarrow{dS}$ for infinitely small area S_i.

$$\Rightarrow \qquad \oint_C \vec{A} \cdot \overrightarrow{dl} = \oint_C \nabla x \vec{A} \cdot \overrightarrow{dS} \qquad *$$

Note. This Equ. * is called Stokes theorem
⟹ this theorem is used to convert the surface integral of the curl of vector field

$\vec{A}$ into the line integral of vector field and vise versa.

Example 1.25: Closed surface is defined is defined in spherical coordinates by 3< r<5, $\pi < \theta < \frac{\pi}{4} < \emptyset < \frac{\pi}{6}$ Find volume enclosed.

$$V = r^2 \sin\theta \ d\theta \ dr \ d\emptyset = \int_{r=3}^{5} r^2 \ dr \ \int_{\pi}^{\frac{\pi}{3}} \sin d\theta \ \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} d\emptyset$$

$$= \left.\frac{r^3}{3}\right|_3^5 \times [-\cos\theta]_{\pi}^{\frac{\pi}{3}} \times \left[\frac{\pi}{6} - \frac{\pi}{4}\right]$$

$$= \left(\frac{125-27}{3}\right) \times \left[\frac{\sqrt{3}}{2} + (-1)\right] \times \left(\frac{4\pi - 6\pi}{24}\right)$$

$$= -\left(\frac{125-27}{3}\right)\left(\frac{2.372}{2}\right)\left(\frac{-2\pi}{24}\right) = \frac{2\pi \times (125-27)(3.732}{144}$$

$$= \frac{2296.82}{144} = 15.95$$

1.18 Cartesian Coordinates

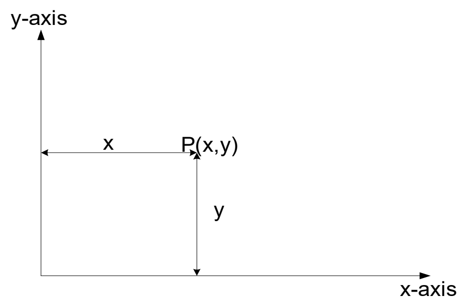


Figure 1.13 (a) Cartesian Coordinate

You are probably familiar with Cartesian Coordinates. In two dimensions, we can specify a point on a plane using two scalar values, generally called x and y as in Fig. 1.13 (a)

We can extend this to three dimensions, by adding a third scalar value z as in Fig. 1.13 (b)

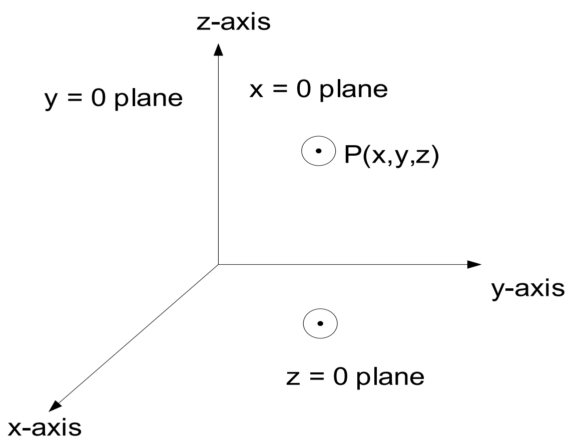


Figure 1.13 (b) Point in Cartesian Coordinate System

Note the coordinate values in the Cartesian system effectively represent the distance from a plane intersecting the origin. For example, $x = 3$ means that the point is 3 units from the $y - z$ plane (i.e, the $x = 0$ plane).

Likewise, the y coordinate provides the distance from the $x - z$ $(y = 0)$ plane, and the z coordinate provides the distance from the $x - y$ $(z = 0)$ plane.

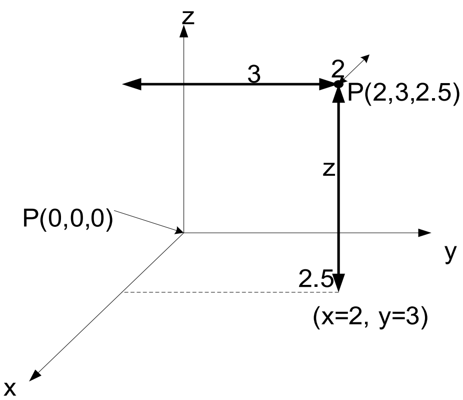


Figure 1.14 P(2,3,2.5) in cartesian coordinate

Once all three distances are specified, the position of a point is uniquely identified as shown in Fig. 1.14

1.18.1 Terms

(a) $\overrightarrow{dl} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$ **differential length** (1.32)

$$|dl| = \sqrt{|dx|^2 + |dy|^2 + |dz|^2}$$

(b) $dv = dx\,dy\,dz$ **differential volume**

(c) $\overrightarrow{ds} = \pm dy\,dz\,\hat{a}_x, \pm dz\,dx\,\hat{a}_y, \pm dx\,dy\,\hat{a}_z$ **differential area**

The three displacements (increments) $dx\,\hat{x}, dy\,\hat{y}, dz\,\hat{z}$ also define three surfaces of infinitesimal areas in three planes intersecting at point P.

1.18.2 Cartesian Base Vectors

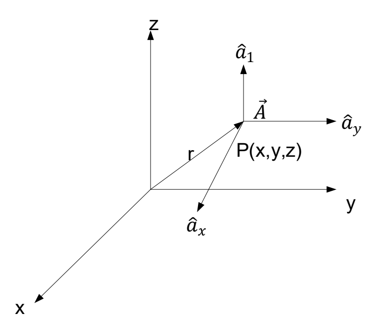


Figure 1.15 Position Vector

As the name implies, the Cartesian Base Vectors are related to the Cartesian coordinates. Specifically, the unit vector $\hat{a}_x$ points in the direction of increasing x. in other words, it points away from the $y - z$ $(x = 0)$ plane. Similarly, $\hat{a}_y and\ \hat{a}_z$ point in the direction of increasing y and z, respectively as in Fig. 1.15.

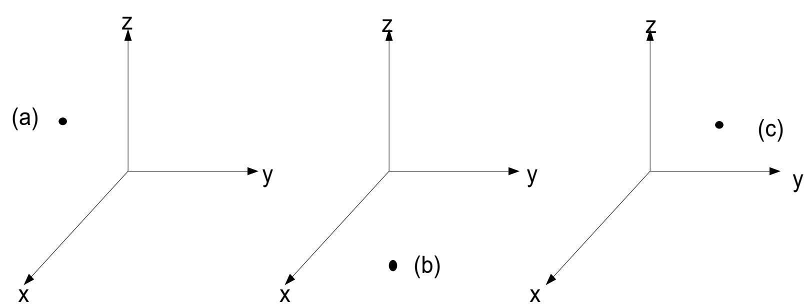
A vector drawn from origin to an arbitrary point $P(x, y, z)$ is called **position vector.**

$$\hat{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$$

Example 1.26: Try and plot the following points in Cartesian coordinate system

(a) (2, -3, 5) (b) (1, 4, -6) (c) (-1, 2, 3)

Solution



1.19 Circular Cylindrical Coordinate System

Although the rectangular coordinate system does appear easier to work with, it nevertheless often presents more work in order to break a given problem down to a more palatable and "digestible" form. So, initial drawn effort needs to be made here to convert rectangular to cylindrical and/or spherical coordinate systems, and vice versa. Therefore, need-life problems than become easier to tackle.

For circular cylindrical coordinates (or cylindrical coordinates), the three "coordinates" are ρ, defining the radial distance from the origin, Ø, which is the angle the vector makes with "X" axis" anticlockwise, and Z, the distance perpendicularly from x – y plane up the z axis. So, any vector located from an imaginary origin ends up at a point which is the meeting point of three planes of: ρ constant (cylindrical radius); Ø constant, (angle with x axis, and z that is the same as the z of the rectangular coordinates.

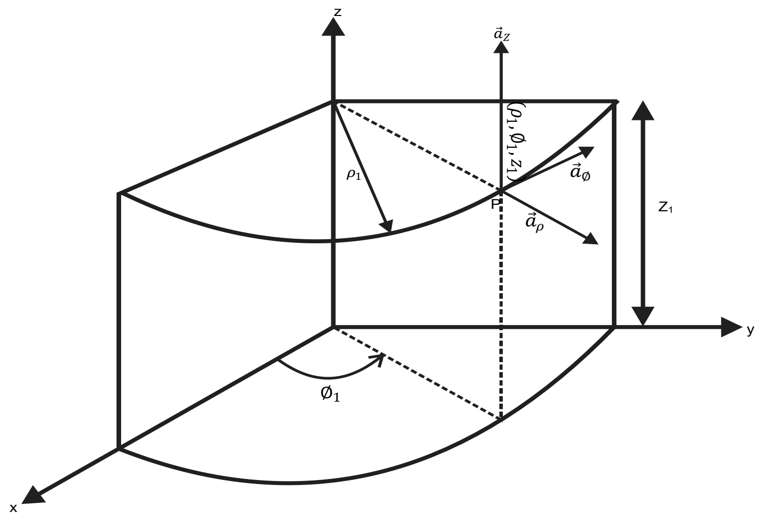


Figure 1.16 Rectangular Vector at point $P(\rho_1, \emptyset_1, z_1)$

In the diagram above, a vector located at $P(\rho, \emptyset, z,)$has the corresponding unit vectors $\hat{a}_e, \hat{a}_\emptyset, \hat{a}_z$ initially perpendicular. Unlike in rectangular coordinates system, whereas $\hat{a}_z$ is also independent of changes along z axis in the cylindrical system, the unit vectors $\vec{a}_\rho$and $\vec{a}_\emptyset$ do vary with changes in ρ and $\emptyset$ respectively. Their lengths remain unity, but their directions constantly vary with varying e and $\emptyset$. Since direction is an integral part of the definition of a vector, these two can no longer be treated as constants in differentiations and integrations! The sketch below vividly shows the above:

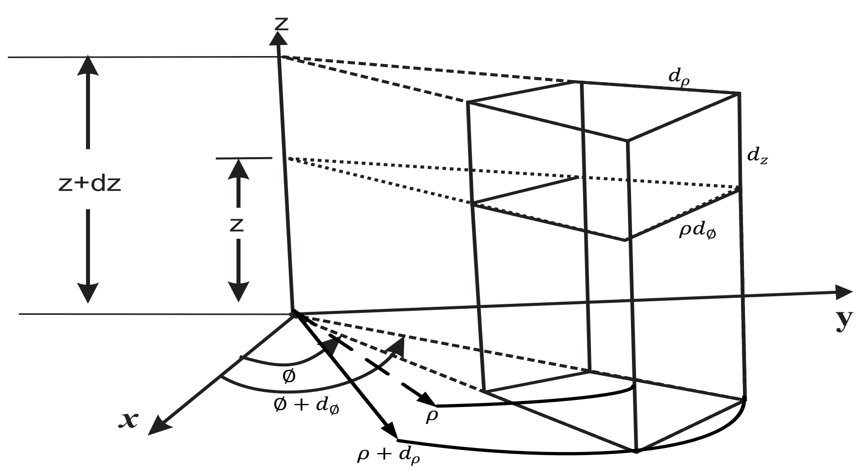


Figure 1.17 Differential Volume of a Cube

Shown above is a differential volume whose sides are the respective increments in $\rho, \emptyset\ and\ z$. Note, however, that the increment counterclockwise, is not just $\emptyset$, but rather

$\rho d\emptyset$, as a quick mental work would readily discern. The incremental volume is $d\rho\ (\rho d\emptyset) dz = \rho d\rho d\emptyset\ dz$. The areas of the differential volume are:

Frontal= $\rho d\emptyset dz$; left ("elevation") = $d\rho\ dz$; right ("elevation") =left =$d\rho\ dz$; projection downward=projection upward=$(d\rho)\rho d\emptyset = \rho d\rho d\emptyset$

1.19.1 Relating Cylindrical and Rectangular Coordinate Systems

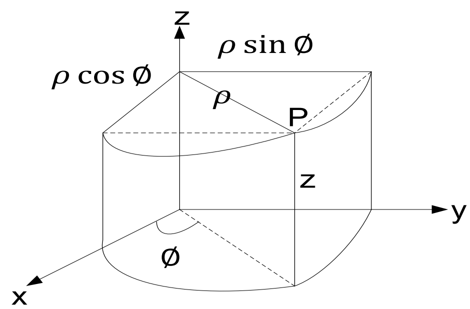


Figure 1.18 (a) Relating Rectangular and Cylindrical coordinate systems

In the Fig. 1.18 above relating rectangular and cylindrical coordinate systems,

$$x = \rho\cos\emptyset,\ \ y = \rho\sin\emptyset, \qquad z = z$$

From the above, $\quad x^2 + y^2 = (\rho\cos\emptyset)^2 + (\rho\sin\emptyset)^2$

$$x^2 + y^2 = \rho^2(cos^2\emptyset + sin^2\emptyset) = \rho^2(1) = \rho^2$$

$$\Longrightarrow\ \rho = \sqrt{x^2 + y^2} \quad (\rho\ \geq 0)\ \ (\rho\ is\ never\ negative)$$

$$\tan\emptyset = \frac{\rho\sin\emptyset}{\rho\cos\emptyset} = \frac{x}{y} \Longrightarrow \emptyset = tan^{-1}\left(\frac{x}{y}\right)\ \ and\ depends\ on\ signs\ of\ x\ and\ y\ \ z = z$$

So, transformations of scalar functions are easily accomplished using the above relationships, from one to the other. However, transformations of vector functions are a bit more demanding, as those require two steps to be taken as opposed to the scalar transformations.

Given a rectangular vector $\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$ to transform to a vector in cylindrical coordinates

$$\vec{A} = A_\rho \hat{a}_\rho + A_\emptyset \hat{a}_\emptyset + A_z \hat{a}_z$$

We know that $\vec{A}.\hat{a}_\rho = |\vec{A}||\hat{a}_\rho|\cos\theta = A_\rho \times 1 \times \cos 0$

So, projection of $\vec{A}$ in $\hat{a}_\rho$ direction= A_ρ

Likewise, $\vec{A}.\hat{a}_\rho = A_\emptyset, \vec{A}.\hat{a}_z = A_z$

$$A_\rho = \vec{A}.\hat{a}_\rho = (A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z).\hat{a}_\rho = A_x\hat{a}_x.\hat{a}_\rho + A_y\hat{a}_y.\hat{a}_\rho$$

$$A_\emptyset = \vec{A}.\hat{a}_\emptyset = (A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z).\hat{a}_\emptyset = A_x\hat{a}_x.\hat{a}_\emptyset + A_y\hat{a}_y.\hat{a}_\emptyset$$

$$A_z = \vec{A}.\hat{a}_z = (A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z).\hat{a}_z = A_z\hat{a}_z.\hat{a}_z = A_z$$

[Recall that dot product of mutually perpendicular vectors is zero $(\cos 90^0 = 0)$, hence the "disappeared" terms].

From the Fig 1.18

$$\hat{a}_x.\hat{a}_\rho = (1)(1)\cos\emptyset = \cos\emptyset$$

$$\hat{a}_y.\hat{a}_\rho = (1)(1)\cos(90^o - \emptyset) = \sin\emptyset$$

$$\hat{a}_z.\hat{a}_\emptyset = (1)(1)\cos(90^o) = 0$$

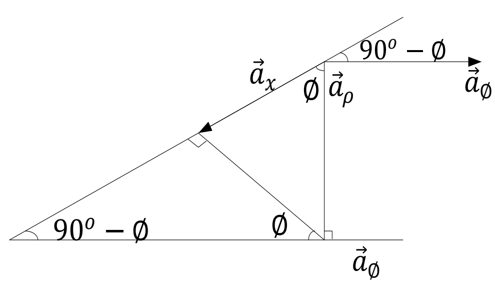


Figure 1.18 (b)

Recall that a vector is completely identified by its magnitude and direction, and not by its actual placement at a point, line, area or space. This thus informs the "sliding" of the unit vector above to join tail-to-tail with the unit vector $\hat{a}_x$, straddling each other with an angle of $180^0 - (90^0 - \emptyset) = 90^0 + \emptyset$

So, $\hat{a}_x - \hat{a}_\emptyset = \cos(90^0 + \emptyset) = \cos 90^0 \cos\emptyset - \sin 90 \sin\emptyset$,

By trigonometric identity $= -\sin\emptyset$

$$\hat{a}_x . \hat{a}_\emptyset = -\sin\emptyset$$

$$\hat{a}_y . \hat{a}_\emptyset = \cos\emptyset$$

Angle between $\hat{a}_x$ and $\hat{a}_\emptyset$ is $90^o + \emptyset$, since $\hat{a}_x \; and \; \hat{a}_y$ are mutually perpendicular, angle between $\hat{a}_y$ and $\hat{a}_\emptyset$ is $(90^o + \emptyset) - 90^o = \emptyset \Longrightarrow \hat{a}.\hat{a}_\emptyset = \cos\emptyset$

Obviously, $\hat{a}_z.\hat{a}_\rho = \hat{a}_z.\hat{a}_\emptyset = \cos 90^o = 0$

SUMMARY

	$\hat{a}_\rho$	$\hat{a}_\emptyset$	$\hat{a}_z$
$\hat{a}_x$	$\cos\emptyset$	$-\sin\emptyset$	0
$\hat{a}_y$	$\sin\emptyset$	$\cos\emptyset$	0
$\hat{a}_z$	0	0	1

From the above, then

$$A_\rho = A_x\hat{a}_x.\hat{a}_\rho + A_y\hat{a}_y.\hat{a}_\rho = A_x\cos\emptyset + A_y\sin\emptyset$$

$$A_\emptyset = A_x\hat{a}_x.\hat{a}_\emptyset + A_y\hat{a}_y.\hat{a}_\emptyset = -A_x\sin\emptyset + A_y\cos\emptyset$$

A_z=……..=Az

Example 1.27: To transform the "rectangular" vector $\vec{A} = z\hat{a}_x + x\hat{a}_y - y\hat{a}_z$ into cylindrical coordinates

Solution

$$A_\rho = \vec{A}\cdot\hat{a}_\rho = \left(z\hat{a}_x + x\hat{a}_y - ya_z\right)\hat{a}_\rho$$

$$= z\hat{a}_x \cdot \hat{a}_\rho + x\hat{a}_y \cdot \hat{a}_\rho - y\hat{a}_z \cdot \hat{a}_\rho$$

$$= z\cos\emptyset + x\sin\emptyset - y \times 0 = z\cos\emptyset + \rho\cos\emptyset\sin\emptyset$$

$$A_\emptyset = \vec{A}\cdot\hat{a}_\emptyset = z\hat{a}_x \cdot \hat{a}_\emptyset + x\hat{a}_y \cdot \hat{a}_\emptyset - y\hat{a}_z \cdot \hat{a}_\emptyset$$

$$= -z\sin\emptyset + x\cos\emptyset = -z\sin\emptyset + \rho cos^2\emptyset$$

Note that $x = \rho\cos\emptyset$

$$A_z = \vec{A}\cdot\hat{a}_z = -y = -\rho\sin\emptyset$$

Finally, $\vec{A} = (z\cos\emptyset + \rho\cos\emptyset\sin\emptyset)\ \hat{a}_\rho + (\rho\cos^2\emptyset - z\sin\emptyset)\ \hat{a}_\emptyset - \rho\sin\emptyset\hat{a}_z$

1.20 Spherical Coordinates

- Geographer specify a location on the earth's surface using three scalar value: longitude, latitude and altitude.
- Both longitude and latitude are angular measure, while altitude is a measure of distance
- Latitude, longitude and altitude are similar so spherical coordinates.
- Spherical coordinates consist of one scalar value (r), with unit of distance, while the other two scalar values $(\theta, \emptyset)$ have angular units (degree or radians), shown in Fig. 1.19

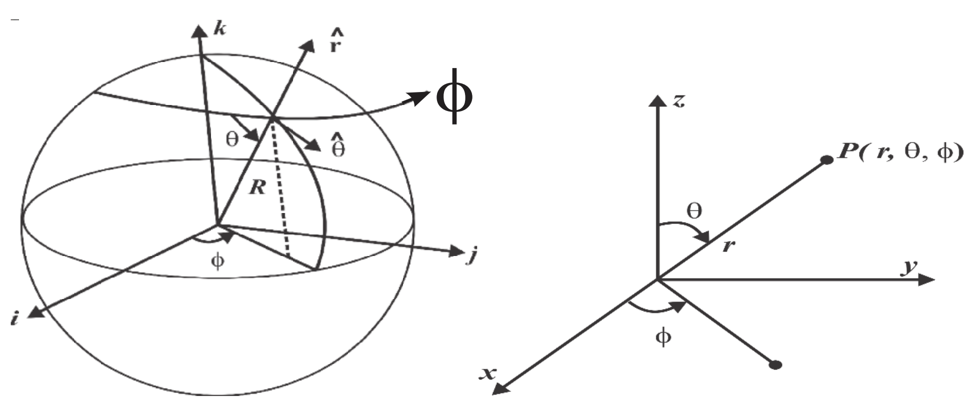


Figure 1.19 Spherical Coordinate System

1. For spherical coordinates, $r(0 \leq r < \infty)$ expresses the distance of the point from the origin (I.e similar to altitude).
2. Angle θ $(0 \leq \theta \leq \pi)$represents the angle formed with the z-axis (i.e similar to latitude). Angle θ $(0 \leq \emptyset \leq 2\pi)$ represents the rotation angle around the z-axis, precisely the same as the cylindrical to longitude) in Fig. 1.20.

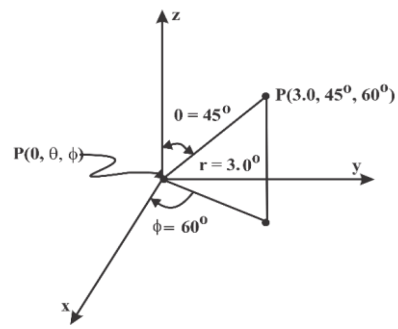


Figure 1.20 $P(3, 45^o, 60^o)$ in Spherical Coordinate System

Thus, using spherical coordinates a point in space can be unambiguously defined by one distance and two angles.

1.20.1 Spherical Base Vector

Spherical base vectors are the "natural" base vectors of a sphere in Fig. 1.21

1. $\hat{a}_r$ points in the direction of increasing r. in other words $\hat{a}_r$ points away from the origin. This is analogous to the direction we call up
2. $\hat{a}_0$ points in the direction of increasing θ. This analogous to the direction we call south

Fig. 1.21 (a) spherical base vectors (b) transformation diagram

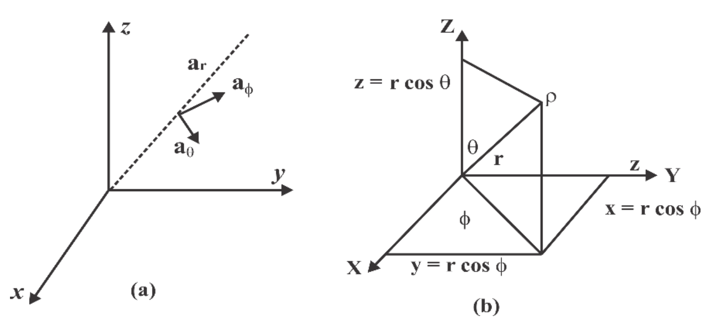


Figure 1.21 (a) Spherical Base Vector (b) Transformation Diagram

3. $\hat{a}_\emptyset$ points in the direction of increasing ∅. This is analogous to the direction we call east.

From 1.21 (b) we get

$$x = r\sin\theta\cos\emptyset, y = r\sin\theta sin\emptyset, z = r\sin\theta$$

Finally, we can write cylindrical base vectors in terms of spherical base vectors, or vice versa, using the following relationships.

$$\hat{a}_{\rho,}\cdot\hat{a}_r = \sin\theta \qquad \hat{a}_\emptyset\cdot\hat{a}_r = 0 \qquad \hat{a}_z\cdot\hat{a}_x = \cos\theta$$

$$\hat{a}_\rho\cdot\hat{a}_\theta = \cos\theta \qquad \hat{a}_\emptyset\cdot\hat{a}_\theta = 0 \qquad \hat{a}_z.\hat{a}_\theta = -\sin\theta$$

$$\hat{a}_{\rho,}\,\hat{a}_\emptyset = 0 \qquad \hat{a}_\emptyset\cdot\hat{a}_\emptyset = 1 \qquad \hat{a}_z\cdot\hat{a}_\emptyset = 0$$

$$\hat{a}_\rho = (\hat{a}_\rho\cdot\hat{a}_r)\hat{a}_r + (\hat{a}_\rho\cdot\hat{a}_\theta)\hat{a}_\theta + (\hat{a}_\rho\cdot\hat{a}_\emptyset)\,\hat{a}_\emptyset$$

$$= \sin\theta\,\hat{a}_r - \cos\emptyset\,\hat{a}_\theta$$

$$\hat{a}_\theta = (\hat{a}_\theta\cdot\hat{a}_\rho)\hat{a}_\rho + (\hat{a}_\theta\cdot\hat{a}_\emptyset)\hat{a}_\emptyset + (\hat{a}_\theta\cdot\hat{a}_z)\,\hat{a}_z$$

$$= \cos\theta\,\hat{a}_\rho - \sin\theta\,a_z$$

$$x = \rho\cos; y = \rho\sin; z = z$$

1.20.2 Spherical Coordinate System Summary

Here, the variables are r (radius); θ (angle of a cone z axis; ∅ (angle as in the foregoing cylindrical coordinate system.

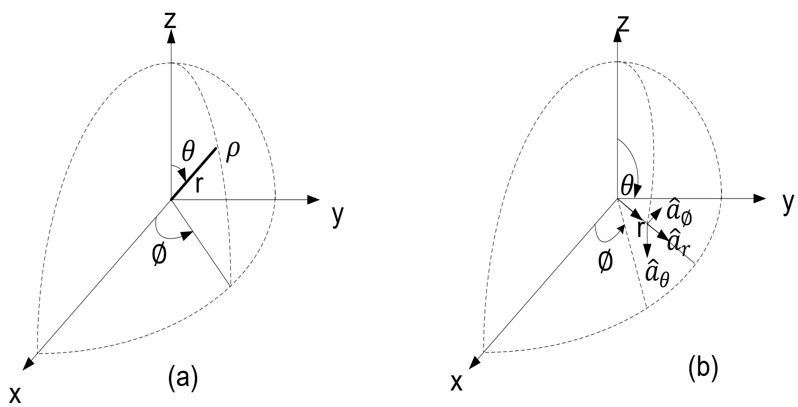


Figure 1.22 (a) Spherical coordinate $r, \theta, \emptyset$ (b) Unit vectors: $\hat{a}_r$, $\hat{a}_\theta$, $\hat{a}_\emptyset$

(1) Surface $r =$ constant = a sphere
(2) Surface θ= constant = a cone
(3) Surface ∅ = constant= a plane passing through the z axis $(\theta = 0)$, same angle as ∅ in the cylindrical coordinate system.

Any point in spherical system is thus an intersection of a sphere, a cone and a plane where are mutually perpendicular (to one another).

The unit vector $\hat{a}_r$, $\hat{a}_\theta$, $\hat{a}_\emptyset$ are mutually perpendicular, where:

(1) $\hat{a}_r$ radiates outward towards increasing r, and normal to the sphere r = constant, lying in the cone θ =constant and the plane ∅ =constant.
(2) Unit vector $\hat{a}_\theta$ lies normal to the surface of the cone, lies in the plane ∅ = constant, and is tangential to the sphere r = constant.
(3) Unit vector $\hat{a}_\emptyset$ is identical to that in the cylindrical, normal o the plane ∅ =constant, and tangential to the cone θ =constant and to the sphere r = constant.
For the transformation of scalars from the rectangular to the spherical coordinate system.
$x = r \sin\theta \cos\emptyset$, $y = r \sin\theta \sin\emptyset$, $z = r\cos\theta$

In the reverse, $x^2 + y^2 + z^2 = r^2 sin^2\theta\ cos^2\emptyset + r^2 sin^2\theta\ sin^2\emptyset + r^2 cos^2\theta$

$$= r^2[sin^2\theta(cos^2\emptyset + sin^2\emptyset) + cos^2\theta]$$

$$r^2(sin^2\theta + cos^2\theta) = r^2 = r = \sqrt{x^2 + y^2 + z^2}, r \geq 0$$

$$\theta = cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}} \ (0^o \leq \theta \leq 180^0)$$

$$\emptyset = tan^{-1}\left(\frac{y}{x}\right)$$

Note that:

$$a_z.a_r = \cos\theta\,, a_z.a_\theta = -\sin\theta\,, a_z.a_\emptyset = 0$$

The dot product of $\hat{a}_x$, $\hat{a}_y, \hat{a}_z$ “against” $\hat{a}_r$ of the three equations above relating x, y, z to their spherical equivalents, namely, $\sin\theta\cos\theta, \sin\theta\sin\emptyset\,, \cos\emptyset$, respectively. The dot products of $\hat{a}_x, \hat{a}_y, \hat{a}_z$ and both $\hat{a}_\theta$ of the spherical can be worked out by borrowing a leaf from the previously performed cylindrical system. Without going through the whole rigmarole, here’s the

	$\hat{a}_r$	$\hat{a}_\theta$	$\hat{a}_\emptyset$
$\hat{a}_x$	$\sin\theta\cos\emptyset$	$\cos\theta\cos\emptyset$	$-\sin\emptyset$
$\hat{a}_y$	$\sin\theta\sin\emptyset$	$\cos\theta\sin\emptyset$	$\cos\emptyset$
$\hat{a}_z$	$\cos\theta$	$-\sin\theta$	0

Example 1.28: Transform the vector field $G = \left(\frac{xz}{y}\right)\hat{a}_x$ into spherical coordinates

Solution: we find the three spherical components by dotting G with appropriate unit vectors and we change variables during the procedure:

$$G_r = \vec{G}\cdot\vec{a}_r = \frac{xz}{y}\hat{a}_x\cdot\hat{a}_r = \frac{xz}{y}\sin\theta\cos\emptyset$$

$$= r\sin\theta\cos\theta\frac{\cos^2\emptyset}{\sin\emptyset}$$

$$G_\theta = \vec{G} \cdot \vec{a}_\theta = \frac{xz}{y} \hat{a}_x \cdot \hat{a}_\theta = \frac{xz}{y} \cos\theta \cos\emptyset = r\, cos^2\theta\ \frac{\cos 2\emptyset}{\sin\emptyset}$$

$$G_\emptyset = \vec{G} \cdot \vec{a}_\theta = \frac{xz}{y} \hat{a}_x \cdot \hat{a}_\emptyset = \frac{xz}{y} (-\sin\emptyset) = -r\cos\theta\cos\emptyset$$

Combining the results

$$G = r\cos\theta\cos\emptyset\cot\emptyset\ \hat{a}_r + r\, cos^2\theta\ cos\emptyset\ cot\emptyset\ \hat{a}_\theta - r\cos\theta\cos\emptyset\ \hat{a}_\emptyset$$

Example 1.29: Given two points, C (-3, 2, 1) and D $(r = 5, \theta = 20^o, \emptyset = -70^0)$ find

(a) The spherical coordinates of $\vec{C}$
(b) The rectangular coordinates of $\vec{D}$
(c) Distance from C to D

Solution: we know that

(a)
$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{9 + 4 + 1} = 3.74$$

$$\emptyset = tan^{-1}\frac{y}{x} = tan^{-1}\left(\frac{2}{3}\right) = 146.3^0$$

$$\theta = cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}} = cos^{-1}\left(\frac{1}{\sqrt{14}}\right) = 74.5^0$$

$The\ Spherical\ Coordinates\ of\ \ \vec{C}\ \ are\ (r, \theta, \emptyset) = (3.74, 74.5^0, 146.3^o).$

(b)
$$x = r\sin\theta\cos\emptyset = 5\sin 20^0 \cos(70^0) = 0.585$$

$$y = r\sin\theta\sin\emptyset = 5\sin 20^0 \sin(-70^0) = -1.607$$

$$z = r\cos\theta = 5\cos(20^0) = 4.70$$

$$\therefore \qquad D(x = 0.585, y = -1.607, z = 4.70)\ Ans$$

(c) Distance $\overrightarrow{CD} = |\vec{D} - \vec{C}|$

$$= \left|(0.585 + 3)\hat{a}_x + (-1.607 - 2)\hat{a}_y + (4.70 - 1)\hat{a}_z\right|$$

$$= |3.585\hat{a}_x - 3.607\hat{a}_y + 3.70\hat{a}_z|$$

$$= \sqrt{3.585^2 + (-3.607)^2 + 3.7^2}$$

$$= 6.28 \; \boldsymbol{Ans}$$

Example 1.30: Transform the vector field $\vec{V} = (\frac{yz}{x})\hat{a}_x$ into its spherical components and variables

Proc: $V_r = \vec{V} \cdot \hat{a}_r = \left(\frac{yz}{x}\right)\hat{a}_x \cdot \hat{a}_r = \left(\frac{yz}{x}\right)\sin\theta\cos\emptyset$

$$= \left(\frac{yz}{x}\right)\sin\theta\cos\emptyset = (\sin\emptyset)\,(r\cos\theta)\sin\theta\cos\emptyset$$

$$= r\cos\theta\sin\theta\sin\emptyset$$

$$V_\theta = \vec{V} \cdot \hat{a}_\theta = \frac{\cos\theta}{\cos\emptyset}$$

$$r\,cos\,^2\theta\sin\emptyset$$

$$V_\emptyset = \vec{V} \cdot \hat{a}_\emptyset = \frac{(r\cos\theta)}{\cos\emptyset}(-\sin\emptyset)$$

$$= -\frac{r\cos\theta\; sin^2\emptyset}{\cos\emptyset} = (-r\cos\theta\sin\emptyset\tan\emptyset)$$

$$\Longrightarrow \qquad \vec{V} = (r\cos\theta\sin\theta\sin\emptyset)\,\hat{a}_r + (r\,cos^2\theta\sin\emptyset)\,\hat{a}_\theta - \frac{(r\cos\theta\; sin^2\emptyset)}{\cos\emptyset}\hat{a}_\emptyset$$

$$= r\cos\theta\sin\emptyset\,(\sin\theta\,\hat{a}_r + \cos\theta\,\hat{a}_\theta - \tan\emptyset\,\hat{a}_\emptyset)$$

1.21 Applications

The geographic coordinates system applies the two angles of the spherical coordinate system to express locations on earth, calling them latitude and longitude. Just as the two-dimensional Cartesian coordinates system is useful to the plane, a two-dimensional spherical coordinate system is useful on the surface of a sphere. In this system, the sphere is taken as a unit sphere, so the radius is unity and can generally be ignored. This simplification can also be very useful when dealing with objects such as rotational matrices.

Spherical coordinates are useful in analyzing systems that are symmetrical about a point; a sphere that has the Cartesian equation $x^2 + y^2 + z^2 + c^2$ has the very simple equation $\rho = c$ in spherical coordinates. An example is in solving a triple integral with a sphere as its domain.

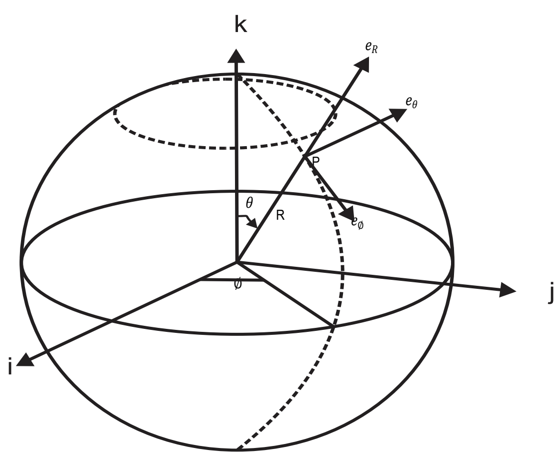
Spherical; coordinates are the natural coordinates for describing and analyzing physical situations where there is spherical symmetry, such as the potential energy surrounding a sphere (or point) with mass or charge. Two important partial differential equations, Laplace's equation and the Helmholtz equation, allow a separation of variables in spherical coordinates. The angular portions of the solutions to such equations take the form of spherical harmonics.

Another application is ergonomic design, where ρ is the arm length of a stationary person and the angles described the direction of the arm as it reaches out.

1.22 Converting Vectors Between Cartesian and Spherical Polar Bases

Let $\vec{a} = a_R\hat{\rho}_R + \hat{a}_\theta\hat{\rho}_\theta + a_\emptyset\hat{\rho}_\emptyset$ be a vector. Find the formula for the components of a in the basis {I, j, k}, i.e, find a_x, a_y, a_zsuch that $a = a_x i + a_y j + a_z k$.

It is easier to do the transformation by expressing each basis vector $\{\rho_R, \rho_{\theta,} \rho_\emptyset\}$ as component in {I, j, k) and then substituting. To do this recall that $r = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$



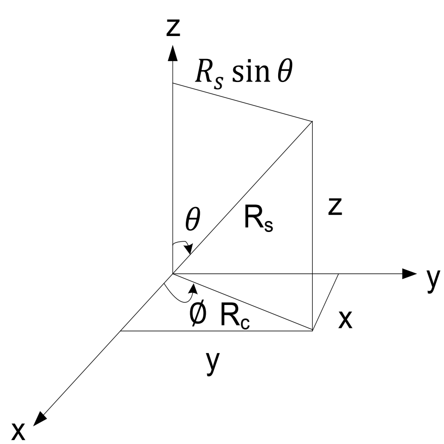


Fig 1.26 conversion between Cartesian and spherical coordinates

$$x = R\sin\theta\cos\emptyset \qquad \Rightarrow R = \sqrt{x^2+y^2+z^2} \qquad 1.33$$

$$y = R\sin\theta\sin\emptyset \qquad \Rightarrow R = \theta = cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \qquad 1.34$$

$$z = R\cos\theta \qquad \Rightarrow \emptyset = tan^{-1}\left(\frac{y}{x}\right) \qquad 1.35$$

And finally recall that by definition

$$\rho_R = \frac{1}{\left|\frac{\partial r}{\partial R}\right|}\frac{\partial r}{\partial R} \qquad \rho_\theta = \frac{1}{\left|\frac{\partial r}{\partial \theta}\right|}\frac{\partial r}{\partial \theta} \qquad \rho_\emptyset = \frac{1}{\left|\frac{\partial r}{\partial \emptyset}\right|}\frac{\partial r}{\partial \emptyset} \qquad 1.36$$

Hence, substituting for x, y, z and differentiating

$$r = R\sin\theta\cos\emptyset i + R\sin\theta\sin\emptyset j + R\cos\theta k$$

$$\Rightarrow \qquad \frac{\partial r}{\partial R} = \sin\theta\cos\emptyset i + \sin\theta\, sin\emptyset j + \cos\theta k$$

Conveniently we find $\frac{\partial r}{\partial R} = 1$ (check this for yourself, recalling the trig simplification ($\sin^2 A + \cos^2 A = 1$)

Therefore $\rho_R = \sin\theta\cos\emptyset i + \sin\theta\, sin\emptyset j + \cos\theta k$

Similarly, $\frac{\partial r}{\partial R} = R\cos\theta\cos\emptyset i + R\cos\theta\, sin\emptyset j - R\sin\theta k$

And $\left|\frac{\partial r}{\partial R}\right| = R, so\ that$

The third basis vector follows as,

$$\frac{\partial r}{\partial R} = -Rsin\,\theta\sin\emptyset i + R\sin\theta\cos\emptyset j$$

$$and\ \left|\frac{\partial r}{\partial R}\right| = R\sin\theta, so\ that$$

$$\rho_\emptyset = -\sin\emptyset i + cos\theta j$$

Finally, substituting

$$a = a_R[sin\ \theta \cos \emptyset i + \sin \theta \sin \emptyset \text{j} + \cos \emptyset k]$$

$$+a_\theta[cos\ \theta \cos \emptyset i + \cos \theta \sin \emptyset - \sin \theta\ k] + a_\emptyset[-sin\emptyset i + cos\emptyset j]$$

Collecting terms in **I, j,** and **k** we see that

$$a_x = sin\ \theta \cos \emptyset\ a_R + \cos \theta \cos \emptyset\ a_\emptyset - \sin \emptyset a_\emptyset \quad 1.37$$

$$a_y = sin\ \theta \sin \emptyset\ a_R + \cos \theta \sin \emptyset\ a_\emptyset + \cos \emptyset a_\emptyset \quad 1.38$$

$$a_z = \cos \theta\ a_R - \sin \emptyset a_\emptyset \quad 1.39$$

If you like matrices, this transformation can be expressed as

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} \sin \theta\ cos\emptyset & \cos \theta \cos \emptyset & -\sin \emptyset \\ \sin \theta \sin \emptyset & \cos \theta \sin \emptyset & \cos \emptyset \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} a_R \\ a_\theta \\ a_\emptyset \end{bmatrix}$$

Conversely, let $a = a_x i + a_y j + a_z k$. Find components

$$a = a_R\ \rho_R + a_\theta\ \rho_\theta + a_\emptyset \rho_\emptyset$$

This time, we can use the formal approach present in section 2. We have

$$a = a_R \rho_R + a_\theta \rho_\theta + a_\emptyset \rho_\emptyset = a_x i + a_y j + a_z k$$

$$\Rightarrow \qquad a.\rho_R = a_R = a_x i.\rho_y j.\rho_R + a_R + a_2 k.a_R$$

(Where we have $\rho_\theta.\rho_\emptyset = \rho_\emptyset.\rho_\theta = 0$). Recall that

$$\rho_R = \sin \theta \cos \emptyset i + \sin \theta \sin \theta j + \cos \theta k$$

$$\Rightarrow \qquad i.\rho_R = sin\theta \cos \emptyset\ j.\rho_R = \sin \theta \sin \emptyset\ k.\rho_R = \cos \theta$$

Substituting, we get

$$a_R = \sin \theta \cos \emptyset a_x + \sin \theta \sin \emptyset a_y + \cos \emptyset a_z \quad 1.40$$

Proceeding in exactly the same way for the other two components

$$a_\emptyset = \cos \theta \cos \emptyset\ a_x + \cos \theta \sin \emptyset a_y - \sin \emptyset a_z \quad 1.41$$

$$a_\emptyset = -\sin\emptyset a_x + \cos\emptyset a_y \qquad 1.42$$

In matrix from

$$\begin{bmatrix} a_R \\ a_\theta \\ a_\emptyset \end{bmatrix} = \begin{bmatrix} \sin\theta\ cos\emptyset & \sin\theta\sin\emptyset & \cos\theta \\ \cos\theta\cos\emptyset & \cos\theta\sin\emptyset & -\sin\theta \\ -\sin\emptyset & \cos\emptyset & 0 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

(Comparing this result to the transformation from spherical to rectangular coordinates, we notice that the matrices involved in the transformation have a neat property-for each matrix, its inverse is equal to its transpose).

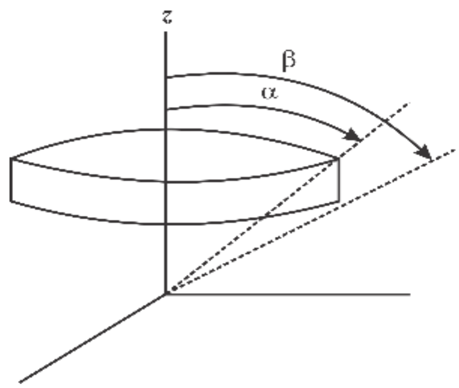


Figure 1.24

Example 1.31: Use spherical coordinate system to find area of strip $a \leq \theta \leq \beta$ on the spherical shell of radius “a” in Fig. 2.24. What will be the result when $\alpha = 0\ and\ \beta = \pi$?

Solution: the differential surface element is

$$\overrightarrow{ds} = r^2 \sin\theta\ d\theta\ d\emptyset$$

$$then\ A = \int_0^{2\pi} \int_\alpha^\beta \alpha^2 \sin\theta\ d\theta\ d\emptyset$$

$$= a^2[-\cos\theta]_0^\beta\ [\emptyset]_0^{2\pi}$$

$$= a^2[\cos\alpha - cos\beta] \times\ 2\pi$$

$$A = 2\pi\alpha^2(\cos\alpha - \cos\beta)$$

Also, when $\alpha = 0\ and\ \beta = \pi\ we\ get\ A = 4\pi a^2$ which is surface area of entire sphere.

Example 1.32: Obtain the expression for the volume of a sphere of radius 'a' from the differential volumes.

Solution: Differential volume element

$$dv = r^2 \sin\theta\ d\theta\ d\emptyset$$

Thus,

$$v = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 \sin\theta\ drd\theta d\emptyset = \left[\frac{r^2}{3}\right]_0^a [-cos\theta]_0^{\pi}[\emptyset]_0^{2\pi}$$

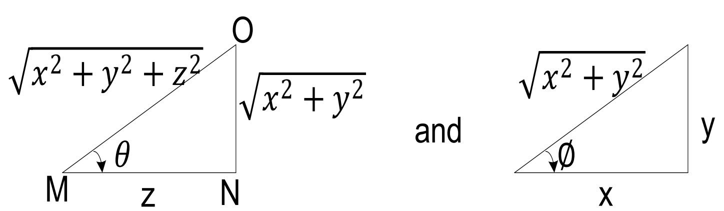
$$= \frac{a^3}{3} \times 2 \times 2\pi = \frac{4}{3}\ \pi\alpha^3$$

Example 1.33: Transform vector $\vec{A} = r\hat{r} + 2\sin\emptyset\ \hat{\theta} + 2\cos\theta\ \widehat{\emptyset}$ in spherical coordinate system to Cartesian coordination system.

Solution: relation between spherical and Cartesian coordinate is given by Equ. (1.40).

$$From \qquad \emptyset = tan^{-1}\left(\frac{y}{x}\right)\ and\ \theta = cos^{-1}\left(\frac{z}{r}\right) \Rightarrow cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

We can represent it on right angled triangle as:



$$\sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}; \sin\emptyset = \frac{y}{\sqrt{x^2+z^2}}; \cos\emptyset = \frac{x}{\sqrt{x^2+y^2}}\ ;\ \cos\theta = \frac{z}{\sqrt{x^2+y^2+}\,z^2}$$

$$A_x = \vec{A}\cdot\vec{a}_x = (r\hat{r}\cdot\hat{a}_x + 2\sin\emptyset\ \hat{\theta}\cdot\hat{a}_x + 2\cos\theta\ \widehat{\emptyset}\cdot\hat{a}_x)$$

$$= r\sin\theta\cos\emptyset + 2\cos\theta\sin\emptyset\cos\emptyset - 2\cos\theta\sin\emptyset$$

$$= \left(x + \frac{2z}{\sqrt{x^2+y^2+z^2}} \times \frac{y}{\sqrt{x^2+y^2}} \times \frac{x}{\sqrt{x^2+y^2}} - \frac{2z}{\sqrt{x^2+y^2+z^2}} \times \frac{y}{\sqrt{x^2+y^2}}\right)$$

$$= x + \frac{2xyz}{(x^2+y^2)\sqrt{x^2+y^2+z^2}} - \frac{2yz}{\sqrt{(x^2+y^2)}\sqrt{x^2+y^2+z^2}}$$

Similarly, $A_y = \vec{A} \cdot \vec{a}_y = r\hat{r}.\hat{a}_y + 2\sin\emptyset\,\hat{\theta}.\hat{a}_y + 2\cos\theta\,\hat{\emptyset}.\hat{a}_y$

$$= r\sin\theta\sin\emptyset + 2\cos\theta\sin^2\emptyset + 2\cos\theta\cos\emptyset$$

$$= y + \frac{2\times z}{\sqrt{x^2+y^2+z^2}} \times \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + \frac{2\times z}{\sqrt{x^2+y^2+z^2}} \times \frac{x}{\sqrt{x^2+y^2}}$$

$$= y + \frac{2zy^2}{(x^2+y^2)\sqrt{x^2+y^2+z^2}} + \frac{2xz}{\sqrt{(x^2+y^2+z^2)(x^2+y^2)}}$$

And $\qquad A_z = \vec{A}.\hat{a}_z = r\hat{r}.\hat{a}_z + 2\sin\emptyset\,\hat{\theta}.\hat{a}_z + 2\cos\theta\,\hat{\emptyset}.\hat{a}_z$

$$= r\cos\theta - 2\sin\theta\sin\emptyset$$

$$= z - \frac{2y}{\sqrt{x^2+y^2+z^2}}$$

$$\therefore \quad \vec{A} = \left[x + \frac{2xyz}{(x^2+y^2)\sqrt{x^2+y^2+z^2}} - \frac{2yz}{\sqrt{(x^2+y^2+z^2)(x^2+y^2)}}\right]\hat{a}_x$$

$$+ \left[y + \frac{2y^2z}{(x^2+y^2)\sqrt{x^2+y^2+z^2}} + \frac{2xz}{\sqrt{(x^2+y^2+z^2)(x^2+y^2)}}\right]\hat{a}_y$$

$$+ \left[z - \frac{2y}{\sqrt{x^2+y^2+z^2}}\right]\hat{a}_z \;\; \boldsymbol{Ans}$$

1.22.1 Converting Between Cylindrical and Rectangular Cartesian Coordinates

The formulas below convert from Cartesian (x, y, z) coordinates to cylindrical polar $r, \emptyset, z$ coordinates and back again.

$$x = r\cos\emptyset \qquad r = \sqrt{x^2 + y^2}$$

$$y = r\sin\emptyset \qquad \emptyset = tan^{-1}\left({}^{y}/_{x}\right)$$

$$z = z \qquad z = z$$

1.22.2 Cylindrical-Spherical Representation of Vectors

	$\hat{r}_s$	$\hat{\theta}$	$\hat{\emptyset}$
$\hat{r}_c$	$\sin\theta$	$\cos\theta$	0
$\hat{\emptyset}$	0	0	1
$\hat{z}$	$\cos\theta$	$-\sin\theta$	0

Refers to Fig. 1.25, when using cylindrical-polar coordinates, all vectors are expressed as components in the basis $(e_r, e_\emptyset, e_z)$ shown. In words.

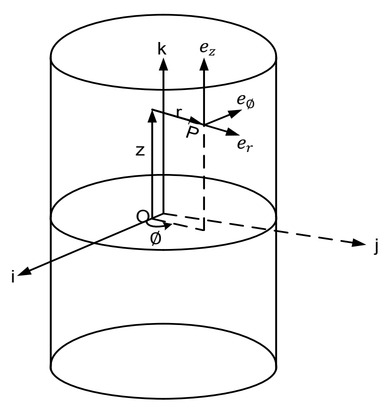


Figure 1.25 Conversion between Base Vectors

e_r is a unit vector normal to the cylinder at P

$e_\emptyset$ is a unit vector circumferential to the cylinder at P, chosen to make $(e_r, e_\emptyset, e_z)$ a right-handed trail.

e_z is parallel to the k vector

You will see that the position vector of point P would be expressed as

$$r = re_r + ze_z = r\cos\emptyset i + r\sin\emptyset j + zk$$

Note also that the basis vectors are intentionally chosen to satisfy

$$e_r = \frac{1}{\left|\frac{\partial r}{\partial r}\right|}\frac{\partial r}{\partial r}\ e_\emptyset = \frac{1}{\left|\frac{\partial r}{\partial \emptyset}\right|}\frac{\partial r}{\partial \emptyset}\ e_z = \frac{1}{\left|\frac{\partial r}{\partial z}\right|}\frac{\partial r}{\partial z} \quad 1.43$$

And there is therefore the natural basis for the coordinate system

1.22.3 Converting Vectors Between Cylindrical and Cartesian Bases

Let $a = a_r e_r + a_\emptyset e_\emptyset + a_z e_z$ be a vector, expressed as components in $(e_r, +e_\emptyset + e_z)$. It is straight forward to show that the component of a in (I, j, k) (a=$a_x i + a_y j, +a_z k$) are (as in Fig. 1.25).

$$a_x = a_r\cos\emptyset - a_\emptyset \sin\emptyset \quad 1.44$$

$$a_y = a_r\sin\emptyset - a_\emptyset \cos\emptyset \quad 1.45$$

$$a_z = a_z \quad 1.46$$

As a matrix

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} \cos\emptyset & -\sin\emptyset & 0 \\ \sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_r \\ a_\emptyset \\ a_z \end{bmatrix}$$

The reverse of this transformation is

$$a_r = a_x\cos\emptyset + a_y\sin\emptyset$$

$$a_\emptyset = -a_x\sin\emptyset + a_y\cos\emptyset$$

$$a_z = a_z$$

In matrix from

$$\begin{bmatrix} a_r \\ a_\emptyset \\ a_z \end{bmatrix} = \begin{bmatrix} \cos\emptyset & \sin\emptyset & 0 \\ -\sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_r \\ a_y \\ a_z \end{bmatrix}$$

Note: remember the cube (wherever solving numerical) on surface of object under consideration in spherical coordinates.

Similarly, we extend point $P(r, \theta, \emptyset)\ to\ Q(r + dr, \theta + d\theta)$ in spherical coordinates as let us first plot P at distance of angles $'\theta'\ and\ \emptyset'with\ 'r'$ as length extend the same by incrementing in all directions. Finally, we obtain cuboid which we generally assume as cube an any objects surface under questions.

Fig. 1.27

Terms:

$$\overrightarrow{dl} = \overrightarrow{dr}\hat{r} + r\, d\theta\, \hat{\theta} + r \sin\theta\, d\emptyset\, \widehat{\emptyset} \quad \text{(Differential length)} \quad 1.47$$

$$dv = (dv)(r \sin\theta\, d\emptyset)\ (r\, d\theta \quad \text{(Differential volume)} \quad 1.48$$

$$\overrightarrow{ds} = \pm(r\, d\theta)\hat{\theta} x\, r \sin\theta\, d\emptyset\, \widehat{\emptyset} = \pm\, r^2 \sin\theta\, d\emptyset\, \hat{r} \quad \text{(Differential area)}$$

$$= \pm(r \sin\theta\, d\emptyset\, \widehat{\emptyset})\ x\ dr\, \hat{r} = \pm r \sin\theta\, dr\, d\emptyset\, \hat{\theta} - 2.32$$

$$= \pm(dr)\hat{r}\ x\ (r\, d\theta)\hat{\theta} = \pm r\, dr\, d\hat{\theta}\, \widehat{\emptyset} \quad 1.49$$

Example 1.34: Express vector $\vec{A} = \cos\emptyset\, \hat{a}_r - 2r\, \hat{a}_\emptyset + \hat{a}_z$ in Cartesian coordinates.

Solution: Using the matrix relation

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\emptyset & -\sin\emptyset & 0 \\ \sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\emptyset \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\emptyset & -\sin\emptyset & 0 \\ \sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\emptyset \\ -2r \\ 1 \end{bmatrix}$$

This can be written as

$$\vec{A} = (cos\,^2\emptyset + 2r \sin\emptyset)\ \hat{a}_x + (\cos\emptyset \sin\emptyset - 2r \cos\emptyset)\ \hat{a}_y + \hat{a}_z$$

From cylindrical to Cartesian coordinates, we have

$$\cos\emptyset = \frac{x}{y} = \frac{x}{\sqrt{x^2 + y^2}}\ and \sin\emptyset = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\therefore \quad \vec{A} = \left(\frac{x^2}{x^2+y^2} + 2y\right)\hat{a}_y + \hat{a}_z. \;\; Ans$$

Example 1.35: Find component of vector $\vec{A} = -2z\,\hat{a}_y + 5y\,\hat{a}_z$ at point (0, -2, 3) which is directed towards the point $Q\left(\sqrt{3}, -30^0, 1\right)$.

Solution: $x = r\cos\theta = \sqrt{3}\cos(-30^0) = \sqrt{3}\times\frac{\sqrt{3}}{2} = \frac{3}{2} = 1.5$

$$y = r\sin\theta = \sqrt{3}\sin(-30^0) = -\sqrt{3}\times\frac{1}{2} = -0.866$$

$$Q = (1.5, -0.866, 1)$$

$$r_{PQ} = (1.5-0)\hat{a}_x + (-0.866+2)\hat{a}_y + (1-3)\hat{a}_z$$

$$r_{PQ} = 1.5\hat{a}_x + 1.13\hat{a}_y - 2\hat{a}_z$$

Unit vector is now:

$$a_{PQ} = \frac{1.5\hat{a}_x + 1.13\hat{a}_y - 2\hat{a}_z}{\sqrt{(1.5)^2 + (1.13)^2 + (2)^2}}$$

Component of vector $\vec{A}$ at point P(0,-2,3) towards point Q

$$\vec{A}\cdot\vec{a}_{PQ} = (-2)(3)\,\hat{a}_y + (2)(-2)\hat{a}_z\cdot\overrightarrow{aQP}$$

$$= \left(-6\hat{a}_y - 4\hat{a}_z\right)\frac{(1.5\hat{a}_x + 1.13\hat{a}_y - 2\hat{a}_z)}{2.74} = \frac{-6.78+8}{2.74} = 0.445$$

Example 1.36: Use cylindrical coordinate system to find the area of curved surface of a right circular cylinder where r=3m, h=4m and $30^o \le \emptyset \le 120^o$

Solution: the different surface element is

$$ds = r\,d\emptyset\,dz$$

$$A = \int_0^4\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} d\emptyset dz = 4\times\left(\frac{2\pi}{3} - \frac{\pi}{6}\right) = 4\times\left[\frac{4\pi-\pi}{6}\right] = 4\times\frac{3\pi}{6}$$

$$\boxed{A = 2\pi m^2}\,Ans$$

Table 1.1 Dot product of unit vectors in three coordinate systems

		Rectangular			Cylindrical			Spherical		
	$\cdot$	$\hat{x}$	$\hat{y}$	$\hat{z}$	$\hat{r}$	$\widehat{\emptyset}$	$\hat{z}$	$\hat{r}$	$\hat{\theta}$	$\widehat{\emptyset}$
Rectangular	$\hat{x}$	1	0	0	$\cos\emptyset$	$-\sin\emptyset$	0	$\sin\theta\cos\emptyset$	$\cos\theta\cos\emptyset$	$-\sin\emptyset$
	$\hat{y}$	0	1	0	$\sin\emptyset$	$\cos\emptyset$	0	$\sin\theta\cos\emptyset$	$\cos\theta\cos\emptyset$	$\cos\theta$
	$\hat{z}$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
Cylindrical	$\hat{r}$	$\cos\emptyset$	$\sin\emptyset$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
	$\widehat{\emptyset}$	$-\sin\emptyset$	$\cos\emptyset$	0	0	1	0	0	0	1
	$\hat{z}$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
Spherical	$\hat{r}$	$\sin\theta\cos\emptyset$	$\sin\theta\sin\emptyset$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
	$\hat{\theta}$	$\cos\theta\cos\emptyset$	$\cos\theta\sin\emptyset$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
	$\widehat{\emptyset}$	$-\sin\theta$	$\cos\emptyset$	0	0	1	0	0	0	1

Note that the unit vectors $\hat{r}$ in the cylindrical and spherical systems are not the same. For example

Spherical	cylindrical	Rectangular
$\hat{r}\cdot\hat{x} = \sin\theta\cos\emptyset$	$\hat{r}\cdot\hat{x} = \cos\emptyset$	$x = \mathrm{r}\sin\theta\cos\emptyset$
$\hat{r}\cdot\hat{y} = \sin\theta\sin\emptyset$	$\hat{r}\cdot\hat{y} = \sin\emptyset$	$y = \mathrm{r}\sin\theta\sin\emptyset$
$\hat{r}\cdot\hat{z} = \cos\theta$	$\hat{r}\cdot\hat{z} = 0$	$z = \mathrm{r}\cos\theta$

FORMULAS

$$\nabla V = \frac{\partial v}{\partial x}\hat{x} + \frac{\partial v}{\partial y}\,\hat{y} + \frac{\partial v}{\partial z}\,\hat{z}$$

Cartesian coordinates

$$= \frac{\partial v}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial v}{\partial \emptyset}\,\widehat{\emptyset} + \frac{\partial v}{\partial z}\,\hat{z}$$

Cylindrical coordinates

$$= \frac{\partial v}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial v}{\partial \emptyset}\,\widehat{\emptyset}$$

Spherical coordinates

$\nabla.J = \frac{\partial Jx}{\partial x} + \frac{\partial Jy}{\partial y} + \frac{\partial Jz}{\partial z}$
Cartesian coordinates

$= \frac{1}{r}\frac{\partial}{\partial r}(rJr) + \frac{1}{r}\frac{\partial J_\emptyset}{\partial \emptyset} + \frac{\partial J_z}{\partial z}$ Cylindrical coordinates

$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 Jr)\frac{1}{r\sin\theta}\frac{\partial v}{\partial \theta}\ (\sin\theta)\, J\theta + \frac{1}{r\sin\theta} + \frac{1}{r\sin\theta}\frac{\partial J\emptyset}{\partial \emptyset}$
Spherical coordinates

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$
Cartesian coordinates

$$= \begin{vmatrix} \frac{\hat{r}}{r} & \widehat{\emptyset} & \frac{\hat{z}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \emptyset} & \frac{\partial}{\partial z} \\ B_r & rB_\emptyset & B_z \end{vmatrix}$$
Cylindrical coordinates

$$= \begin{vmatrix} \frac{\hat{r}}{r^2\sin\emptyset} & \frac{\widehat{\emptyset}}{r\sin\emptyset} & \frac{\widehat{\emptyset}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \emptyset} \\ B_r & rB_r & r\sin\theta\, B_\theta \end{vmatrix}$$
Spherical coordinates

1.23 Exercise

1. Given three vectors

$$\vec{A} = 2\hat{x} + 2\hat{y} - \hat{z}$$

$$\vec{B} = \hat{x} - 3\hat{y} - 4\hat{z}$$

$$\vec{C} = \hat{x} - \hat{y} + \hat{z}$$

Find: (a) $\vec{A}-\vec{B}+2\vec{C}$ (b) The unit vector along $\vec{A}-2\vec{C}$ (c) $\vec{B}.\vec{C}$ (d) $\vec{A}\times\vec{B}$ (e) $\vec{A}\times\vec{B}.\vec{C}$

2. Three points P_1, P_2 and P_3 are given by (2, 3-2), (5,8,3) and (7,6,2) respectively obtain.
   (a) Vector drawn from P_1, to P_2
   (b) Unit vector along the line from P_1, to P_3
3. A vector field is given by $\vec{E}=y\hat{x}-2.5x\hat{y}+3\hat{z}$ at a point P(4, 5, 2). Calculate:
   a. The field $\vec{E}$ at point P
   b. A scalar component of $\vec{E}$ in dir^n of vector $\vec{A}=\frac{1}{3}(2\hat{x}+\hat{y}+2\hat{z})$ at point P
   c. The angle between $\vec{E} and\ \vec{A}\ at\ P$.
4. Given the points $P(r=5,\emptyset=60^o,z=2) and\ Q(r=2,\emptyset=110^o,z=-1)$:
   a. Find distance from P to Q
   b. Find unit vector towards the direction P to Q
5. Two points are given as P(2, -1, -3) and Q(1,3,4). Give the vector that extends from P to Q in
   a. Cartesian
   b. Cylindrical
   c. Spherical
6. An electric field intensity is given as $\vec{E}=\left(\frac{100\cos\theta}{r^3}\right)\hat{r}+\left(\frac{50\sin\theta}{r^3}\right)\hat{\theta}$. Calculate $|\vec{E}|$ and a unit vector in Cartesian coordinates in direction of $\vec{E}$ at point (r=2, $\theta=60^o,\emptyset=20^o$).
7. Give that $\vec{A}=\left(\frac{5r^3}{4}\right)\hat{r}\ c/m^2$ in spherical coordinates, evaluate both sides of divergence theorem for volume enclosed by r=1m, r=2m.
8. Find the rate at which the scalar function $v=r^2\sin 2\emptyset$, in cylindrical coordinates, increases in the direction of vector $\vec{A}=\hat{r}+\hat{\emptyset}$ at the point $\left(2,\frac{\pi}{4},0\right)$.
9. $\vec{A}=2r\cos\emptyset\hat{r}+r\hat{\emptyset}$. Verify Stokes theorem, in cylindrical coordinates in region between closed curves C_1 and C_2
10. A scalar function is given by V(X,Y,Z)= xy. Find a unit vector normal to constant V surface of value 2 at point (2,1,0)
11. For a vector field $F=xy^2\hat{x}+yz^2\hat{y}+2xz\hat{z}$; calculate the line integral $\int_c\vec{F}dl$, where c is a straight line between points (0,0,0) and (1, 2, 3).
12. In cylindrical coordinates (4<r<6), (30 <$\emptyset<60),(2<z<5)$. Find
    a. Volume defined by these parameters

b. Length of longest straight line that lies entirely with in volume
c. Total surface area

13. given $\vec{A} = r^2\hat{r} + r\sin\theta\,\hat{\theta}$ in spherical coordinates: Evaluate $\oint_s \vec{A}.ds$ over the following:
    a. the surface of that part of spherical volume of radius unity lying in the first octant.
    b. The surface of solid spherical shell lying between r=a and r=b where a>b.
14. Given:
    $\vec{A} = r\sin\theta\cos\emptyset\,\hat{r} - \cos 2\theta\sin\emptyset\,\hat{\theta} + \tan\frac{\theta}{2}\ in\ r\,\hat{\emptyset}\ \ at\ point\ p\left(2,\frac{\pi}{2},\frac{3\pi}{2}\right)$,
    determines the vector component of $\vec{A}$ that is
    a. Parallel to $\hat{z}$
    b. Normal to surface $\emptyset = \frac{3\pi}{2}$
    c. Tangent to surface at r=2
    d. Parallel to line y=-2, z=0
15. (a) Show that point transformation between cylindrical and spherical coordinates is given by
    (i) r=f(ρ, z); (ii) $\theta = tan^{-1}f(\rho, z)$; (iii) $\emptyset = f(\emptyset)$; (iv) $\rho = f(r,\theta)$; (v) $z = f(r,\theta)$; (vi)$\emptyset = f(\emptyset)$
    (b) (i) Given $/A/= xz - xy + yz$, express /A/ in cylindrical coordinates.
    (ii) Given /B/=x^2-2y^2+3z^2, express /B/ in spherical coordinates.
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16. (a) Convert point A (0,-4,3) from Cartesian to cylindrical and spherical coordinates.
    (b) Describe the intersection of the following surface: (i)x=1, y=2 (ii) x=3, y=-2, z=5 (iii) r=10, θ=30^0 (iv) $p = 10, \emptyset = 50^0$ (v) $\emptyset = 40^0, z = 8$ (vi) r=4, $\emptyset = 30^0$
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CHAPTER 2

INTEGRAL THEOREMS

2.0 Stokes Theorem

Consider a vector field $B(\bar{r})$; where:

$$\overline{B(\bar{r})} = \vec{\nabla} \times \overrightarrow{A(\bar{r})}$$

Say we wish to integrate this vector field over an open surface S:

$$\iint_S \overline{B(\bar{r})}\,.\,\overline{ds} = \iint_S \overline{B(\bar{r})}.\overrightarrow{\mathrm{ds}} = \iint_S \bar{\nabla} \times \overrightarrow{A(\bar{r})}.\overline{ds}$$

We can likewise evaluate this integral using Stokes theorem.

In this case, the contour C is a closed contour that surrounds surface S. the direction of C is defined by $\overline{ds}$ and the right-hand rule. In other words, C rotates counterclockwise around $\overline{ds}$, as in Fig 2.1

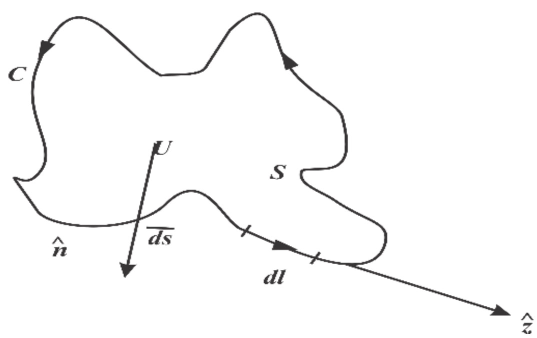


Figure 2.1 Surface with Contour C

$$\iint_S \nabla \times A(\bar{r}).\overline{ds} = \oint_C A(\bar{r}).\overline{dl} \qquad 2.1$$

$$(\overrightarrow{ds}) = (\hat{n}ds)$$

$$(\overline{dl}) = (\hat{z}dl)$$

- Stokes's theorem allows us to evaluate the surface integral of a curl as simply a contour integral!

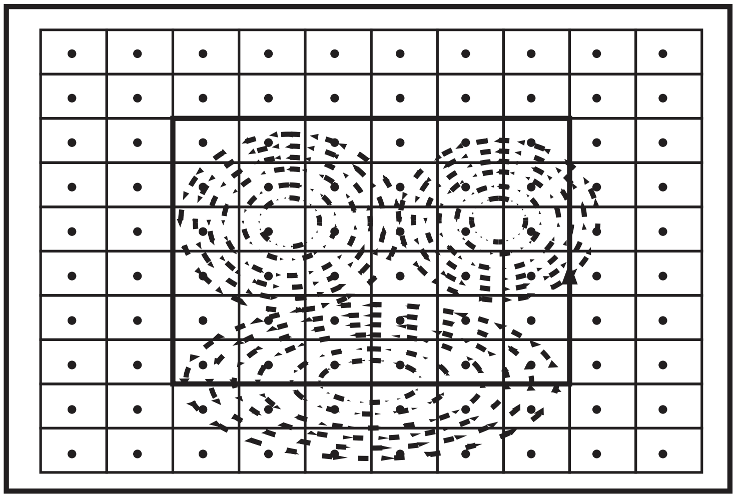


Fig 2.2 Vector field

- Stokes's theorem state that the summation (i.e. integration) of the circulation at every point on a surface is simply the total "circulation" around the closed contour surrounding the surface.

In other words, if the vector field is rotating counterclockwise around some point in the volume, it must simultaneously be rotating clockwise around adjacent points within the volume-the net effect is therefore zero! As in Fig. 2.2.

Thus, the only values that make any difference in the surface integral is the rotation of the vector field around points that lie on the surrounding contour (i.e. the very edge of the surface S). these vectors are likewise rotating in the opposite direction around adjacent points- but these points do not lie on the surface, (thus, they are not included in the integration). The net effect is therefore non-zero!

Note that if S is a closed surface, then there is contour C that exists! In other words.

$$\oiint_S \nabla \times A(\bar{r}).\overline{ds} = \oint_o A(\bar{r}).\overline{dl} = 0 \qquad 2.2$$

Therefore, integrating the curl of any vector field over a closed surface always equals zero.

Example 2.1: A numerical example may help you to illustrate the geometry involved in Stokes theorem. Consider the portion of a sphere shown in Fig. 2.1. the surface is specified by $r = 4, 0 \leq \theta \leq 0.1\pi, 0 \leq \emptyset \leq 0.3\pi$, and the closed path forming its perimeter is composed of three circular arcs. We are given the field $\vec{H} = 6r\sin\emptyset\,\hat{a}_r + 18r\sin\theta\cos\emptyset\,\hat{a}_\emptyset$ and hence evaluate each side of Stokes theorem.

Solution: the first path segment is described in spherical coordinates by $r = 4, 0 \leq \theta \leq 0.1\pi, \emptyset = 0$; the second one by $r = 4, \theta = 0.1\pi, 0 \leq \emptyset \leq 0.3\pi$; and the third by $r = 4, 0 \leq \theta \leq 0.1\pi, \emptyset = 0.3\pi$.

The differential path element $\overline{dl}$ is.

$$\overline{dl} = dr\,\hat{a}_r + rd\theta\,\hat{a}_\theta + r\sin\theta\,d\emptyset\,\hat{a}_\emptyset$$

I=0 on all three segments as $r = 4\ and\ dr = 0$

II=0 on segment 2 since θ =constant

III=0 on segment 1 and 3

$$\oint \vec{H}.\overline{dl} = \int_1 H_\theta\,rd\theta + \int_2 H_\emptyset\,r\sin\theta\,d\emptyset + \int_3 H_\theta\,rd\theta$$

Since, $H_\theta = 0$, we have to evaluate only second

$$\oint \vec{H}.\overrightarrow{dl} = \int_0^{3\pi} [18(4)\sin(\theta.1\pi)\cos\emptyset]\,4\sin(0.1\pi)d\emptyset$$

$$= 288\,sin^2 0.1\pi\sin 0.3\pi = 22.2A$$

Next, evaluate surface integral

$$\vec{\nabla}\times\vec{H} = \frac{1}{r\sin\theta}\left(\frac{dH_\emptyset\sin\theta}{d\theta} - \frac{dH_\theta}{d\emptyset}\right)\hat{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{dH_r}{d\emptyset} - \frac{d(rH_\emptyset)}{dr}\right)\hat{a}_\theta + \frac{1}{r}\left(\frac{d(rH_\theta)}{dr} - \frac{dH_r}{d\theta}\right)\hat{a}_\emptyset$$

$$= \frac{1}{r\sin\theta}(36r\sin\theta\cos\theta\cos\emptyset)\,\hat{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\ 6r\cos\emptyset - 36r\sin\theta\cos\emptyset\right)\hat{a}_\theta$$

$$dS = r^2\sin\theta\, d\theta\, d\emptyset\, \hat{a}_r, the\ integral\ is$$

$$\int_S (\vec{\nabla} \times \vec{H}\,.\,\overrightarrow{ds}) = \int_0^{0.3\pi}\int_0^{0.1\pi}(36\cos\theta\cos\emptyset)\,(16\sin\theta)\,d\theta\,d\emptyset$$

$$= \int_0^{0.3\pi} 576\left(\frac{1}{2}sin^2\theta\right)|_0^{0.1\pi}\ \cos\emptyset\, d\emptyset$$

$$= 288\ sin^2\ 0.1\pi\ 0.3\pi = 22.2\ A$$

Thus, the results check Stokes Theorem and we note in passing that a current of 22.2A is flowing upwards through this section of a spherical cap.

Example 2.2: A vector $\vec{A}$ is represented in X-Y plane as $\vec{A} = -y\,\hat{a}_x + \hat{a}_y$.

Calculate curl $\vec{A}$ and line integral $\oint \vec{A}.\overrightarrow{dl}$ for the closed curve $x^2 + y^2 = r^2, z = 0$. Hence verify Stokes theorem.

Solution:

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ -y & x & 0 \end{vmatrix}$$

$$= \hat{a}_x\left[0 - \frac{d}{dz}\,(x)\right] + \hat{a}_y\left[\frac{d}{dz}(-y) - 0\right] + \hat{a}_z\left[\frac{d(x)}{dx} - \frac{d}{dy}(-y)\right]$$

$$\boxed{= \vec{\nabla} \times \vec{A} = 2\hat{a}_z}$$

As, $\vec{A} = -y\hat{a}_x + x\hat{a}_y$

$$|A| = \sqrt{x^2 + y^2} = r$$

$$\oint \vec{A}.\overrightarrow{dl} = \oint \vec{r}.\overrightarrow{dl} = r\oint dl = r.2\pi r = 2\pi r^2$$

In X-Y plane, normal to surface will be along Y-axis, so that $\overrightarrow{ds} = \hat{a}_z.ds$

$$\int_S \vec{\nabla} \times \overrightarrow{\boldsymbol{A}}.\overrightarrow{\boldsymbol{ds}} = \oint_S 2\hat{a}_z.\hat{a}_z ds = 2\oint_S ds = 2.\pi r^2$$

Comparing Equs (i) and (ii) Stokes theorem is verified.

Example 2.3: Evaluate $\iint_S \vec{F}.\hat{n}\,\overrightarrow{ds}$ where $\vec{F} = x^2\hat{a}_x + y^2\hat{a}_y + z^2\hat{a}_z$ and S is surface of cube bounded by $x = 0, x = 5, y = 0, y = 5, z = 0, z = 5$ as shown in the Fig. 2.2

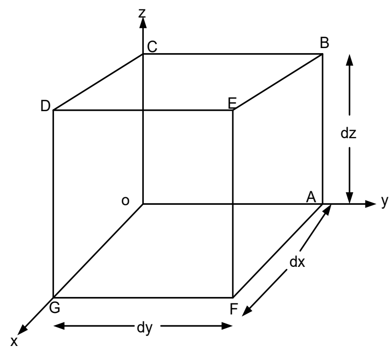


Figure 2.2

Solution: for face DEFG

$$\hat{x} = \hat{a}_x$$

$$x = 5$$

$$dx = 0$$

$$ds = dydz$$

And

$$\iint_{DEFG} \vec{F}\cdot\hat{n}\,ds = \int_0^5\int_0^5 \left(x^2\hat{a}_x + y^2\hat{a}_y + z^2\hat{a}_z\right).\hat{a}_x\,dydz = \int_0^5\int_0^5 dydz = 25$$

For face OABC; $x = 0\ dx = 0\ \hat{n} = -\hat{a}_x\ \ and\ \ ds = dydz$

$$\iint_{OABC} \vec{F}\cdot\hat{n}\,ds = \int_0^5\int_0^5 \left(y^2\hat{a}_y + \hat{z}\hat{a}_z\right)(-\hat{a}_x dydz) = 0$$

For ABEF, $y = 1, dy = 0, \hat{n} = ay \;\; and\; ds = dxdz$

$$\iint\limits_{ADEF} \vec{F}\cdot\hat{n}\, ds = \int_0^5\int_0^5 \left(x^2\hat{a}_x + y^2\hat{a}_y + z^2\hat{a}_z\right).\hat{a}_y\, dzdx = \int_0^5\int_0^5 dydz = 25$$

For face OCDG, $y = 0, dy = 0, \hat{n} = -\hat{a}_y \;\; and\; ds = dxdz$

$$\iint\limits_{OCDG} \vec{F}\cdot\hat{n}\, ds = \int_0^5\int_0^5 (x^2\hat{a}_x + z^2\hat{a}_z)(-\hat{a}_y)\, dxdz = 0$$

For face EBCD, $z = 1\; dz = 0, \hat{n} = \hat{a}_z \;\; and\; ds = dxdy$

$$\iint\limits_{EBCD} \vec{F}.\hat{n}\, ds = \int_0^5\int_0^5 \left(x^2\hat{a}_x + y^2\hat{a}_y + z^2\hat{a}_z\right)\cdot\; \hat{a}_x\, dxdz = \int_0^5\int_0^5 dx\, dz = 25$$

For face $OAFG, Z = 0, dz = 0\;\hat{n} = \hat{a}_z \;\; and\; ds = dxdy$

$$\iint\limits_{OAFG} \vec{F}\cdot\hat{n}\, ds = \int_0^5\int_0^5 \left(x^2\hat{a}_x + y^2\hat{a}_y\right)(-\hat{a}_z)\, dxdy = 0$$

The total surface integral about surface S of cube will be obtained on adding equations.

$$\iint\limits_{S} \vec{F}\cdot\hat{n}\, ds = 25 + 0 + 25 + 0 + 25 + 0 = 75$$

2.1 Divergence Theorem

Recall the studied volume integrals of the form:

$$\iiint\limits_{V} g(\bar{r})dv$$

It turns out that any and every scalar field can be written as the divergence of some vector field, *i.e.*

$$g(\bar{r}) = \nabla\cdot A\,(\bar{r})$$

Therefore, we can equivalently write any volume integrals as:

$$\iiint_V \nabla \cdot A(\bar{r})dv$$

The divergence theorem states that these integrals is equal to:

$$\iiint_V \nabla \cdot A(\bar{r})dv = \oiint_S A(\bar{r})ds \qquad 2.3$$

Where S is the closed surface that completely surrounds volume V and vector $\overline{ds}$ points outward from the closed surface. For example, if volume V is a sphere, then S is the surface of that sphere.

The divergence theorem states that the volume integrals of a scalar field can be likewise evaluated as a surface integral of a vector field!

What the divergence theorem indicates is that the total "divergence" of a vector field through the surface of any volume is equal to the sum (i.e integration) of the divergence at all points within the volume.

In other words, if the vector field is diverging from some point in the volume, if must simultaneously be converging to another adjacent point within the volume the net effect is therefore zero as in Fig. 2.3

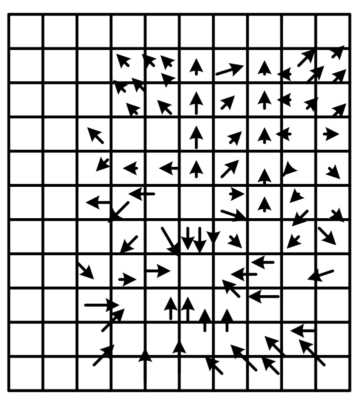


Figure 2.3 Vector Field

Thus, the only values that make any difference in the volume integral are the divergence and convergence of the vector field across the surface surrounding the volume vectors that will be converging or diverging to adjacent points outside the volume (across the surface) from points inside the volume. Since these points just outside the volume are not included in the integration. Their net effect is non-zero!

2.2 Proof of Divergence Theorem

Fig. 2.4 shows a closed surface enclosing a volume V that contains charges (or a charge density) that produce an electric flux density D.

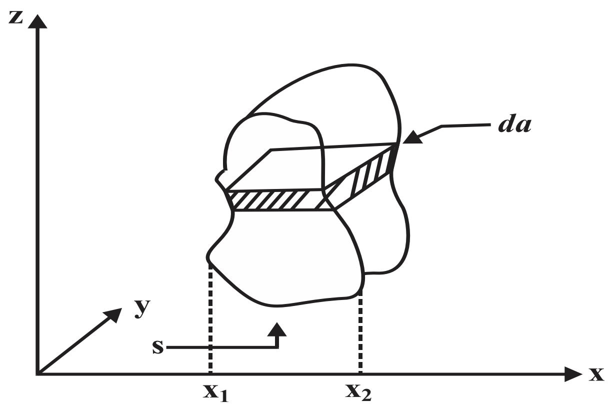


Figure 2.4 Proof of Divergence Theorem

Divergence

$$\vec{\nabla} \, . \, \vec{D} = \frac{\partial Dx}{\partial x} + \frac{\partial Dy}{\partial y} + \frac{\partial Dz}{\partial z}$$

$$\text{So, that} \quad \int_V \vec{\nabla} \cdot \vec{D} dV = \iiint \left(\frac{\partial Dx}{\partial x} + \frac{\partial Dy}{\partial y} + \frac{\partial Dz}{\partial z} \right) dxdydz \qquad 2.4$$

Let D_{x1} and D_{x2} respectively be x component of electric flux entering LHS and leaving RHS of rectangular volume.

The total flux emerging is the algebraic difference of these two.

$$D_{x2} - D_{x1} = \int_{x_1}^{x_2} \frac{\partial Dx}{\partial x} \, dx$$

$$or \quad \iiint \frac{\partial Dx}{\partial x} \, dxdydz = \iint (D_{x2} - D_{x1}) dydz \qquad 2.5$$

$dydz$ is x component of surface elements $\overrightarrow{da}$

∴ Its integration of product of D_x times $\overrightarrow{da}$

Putting (2.5) in (2.4), RHS is

$$\int_V \vec{\nabla} \, . \, \vec{D} \, dV = \oint_S \left(D_a da_x + D_y da_y + D_z da_z\right) = \oint_S D \cdot \overrightarrow{da}$$

HENCE DIVERGENCE THEOREM IS PROVED

2.3 Integral Definition of Divergence Theorem

$$\int_V \vec{\nabla} \, . \, \vec{D} \, dV = \oint_S \vec{D}. \, \overrightarrow{da} \text{ (from divergenc theorem)}$$

$$\nabla \cdot D = \lim_{s \to 0} \frac{\int_S \vec{D} \cdot \overrightarrow{da}}{V} \qquad \text{Net outward flux per unit volume} \qquad 2.5$$

RHS net outward electric flux through the closed surfaces S.

LHS average divergence of D multiplied by volume V that is enclosed S.

Thus, the average divergence of a vector is the net outward flux of vector through a closed surface S divided by volume V enclosed.

He limits of the average divergence as S is allowed to shrink to zero about a point is divergence of vector at that point.

Example 2.4: The volume charge density of a spherical body of radius "a" centered at origin is given by $\rho_v(r, \theta, \emptyset) = \frac{\rho_0}{r} \; C/m^3$ where ρ_0 is constant, calculate the total charge in sphere.

Solution: $dv = r^2 \, sin \, \theta \; dr \; d\theta \; d\emptyset$

$$\rho_v = \frac{\rho_0}{r}$$

$$Q = \int_v \rho_v \, dv$$

$$\Rightarrow \qquad Q = \int_v \frac{\rho_0}{r}\, x\, r^2 \sin\theta\, dr d\theta d\emptyset = \int_0^a \rho_0\, r\, dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\emptyset$$

$$Q = 2\rho_0\, a^2\, C$$

Example 2.5: Given that $D = \left(\frac{10r^3}{4}\right) \hat{a}_r \frac{C}{m^2}$ in cylindrical coordinates, evaluate both sides of divergence theorem for volume enclosed by $r = 1m, r = 2m, z = 0\ and\ z = 10m$.

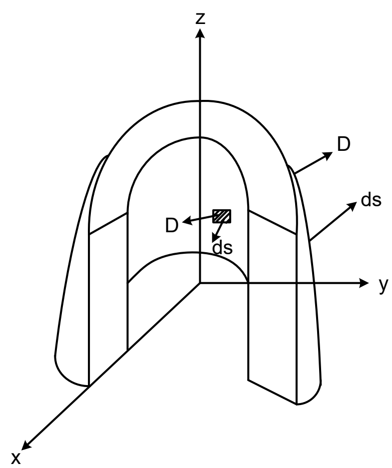


Figure 2.5

Solution: $\oint \vec{D}.\overrightarrow{ds} = \int \vec{\nabla}.\vec{D}\, dv$

Since D has no Z component, $\vec{D}.\overrightarrow{ds}$ is zero for top and bottom. On the inner cylindrical surface ds is in the direction $\hat{a}_r$

$$\oint D.ds = \int_0^{10}\int_0^{2\pi} \frac{10}{4}(1)^3\, a_r\, (1) d\emptyset\, dz\, (-a_r)$$

$$+ \int_0^{10}\int_0^{2\pi} \frac{10}{4}(2)^3\, a_r\, (2) d\emptyset\, dz\, a_r$$

$$= \frac{-200\pi}{4} + \frac{16\ x\ 200\pi}{4} = 750\pi\ C$$

From R.H.S of divergence theorem

$$\nabla \cdot D = \frac{1}{r}\frac{d}{dr}\left(\frac{10r^4}{4}\right) = 10r^2$$

$$\int_v \nabla \cdot D \cdot dV = \int_0^{10}\int_0^{2\pi}\int_1^{2}(10r^2)r\, dr d\emptyset dz = 750\pi \text{ C } proved$$

Example 2.6: Given that $D = \left(\frac{5r^2}{4}\right)\hat{a}_r\ \frac{C}{m^2}$ in spherical coordinates evaluates both sides of divergence theorem for volume enclosed by $r = 4\ and\ \theta = \frac{\pi}{4}$.

Solution: $\int_S \vec{D} \cdot \vec{ds} = \int_v \nabla \cdot D\, dv$

Since D has only a radical component, $\vec{D} \cdot \vec{ds}$ has non-zero value only on surface r = 4m.

$$\oint \vec{D} \cdot \vec{ds} = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{5(4)^2}{4} a_r (4) \sin\theta\, d\theta\, d\emptyset\, a_r$$

$$= 589.1 \text{ C}$$

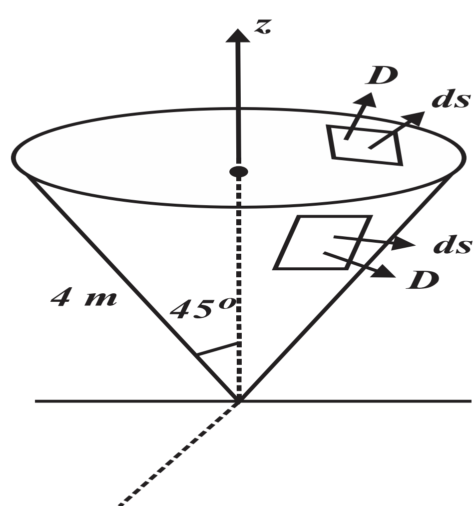


Figure 2.6

For R.H.S of divergence theorem

$$VD = \frac{1}{r^2}\frac{d}{dr}\left(\frac{5r^4}{4}\right) = 5r$$

$$\int_v \nabla \cdot D\, dv = \int_0^{2\pi}\int_0^{\pi/4}\int_0^4 (5r)(r^2 \sin\theta\, drd\theta d\emptyset)$$

$$= 589.1\text{ C } Hence\ proved$$

2.4 Magnetostatics

We have just dealt with the study of electrostatics: fields due to static charges, $E\left(E_x, E_y, E_z\right)$. We shall now deal with the concept of magnetostatics: field due to moving charges (or steady current) $H\left(H_x, H_y, H_z\right)$

The interaction of electrostatic and magnetostatic fields gives rise to electromagnetic field

$\left(E_x, E_y, E_z\right)$ and $\left(H_x, H_y, H_z\right) \Longrightarrow EM$ field

Sources of steady magnetic field are:

i. Permanent magnet
ii. Charging electric field
iii. Direct current

We shall go straight to the basic laws that govern magneto-static fields which include Biot-savart law and Ampere's circuital law.

2.5 Biot-Savart Law

The law states that at any point P the magnitude of the magnetic field intensity dH produced by a differential current element Idl at rl is proportional to the product of the current, the magnitude of the differential length and the sine of the angle lying between the filament and a line connecting the filament to the point P at which the field is to be determined. Also, this magnetic field Intensity is inversely proportional to the surface of the distance from the differential element to the point, P.

$$dH = \frac{I\, dl}{4\pi R^2} a_R \qquad 2.6$$

$$= \frac{I\,dl \times R}{4\pi R^3} \ \ {}^{\mathrm{A}}\!/_{\mathrm{m}}$$

Where $$R = r - r'$$

$$dH = \frac{I\,dl \ \times (r - r')}{4\pi |r - r'|^3} \qquad 2.7$$

For surface or volume current distribution, we simply replace $I\,dl$ in Eq. 2.6 with $K\,ds$ or $J\,dv$ respectively, i.e.

$$I\,dl = K\,ds = J\,dv \qquad 2.8$$

Hence,

$$H = \oint_L \frac{I\,dl \times a_R}{4\pi R^2}$$

$$H = \int_S \frac{K\,ds \times a_R}{4\pi\,R^2}$$

$$= \int_V \frac{J\,dv \times a_R}{4\pi R^2} \qquad 2.9$$

Example 2.7: Find the incremental contribution dH to the magnetic field intensity at the origin caused by a differential current element in free space $I\,dl$ equal to

(a) $3\pi a_z$ $\mu\mathrm{A.m}$ Located at $(3, -4, 0)$ and

(b) $\pi(a_x - 2a_y + 2a_z)$ $\mu\mathrm{A.m}$ located at $(5,0,0)$

Solution

a) Given $I\,dl = 3\pi a_z \times 10^{-6}$ Am

$$r = (0, 0, 0)$$
$$r = (3, -4, 0)$$

Now, $$r - r' = (-3, 4, 0)$$

$$|r - r'| = \sqrt{3^2 + 4^2} = 5$$
$$|R| = |r - r'| = 5$$
$$a_R = \frac{R}{|R|} = \frac{r - r'}{|r - r'|} = \frac{-3a_x + 4a_y}{5}$$

$$dH = \frac{Idl \times a_R}{4\pi R^2}$$

$$= \frac{3\pi \times 10^{-6}\ A.ma_z \times (-3a_x + 4a_y)}{4 \times \pi \times 25 \times 5}$$

$$= 9.4 \times 10^{-6}\ A.m\ a_z \times (-3a_x + 4a_y)$$

$$= \begin{vmatrix} a_x & a_y & a_z \\ 0 & 0 & 9.4 \times 10^{-6} \\ -3 & 4 & 0 \end{vmatrix}$$

$$dH = (0 - 3.769 \times 10^{-5})a_x - (0 + 2.827 \times 10^{-5})\ a_y + 0$$

$$= -3.769 \times 10^{-5}\ a_x - 2.827 \times 10^{-5}\ a_y\ \text{A/m}$$

$$dH = -38\ a_x - 28\ a_y\ \mu\text{A/m} \quad \textbf{\textit{Ans}}$$

(b) $Idl = \pi(a_x - 2a_y + 2a_z)\ \mu\text{A.m}$

$$r = (0,0,0)$$

$$r' = (5,0,0)$$

$$r - r' = (-5, 0,0)$$

$$dH = \frac{I\ dl \times a_R}{4\pi R^2}$$

$$= \frac{\pi(a_x - 2a_y + 2a_z) \times -5a_x}{4\pi \times 5^2 \times 5}\ \mu\text{A.m}$$

$$= \frac{\pi(a_x - 2a_y + 2a_z) \times -5a_x}{4\pi \times 25 \times 5}\ \mu\text{A.m}$$

$$= \frac{a_x - 2a_y + 2a_z \times -5a_x}{500}\ \mu\text{A.m}$$

$$= (a_x - 2a_y + 2a_z)(-0.01a_x)\ \mu\text{A.m}$$

$$= \begin{vmatrix} a_x & a_y & a_z \\ -0.01 & 0 & 0 \\ 1 & -2 & 2 \end{vmatrix} \times 10^{-6}$$

$$= [-(-0.02 - 0)a_y + (0.02 - 0)a_z] \times 10^{-6}\ ^{\mathrm{A}}/_{\mathrm{m}}$$
$$dH = 20a_y + 20a_z \text{ nA/m}$$

2.6 Ampere's Circuit Law

The law states that the circulation by the magnetic field intensity, H around a closed path is equal to the current enclosed by the path, i.e.

$$\oint H \bullet \mathrm{dl} = \mathrm{I} \qquad 2.10$$

This law is similar to Gauss's law which we have earlier treated; $\oint_S Ds \bullet \mathrm{ds} = \mathrm{Q}$. Ampere's circuital law can be explained using the Fig. 2.7;

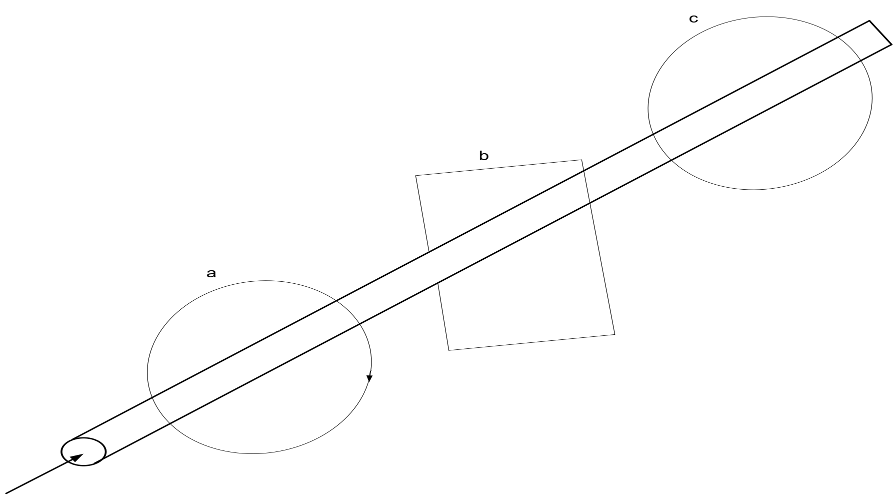


Figure 2.7

The conductor has a total current I. The fine integral of the magnetic field intensity, H around the paths a and b is just equal to the total current I. The line integral of H around the path C is obviously less than I because the entire current is not enclosed by the path.

Example 2.8: Each of the three coordinate axes carries a filamentary current of $2\ \mathrm{A}$ in the a_x, a_y, or a_z, direction. Find the magnetic field intensity H at the point (2,3,4).

Solution:

$$I = \oint H \bullet dl$$

Now $\int dl$, is simply the circumference of the path enclosing the current. Hence $\int dl = 2\pi r$

$$I = H \bullet 2\pi r$$

$$H = \frac{I}{2\pi R}\hat{a}_r$$

I is the a_x, a_y, a_z, directions. We therefore calculate Hin the 3 directions

$$H = Ha_x + Ha_y + Ha_z$$

In the $a_x$direction, we have:

$$Idl = 2 \times a_x = 2a_x$$

$$P_2 = (2,3,4) = (2a_x + 3a_y + 4a_z)$$

$$P_1 = (0,0,0)$$

$$r_{12} = r_2 - r_1 = (2,3,4) - (0,0,0) = (2,3,4)$$

$$|r_2 - r_1|_{\hat{a}_x} = \sqrt{3^2 + 4^2} = 5$$

$$Ha_x = \frac{2a_x}{2\pi(5)} \times \frac{(2a_x + 3a_y + 4a_z)}{5}$$

$$Ha_x = 12.7 \times 10^{-3}a_x \times \left(2a_x + 3a_y + 4a_z\right)$$

$$= \begin{vmatrix} a_x & a_y & a_z \\ 12.7 \times 10^{-3} & 0 & 0 \\ 2 & 3 & 4 \end{vmatrix}$$

$$= -(4 \times 12.7 \times 10^{-3})a_y + (3 \times 12.7 \times 10^{-3})a_z$$

$$Ha_x = -0.0509a_y + 0.0382a_z$$

In the a_y direction, we have:

$$Idl = 2 \times a_y = 2a_y$$

$$P_2 = (2,3,4) = (2a_x + 3a_y + 4a_z)$$

$$P_1 = (0,0,0)$$

$$r_{12} = r_2 - r_1 = (2,3,4) - (0,0,0) = (2,3,4)$$

$$|r_2 - r_1|_{\hat{a}_y} = \sqrt{2^2 + 4^2} = 4.47$$

$$Ha_y = \frac{2a_y}{2\pi(4.47)} \times \frac{(2a_x + 3a_y + 4a_z)}{4.47}$$

$$Ha_y = 15.9 \times 10^{-3}a_y \times \left(2a_x + 3a_y + 4a_z\right)$$

$$= \begin{vmatrix} a_x & a_y & a_z \\ 0 & 15.9 \times 10^{-3} & 0 \\ 2 & 3 & 4 \end{vmatrix}$$

$$= (4 \times 15.9 \times 10^{-3})a_x - (2 \times 15.9 \times 10^{-3})a_z$$

$$Ha_y = 0.0637a_x - 0.0318a_z$$

In the a_z direction;

$$Idl = 2 \times a_z = 2a_z$$
$$P_2 = (2{,}3{,}4) = (2a_x + 3a_y + 4a_z)$$
$$P_1 = (0{,}0{,}0)$$
$$r_{12} = r_2 - r_1 = (2{,}3{,}4) - (0{,}0{,}0) = (2{,}3{,}4)$$
$$|r_2 - r_1|_{\hat{a}_z} = \sqrt{2^2 + 3^2} = 3.605$$
$$Ha_z = \frac{2a_z}{2\pi(3.605)} \times \frac{(2a_x + 3a_y + 4a_z)}{3.605}$$
$$Ha_z = 24.5 \times 10^{-3} a_z \times \left(2a_x + 3a_y + 4a_z\right)$$
$$= \begin{vmatrix} a_x & a_y & a_z \\ 0 & 0 & 24.5 \times 10^{-3} \\ 2 & 3 & 4 \end{vmatrix}$$
$$= -(3 \times 24.5 \times 10^{-3})a_x - (-2 \times 24.5 \times 10^{-3})a_y$$
$$Ha_z = -0.0735a_x + 0.049a_y$$

Hence H in all $a_x a_y a_z$, directions are:

$$H = -0.0509a_y + 0.0382a_z + .0637a_x - 0.0318a_z - 0.0735a_x + 0.049a_y$$
$$H = -9.8a_x - 1.9a_y + 6.4a_z \ {}^{\mathrm{A}}/_{\mathrm{m}} \quad \boldsymbol{Ans}$$

2.7 Forces Due to Magnetic Fields

There are at least three ways in which forces due to magnetic field can be experienced:

a. Force on a moving charge particle in a magnetic field.

b. Force on a differential current element in an external magnetic field.

c. Force between two differential current elements.

2.7.1 Force on a Moving Charge Particle

We know that the electric force F_e on a stationary or moving electric charge Q in an electric field E is given by Coulombs law and is related to the electric field intensity as

$$F_e = QE \quad (N)$$

If Q is positive, then F_e and E arc in the same Irection. A magnetic field, B can equally exert force on a charge particle only if the particle is in motion. The magnetic force F_m

experienced by a charge Q moving with a velocity, V, In a uniform magnetic field B is given by;

$$F_m = Q(V \times B) \qquad 2.11$$

Note: E, V, B are all vectors

F_m is perpendicular to both V and B. Observe that F_e is independent of velocity while F_m is dependent on it. For a moving charge Q in the presence of both electric and magnetic fields, the total force on the charge is given by Lorentz force equation;

$$F = F_e + F_m \qquad 2.11a$$

$$= QE + Q(V \times B)$$

$$= Q(E + V \times B)$$

which relates mechanical force to electric force. If the mass of the charge particle moving in both electric and magnetic fields is $m(kg)$ then by Newton's second law of motion;

$$F = m\frac{du}{dt} = ma = Q(E + V \times B) \qquad 2.11b$$

The solution of the above Equ. (2.11) is very crucial in finding the motion of charged particles in electric and magnetic fields. We should bear in mind that in such fields, the bulk of the energy transfer is due to the electric field.

Examples 2.9: A point charge of -1.8 C has a velocity of $4a_x + 3a_y - 2a_z$ m/s. Find the magnitude of the force exerted on it by the field; (a) $E = 9a_x + 4a_y - 6a_z$ V/m (b) $B = -4a_x + 4a_y + 3a_z$ Wb/m^2 (c) both E and B

Solution

(a) $F_e = QE$

$$= -1.8(9a_x + 4a_y - 6a_z)$$

$$= -16.2a_x - 7.2a_y + 10.8a_z$$

$$|F_e| = \sqrt{16.2^2 + 7.2^2 + 10.8^2}$$

$$|F_e| = 20.76 \text{ N } \boldsymbol{Ans}$$

(b) $F_m = Q(V \times B) = -1.8 \begin{vmatrix} a_x & a_y & a_z \\ 4 & 3 & -2 \\ -2 & -2 & 1 \end{vmatrix}$

$$-1.8[(3-4)a_x - (4-4)a_y + (-8+6)a_z]$$

$$= -1.8\,(-a_x - 2a_z)$$

$$= 1.8a_x + 3.6a_z$$

$$|F_m| = \sqrt{1.8^2 + 3.6^2}$$

$$|F_m| = 4.02 \text{ N} \;\; \boldsymbol{Ans.}$$

(c) $F = Q(E + V \times B)$

$$= -1.8\left((9a_x + 4a_y - 6a_z) + \begin{vmatrix} a_x & a_y & a_z \\ 4 & 3 & -2 \\ -2 & -2 & 1 \end{vmatrix} \right)$$

$$= -1.8\,(9a_x + 4a_y - 6a_z + -a_x - 2a_z)$$

$$= -1.8(8a_x + 4a_y - 8a_z)$$

$$F = -14.4a_x - 7.2a_y + 14.4a_z$$

$$|F| = \sqrt{14.4^2 + 7.2^2 + 14.4^2}$$

$$|F| = 21.6 \text{ N}$$

Example 2.10: A charge particle of mass 4 kg and charge 2.5 C starts from point $(-1,2,0)$ with velocity $2a_x + 3a_z$ m/s , in an electric field $6a_x + 5a_y$ V/m. At time 1 s. Determine (a) the acceleration of the particle. (b) the velocity. (c) the kinetic energy. (d) its position.

Solution

(a) $F = ma = QE \Longrightarrow a = \frac{QE}{m}$

$$a = \frac{2.5}{4}(2a_x + 3a_z)$$

$$1.25a_x + 1.875a_y \text{ m/s}^2$$

(b) acceleration = rate of change of velocity i.e.

$$a = \frac{du}{dt} = \frac{d}{dt}(U_x, U_y, U_z) = 1.25a_x + 1.875a_y \text{ m/s}^2$$

Equating components to find U at time t

$$\frac{d}{dt}U_x = 2.5 \Rightarrow U_x = \int 1.25dt = 1.25t + L$$

$$\frac{d}{dt}U_y = 3.75 \Rightarrow U_y = \int 1.875dt = 1.875t + M$$

$$\frac{d}{dt}U_z = 0 \Rightarrow U_z = \int 0dt = 0 + N$$

We now solve for the constants L, M, N as follows: at t=0,

$U = 2a_x + 3a_z$ m/s which shows that

$$U_x|_{t=0} = 2 = 1.25(0) + L \Rightarrow L = 2$$

$$U_y\Big|_{t=0} = 0 = 1.875(0) + M \Rightarrow M = 0$$

$$U_z|_{t=0} = 3 = 0 + N \Rightarrow N = 3$$

$$U(t) = \left(U_x, U_y, U_z\right)$$

$$U(t) = (1.25t + 2, 1.875t, 3\,)$$

Which can be expressed as,

$$\boldsymbol{U}(t) = (1.25t + 2)a_x + (1.875t)a_y + 3a_z$$

Then the velocity at $t = 1$ s is thus,

$$\boldsymbol{U}(1) = (1.25 + 2)a_x + (1.875)a_y + 3a_z$$

$$= 3.25a_x + 1.875a_y + 3a_z \ \text{ m/s}$$

(c) $K.E = \frac{1}{2}mu^2$

$$= \frac{1}{2} \times 4 \times (3.25^2 + 1.875^2 + 3^2)$$

$$= 46.2\text{J}$$

(d) $U = \frac{dl}{dt} = \frac{d}{dt}(x, y, z)$

$$= (1.25t + 2, 1.875t, 3)$$

Equating coefficients of components gives

$$\frac{dU_x(t)}{dt} = 1.25t + 2 \Rightarrow U_x(t) = \int (1.25t + 2)dt = 0.65t^2 + 2t + L_1$$

$$\frac{dU_y(t)}{dt} = 1.875t \Rightarrow U_y(t) = \int (1.875t)dt = 0.9375t^2 + M_1$$

$$\frac{dU_z(t)}{dt} = 3 \Rightarrow U_z(t) = \int (3)dt = 3t + N_1$$

At $t = 0, (x, y, z) = (-1, -2{,}0)$

From where

$$U_x(0) = 0.65(0)^2 + 2(0) + L_1 \Rightarrow L_1 = -1$$

$$U_y(0) = 0.9375(0)^2 + M_1 \Rightarrow M_1 = -2$$

$$U_z(0) = 3t + N_1 \Rightarrow N_1 = 0$$

Substituting,

$$\left(U_x(t), U_y(t), U_z(t)\right) = (0.65t^2 + 2t - 1, 0.9375t^2 - 2{,}3t) = l_{(t)}$$

∴ at $t = 1$ $l_{(t)} = (1.65, -1.0625{,}3)$

∴ the position is $1.65a_x - 1.0625a_y + 3a_z$ $\boldsymbol{Ans}$.

Example 2.11: A charged particle of mass 4 kg and charge 2 C starts at the origin with velocity $6a_y$ and travels in a region of uniform magnetic field $B = 20a_z\ {}^{\text{Wb}}/_{\text{m}^2}$. At $time = 6$ s, calculate, (a) the velocity and acceleration of the particle. (b) the magnitude force on it. (c) its kinetic energy and location. (d) the particle's trajectory. Show that the kinetic energy is constant.

<u>Solution:</u>

(a) $F = ma = m\frac{du}{dt} = Q\,(U \times B)$

$\Longrightarrow$ $a = \frac{du}{dt} = \frac{Q}{m}(U \times B)$

$\therefore$ $\frac{d}{dt}\left(U_x a_x + U_y a_y + U_z a_z\right) = \frac{1}{2}\begin{vmatrix} a_x & a_y & a_z \\ U_x & U_y & U_z \\ 0 & 0 & 20 \end{vmatrix}$

$$= \frac{1}{2}\left(20U_y a_x - 20U_x a_y + 0a_z\right)$$

$$= 10U_y a_x - 10U_x a_y$$

Equating components,

$$\frac{dU_x}{dt} = 10U_y$$

$$\frac{dU_y}{dt} = -10U_x$$

$$\frac{dU_z}{dt} = 0 \Longrightarrow U_z = C_0$$

$$\frac{d^2U_x}{dt^2} = 10\frac{dU_y}{dt} = 10 \times (-10U_x)$$

i.e $\frac{d^2U_x}{dt^2} = -100U_x$

$\Longrightarrow$ $\frac{d^2U_x}{dt^2} + 100U_x = 0$

The solution is of the form,

$$U_x = C_1 \cos 10t - C_2 \sin 10t$$

But, $10U_y = \frac{dU_x}{dt} = -10C_1 \sin 10t + 10C_2 \cos 10t$

i.e, $U_y = \frac{dU_x}{dt} = -C_1 \sin 10t + C_2 \cos 10t$

initial conditions at $t = 0$ and $U = 6a_y$, from

where,

$$U_x = 0 = -C_1 \cos 10(0) + C_2 \sin 10\,(0)$$

$$0 = C_1 + 0$$

$\therefore \qquad C_1 = 0$

$$U_y = 6 = -C_1 \cos 10(0) + C_2 \cos 10\,(0)$$

$$= 6 = 0 + C_2$$

$\therefore \qquad C_2 = 0$

$$U_z = 0 = C_0$$

$\therefore \qquad U = \left(U_x, U_y, U_z\right) = (6 \sin 10t,\ 6 \cos 10t, 0)$

$$U_{t=6\,s} = (6 \sin 60,\ 6 \cos 60{,}0)$$

$$\text{Required velocity } = -1.8289a_x - 5.7145a_y\, {}^{m}/_{s}$$

The acceleration therefore, is

$$a = \frac{du}{dt} = (60 \sin 10t,\ 60 \cos 10t, 0)$$

$$a_{t=6\,s} = (60 \cos 60, -60 \sin 60{,}0)$$

$$= -57.1448a_x + 18.2886a_y + 0a_z\ {}^{m}/_{s^2}$$

(b) $\qquad F_m = Q(U \times B)$

But, $\qquad F = ma$

$$= 4(60 \cos 60, -60 \sin 60{,}0)N$$

$$= -228.58a_x + 73.15a_y\ N$$

$$F = 2\left(-1.8289a_x - 5.7145a_y\right) \times 20a_z$$

$$= \begin{vmatrix} a_x & a_y & a_z \\ -1.8289 & -5.7145 & 0 \\ 0 & 0 & 20 \end{vmatrix}$$

$$F = 2(-114.29 - 0)a_x - 2(-36.578 - 0)a_y + 0$$

$$= -228.58a_x + 73.15a_y\ N$$

(c) $\qquad K.E = \frac{1}{2}m|u|^2$

$$= a_x - a_y$$

$$= \frac{1}{2} \times 4(-1.8289)^2 + (-5.7145)^2 + 0^2)$$

$$= 72 \text{ J}$$

To find the location,

$$U = (U_x, U_y, U_y)$$

$$= (6 \sin 10t , 3 \cos 10t , 0)$$

$$U_x = \frac{dx}{dt} = 6 \sin 10t \Longrightarrow x = \frac{-6}{10} \cos 10t + b_1$$

$$U_y = \frac{dy}{dt} = 6 \cos 10t \Longrightarrow y = \frac{-6}{10} \sin 10t + b_2$$

$$U_z = \frac{dz}{dt} = 0 \Longrightarrow z = b_3$$

At $t = 0, (x, y, z) = (0,0,0)$

$$X_{t=0} = 0 = \frac{-6}{10} \cos 10\,(0) + b_1 \Longrightarrow b_1 = \frac{6}{10}$$

$$Y_{t=0} = 0 = \frac{6}{10} \cos 10\,(0) + b_2 \Longrightarrow b_2 = 0$$

$$Z_{t=0} = 0 = b_3$$

$$(x, y, z)_t = \left(\frac{-6}{10} \cos 10t + \frac{6}{10}, \frac{6}{10} \sin 10t , 0\right)$$

At $t = 4$ s

$$(x, y, z) = (0.6 - 0.6 \cos 60 , 0.6 \sin 60,0)$$

$$x = 0.6 - 0.6 \cos 5t$$

∴ $\cos 10t = \frac{-0.6 - x}{0.6}$

$$y = 0.6 \sin 10t$$

∴ $\sin 10t = \frac{y}{0.6}$

$$\cos^2 10t + \sin^2 10t = 1 = \left(\frac{0.6 - x}{0.6}\right)^2 + \left(\frac{y}{0.6}\right)^2$$

$\Longrightarrow$ $$0.6^2 = (0.6 - x)^2 + y^2$$

$$0.6^2 = (x - 0.6)^2 + (y - 0)^2$$

This is the form $x^2 + y^2 = r^2$, the equation of a circle with radius $= 0.6$

$$K.E = \frac{1}{2} m\, |u|^2$$

Where $m = 4$, at $t = 0, u = 6a_y$

$$|U| = (0^2 + 6^2 + 0^2)^{1/2}$$

$$= (6^2)^{1/2}$$

$$= 6$$

$$U^2 = a$$

$\therefore$ $$K.E = \frac{1}{2} \times 4 \times 36 = 72 \text{ J}$$

Example 2.12: An electron has a velocity of $10^6 \ ^{\text{m}}/_{\text{s}}$ in the a_x direction in a magnetic field $B = 0.2a_x - 0.3a_y + 0.5a_z \ ^{\text{Wb}}/_{\text{m}^2}$,

(a) What electric field is present if no not force is being applied to the electron?

(b) If $E = E_0(a_x + a_y + a_z)$, Where $E_0 > 0$, determine E_0 so that the not force on the electron is $0.2\ p\text{N}$.

Solution

$$V = 10^6 \ ^{\text{m}}/_{\text{s}}$$

$$B = 0.2a_x - 0.3a_y + 0.5a_z$$

If no not force, $$F_e + F_m = 0$$

$$F_e = -F_m$$

i.e $QE = -Q(V \times B)$

or $E = -(V \times B)$

$$= 10^6 a_x \left(0.2a_x - 0.3a_y + 0.5a_z\right)$$

$$= \begin{vmatrix} a_x & a_y & a_z \\ 10^6 & 0 & 0 \\ 0.2 & -0.3 & 0.5 \end{vmatrix}$$

$$E = -\left[0a_x - (0.5 \times 10^6 - 0)a_y + (-0.3 \times 10^6 - 0)a_z\right]$$

$$= -(-500a_y - 300a_z)$$

$$= 500a_y + 300a_z\ k\mathrm{V/m}$$

(b) $\quad Net\ force = 0.2 \times 10^{-12}\ \mathrm{N}$

$$= Q(E + V + B)$$

Where Q =electron charge = 1.6×10^{-19} C

$$(E + V \times B) = \frac{0.2 \times 10^{-12}}{1.6 \times 10^{-19}} = 1.25 \times 10^6$$

$$[E + (V \times B)] = 1.25 \times 10^6$$

Substituting

$$E = E_0\left(a_x + a_y + a_z\right)$$

$$= E_0 a_x + E_0 a_y + E_0 a_z$$

So,

$$E_0 a_x + E_0 a_y + E_0 a_z + (V \times B) = 1.25 \times 10^6$$

But, $\quad V \times B = \left(-500a_y - 300a_z\right) k\mathrm{V/m}$

Hence,

$$E_0 a_x + E_0 a_y + E_0 a_z + \left(-500a_y - 300a_z\right) = 1250000$$

Select like terms

$$E = E_0 a_x + (E_0 - 500000)a_y + (E_0 - 300000)a_z = 1250000$$

$$|E| = [E_0^2 + (E_0 - 500000)^2 + (E_0 - 300000)^2]^{\frac{1}{2}} = 1250000$$

$$E_0^2 + (E_0 - 500000)^2 + (E_0 - 300000)^2 = 1.562 \times 10^2$$

Expanding

$$E_0^2 + E_0^2 + E_0^2 + 25 \times 10^{10} + 9 \times 10^{10} - 16 \times 10^5 E_0 = 1.562 \times 10^{12}$$

$$3E_0^2 + 34 \times 10^{10} - 16 \times 10^5 E_0 = 1.562 \times 10^{12}$$

$$3E_0^2 - 16 \times 10^5 E_0 - 1.222 \times 10^{12} = 0$$

Apply the quadratic function

$$E_0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where $b = -16 \times 10^5, a = 3, a = 1.222 \times 10^{12}$, and bearing in mind that $b^2 - 4ac$ must be positive

$$E_0 = \frac{16 \times 10^5 + 4150180}{6}$$

$$= 957{,}888\, Volts$$

$$= 957\, k\text{V}$$

2.8 Force on a Differential Current Element

To determine the force on a current element Idl of a current carrying conductor due to magnetic field B, using the field that for connection current;

$$Current\ density\ J = p_v u \qquad 2.12$$

Also, recall that the current element, $I, dl = K\ ds = JdV$, so that combining the equations, we have,

$$Idl = \rho_v udu = dQu \qquad 2.12a$$

Alternatively,

$$Idl = \frac{dQ}{dt} dl = dQ \frac{dl}{dt} = dQu \qquad 2.12b$$

Hence $\quad Idl = dQu$

Equ. (2.12b) shows that a charge dQ moving with velocity u, (thereby producing connection current element dQu,) is equivalent to a conduction current element Idl.

Thus, the force on a current element Idl, in a B-field is found from Equ 2.11 by simply replacing Qu by Idl i.e

$$dF = Idl \times B$$

If the current **I** is through a closed path L or circuit, the force on the circuit is given by,

$$F = \oint_L Idl \times B \qquad 2.13$$

In using the last two Equ. 2.13 we should bear in mind that the magnetic field producing by the current element Idl, does not exert force on the element itself just as a point charge does not exert force on itself. The magnetic field that exerts force on Idl, must be due to another element. In other words, the magnetic field B in the equations is external to the current element Idl. If instead of the line current element Idl, we have surface current $K\ ds$, or volume current element $J\ dV$, we simply write;

$$dF = K\ ds \times B \ or \ dF = J\ dV \times B$$

So that,

$$F = \int_s K\ ds \times B \qquad 2.14$$

$$= \int_v J\ dV \times B \qquad 2.15$$

2.9 Force between Two Current Elements

Let us now consider the force between two differentials current .elements $I_1 dL_1$ and $I_2 dL_2$. According to Biot-Savart law, both current elements produce magnetic fields. So, we may find the force $d(dF_1)$, on an element $I_1 dL_1$, due to the field produced by element $I_2 dL_2$, as shown below.

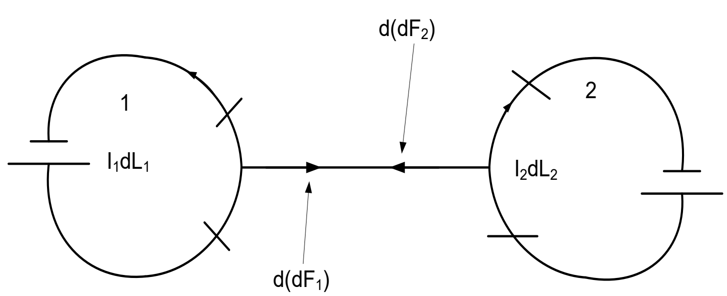


Figure: 5.2. Two differential current elements

$$d(dF_1) = I_1 dL_1 \times B_2 \qquad 2.16$$

From Biot-Savart law,

$$dB_2 = \frac{\mu_0 I_2 dL_2}{4\pi R_{2l}^2}\ a_{R_{21}}$$

Therefore,

$$d(dF_1) = \frac{\mu_0 I_1 dL_1 \times I_2 dL_2}{4\pi R_{2l}^2} \qquad 2.17$$

Equ 2.17 is essentially the law of force between two differential current elements and its analogous to Coulomb's law which expresses the force between two stationary point charges. From the above Equ 2.17, we obtain the total force Fl on current loop I due to loop 2. The force F_2 on loop 2 due to the field from loop I is obtained from the same Equ 2.17 by simply Interchanging subscripts I and 2. The total force between two is obtained by integrating Equ 2.17 twice.

$$F_1 = \frac{\mu_0 I_1 L_2}{4\pi} \oiint \frac{dL_1 \times dL_2}{R_{2l}^2} a_{R_{21}} \qquad 2.18$$

Example 2.13: A filamentary conductor of infinite length on the z axis of $2\ \mathrm{A}$ in the a_z direction. Find the magnitude of the force on a 2.5 cm length of the conductor in the magnetic field $B = 0.1a_x - 0.2a_z\ {}^{\mathrm{Wb}}/_{\mathrm{m}}$.

Solution

$$F = Q(V \times B)$$

$$= lt\ (V \times B)$$

$$= lt\left(\frac{l}{t} \times B\right)$$

$$= I\ (L \times B)$$

$$= BIL$$

$$= (0.1a_x - 0.2a_z) \times 2.5 \times 10^{-2} a_z \times 2$$

$$= (0.1a_x - 0.2a_z) \times 5 \times 10^{-2} a_z$$

$$= \begin{vmatrix} a_x & a_y & a_z \\ 0.1 & 0 & -0.2 \\ 0 & 0 & 5 \times 10^{-2} \end{vmatrix}$$

$$= -5 \times 10^{-3} a_y$$

$$|F| = \sqrt{25 \times 10^{-3}} = 5\ m\text{N}$$

Example 2.14: A current filament passing through $P_1(0,0,0)$ carries a current of $6\ \text{A}$ in the a_x direction, and a second filament goes through $P_2(4,8,2)$, also carrying $6\ \text{A}$, but in the a_y direction. (a) Find the vector force exerted on an incremented length dL_2 of the second filament located at P_2, by an incremented length dL_1 of the first conductor at P_1 (b) Find the force on dL_1 at P_1 caused by dL_2 at P_2

Solution

Given $P = (0,0,0)$

$$I_1 = 6\ \text{A}$$

$$P_2 = (4,8,2)$$

$$I_2 = 6\ \text{A}$$

$$\mu_0 = 4\pi \times 10^{-7}$$

$$dF_2 = \frac{\mu_0 I_1 I_2 dL_1 dL_2}{4\pi R^2} a_R$$

$$a_R = \frac{(4,8,2) - (0,0,0)}{(4^2 + 8^2 + 2^2)^{1/2}}$$

$$dF_2 = \frac{\mu_0 I_1 I_2 dL_1 dL_2}{4\pi R^3} (4a_x + 8a_y + 2a_z)$$

$$= \frac{4\pi \times 10^9 (6a_x \times 6a_y) \times dL_1 dL_2 (4a_x + 8a_y + 2a_z)}{4\pi (4^2 + 8^2 + 2^2)^{3/2}}$$

$$= \frac{10^{-9} (6a_x \times 6a_y) \times dL_1\, dL_2 (4a_x + 8a_y + 2a_z)}{769.87}$$

We resolve $6\ A\ a_x$ and $6\ A\ a_y$

$$= \begin{vmatrix} a_x & a_y & a_z \\ 6 & 0 & 0 \\ 0 & 6 & 0 \end{vmatrix} = 36a_z$$

$$dF_2 = \frac{36a_z \times \ dL_1 dL_2\left(4a_x + 8a_y + 2a_z\right) \times 10^{-7}}{769.87}$$

$$now, \quad = \begin{vmatrix} a_x & a_y & a_z \\ 4 & 8 & 2 \\ 0 & 0 & 36 \end{vmatrix}$$

$$= -144a_y + 288a_x$$

$$dF_2 = \frac{(288a_x - 144a_y)dL_1 dL_2 \times 10^{-7}}{769.87}$$

$$= 0.374a_x - 0.187a_y dL_1 dL_2 \times 10^{-7}$$

$$= 37.4a_x \ dL_1 dL_2 \ n\text{N}$$

$$dF_1 = 18.7a_y dL_1 \ dL_2 \ n\text{N}$$

State of particles	**Electric field**	**Magnetic field**	**Combined field**
Stationary	$F = QE$	$F = 0$	$F = QE$
Moving	$F = QE$	$F = Q(V \times B)$	$F = Q(E + V \times B)$

2.10 Exercise

1. Given a vector function $\vec{A} = (2x + C_1 \ z)\hat{z} + (C_2 x - 3z)\hat{y} + (2x + C_3 y + C_4 z)\hat{z}$
   a. Calculate the value of constants $C_{1,} C_{2,} C_3 \ if \ \vec{A}$ is irrotational.
   b. Determine constant $C_4 \ if \ \vec{A}$ is solenoidal.
   c. Determine scalar potential function V, whose negative gradient equals $\vec{A}$
2. A vector field $\vec{A}\ (r, \emptyset, z) = 30e^{-r}\hat{r} - 2z\hat{z}$. Verify divergence theorem for the volume enclosed by $r = 2, z = 0, z = 5$.
3. Given $\vec{A} = r\cos\emptyset\,\hat{r} - r\sin\emptyset\,\hat{\emptyset}$ in cylindrical coordinates. evaluate $\oint \vec{A}\,.\,ds$ over the surface of the box bounded by planes $z = 0 \ and \ z = 1, \emptyset = 0 \ \ and \ \emptyset = \pi/2$ and cylinder r = a.

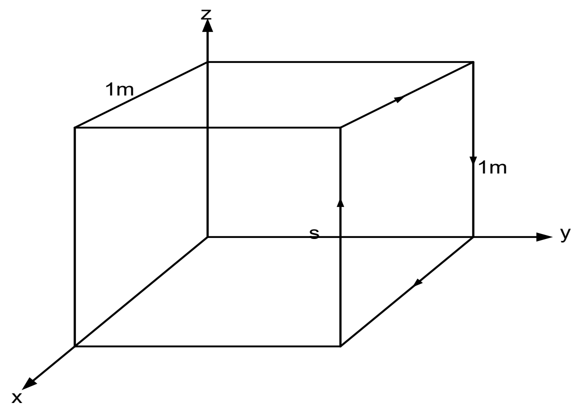


Figure 2.7

4. Integrate vector $\vec{D} = x^2y^3z^4\hat{z}$ over the plane square surface bounded by the points $(x, y, z) = (1,1,2); (5, 1, 2); (1, 5, 2); (5,5,2)$.

5. Express the vector $\vec{A} = xy^2z\ a\ \hat{x} + x^2yz\ a\hat{y} + xyz^2\ \hat{a}_z$ in cylindrical and spherical polar coordinate.

6. Given that $= \frac{10x^3}{3}\ a_x C/m^2$, evaluate both sides of the divergence theorem for volume of a cube, 2m on an edge, centered at origin and with edges parallel to axes.

7.

8. Given a vector field: $\vec{F}(x, y, z) = 5\ 5\ x\ yz\ \hat{x} + y^2\hat{y} + yz\hat{z}$ verify Stokes theorem by evaluating suitable line and surface integrals over the open surface defined to five sides of a cube measuring 1m on a side and about the closed contour boundary S as shown in Fig. 2.7

9. (a) State the Divergence theorem.

(b) Given A(r)=10e^{-2z} $(pa_\rho + a_z,$ (i) determine the flux of A over the closed surface of the cylinder 0$\leq z \leq 1, \rho = 1$.

(ii) Verify the divergence theorem for same.

FEDERAL POLYTECHNIC OKO, HND 2 SECOND SEMESTER 2014 EXAM

10. Describe Stokes theorem

11. Conclude irrotational field. With mathematical proof

12. Conclude solenoidal field, with mathematical proof

13. Describe divergence theorem

14. How to treat discrete sources? Discuss

15. Discuss properties of Dirac-Delta function

16. Where can you use matrices?

17. Explain solving simultaneous equations using matrices

18. Find out inverse of matrix

19. Give physical significant of divergence and Stokes theorem.

20. An infinitely long filament on the x-axis carries a current of $10\ m\text{A}$ in the a_x direction. Find the magnetic field intensity H and its magnitude at $P(3,2,1)$.

Ans: $-0.31a_y + 0.637a_y = 0.712\ m\,{}^{\text{A}}/_{\text{m}}$

a) A current filament of $3a_xA$ lies along the x-axis. Fine the magnetic field intensity H at $P(-1, 3, 2)$

b) A current sheet $K = 8a_x\,{}^{\text{A}}/_{\text{m}}$ flows in the region $-2 < y < 2$ meters in the plane $z = 0$.calculate H at $P(0,0,3)$

c) A current filament on the z-axis carries a current of $7\ m\text{A}$ in the a_z direction, and current sheets of $0.5a_z\ \text{A/m}$ and $-0.2a_z\ \text{A/m}$ are located at $P = 1\ \text{cm}$ and $P = 0.5cm$ respectively. Calculate H at $P =$ (a) $0.5\ \text{cm}$ (b) $1.5\ \text{cm}$ (c) $4\ \text{cm}$ (d) what current sheet should be located at $P = 4\ \text{cm}$ so that $H = 0$ for all $p > 4\ \text{cm}$

d) Express the value of H in cartesian components at $P(0,0,2)$ in the field of a current filament $2.5\ \text{A}$ in the a_z direction at $x = 0.1, y = 0.3$.

21. A 4 coulomb point charge is moving through a uniform electric field $E = 3U_x\,{}^{V}/_{m}$ at $t = 0$, the point charge is located at the origin and has a velocity of $5a_z\ m/s$, Assuming a mass 1kg, use the force equation and Newton's laws to obtain the appropriate differential equations. Calculate at $t = 2$sec

(a) The position of the charge $P(x, y, z)$, (b) its velocity (c) its kinetic energy

Ans. (a) $(24a_x + 10a_z)$ meters, (b) $(24a_x + 5a_z)m/s$, (c) $(300.5\ Joules)$

22. A point charge $Q = 18nC$ has a velocity of $5 \times 10^5\ {}^{m}/_{s}$ in the direction, $a_v = 0.6a_x + 0.75a_y + 0.30a_z$. Calculate the magnitude of the force exerted on the charge by the field

(a) $B = -3a_x + 4a_y + 6a_z mT$ (b) $E = -3a_x + 4a_y + 6a_z\,{}^{kV}/_{m}$ (c) both E and B acting

Ans. $660 \times 10^{-6}N, 140 \times 10^{-6}N, 670 \times 10^{-6}N$

23. A point charge $Q = -0.3\mu C$ and $m = 3 \times 10^{-16}kg$, is moving through the field $E = 30a_z\,{}^{V}/_{m}$, Develop the appropriate differential equations and solve them, subject to the initial conditions at $t = 0; V = 3 \times 10^5 a_x\,{}^{m}/_{s}$, at the origin. At $t = 3\mu s$ find (a) the position $P(x, y, z)$ of the charge. (b) The velocity, V. (c) the kinetic energy of the charge.

24. If the charge stated in equation 3 above is moving through the magnetic field $B = 30a_z mT$ solve as in 3a, b, c above given the same data.

25. A point charge for which $Q = 2 \times 10^{-16} C$ and $m = 5 \times 10^{-26}\ kg$ Moving in the combined fields $E = 100a_x - 200a_y + 300a_z\ {}^{V}/_{m}$ and $B = -3a_x + 2a_y - a_z\ mT$. If the charge velocity at $t = 0$ is $V(0) = 2a_x - 3a_y - 4a_z\ {}^{m}/_{s}$ (a) give the unit vectors showing the direction in which the charge is accelerating at $t = 0$. (b) find the kinetic energy of the charge at $t = 0$

26. Two differential current elements

$$I_1 dL_1 = 3 \times 10^{-6} A.m\ at\ P_1(1,0,0)$$

And $I_2 dL_2 = 3 \times 10^{-6}(-0.5a_z + 0.4a_y + 0.3a_z) A.m\ at\ P_2(2,2,2)$ are located in free space. Find the vector force exerted on

(a) $I_2 dL_2$ by $I_1 dL_1$ (b) $I_1 dL_1$ by $I_2 dL_2$

Ans (a) $(-1333a_x + 0.333a_y - 2.67a_z) \times 10^{-20} N$

(b) $(4.67a_x + 0.667a_z) \times 10^{-20} N$

CHAPTER 3

ELECTROSTATICS

3.0 Coulomb's Law and Electric Field Intensity

Coulomb's law: the force between two very small objects separated in a vacuum or free space by a distance, which is large compare to their size, is proportional to the charge on each and inversely proportional to the square of the distance between them.

$$F = k\,\frac{Q_1 Q_2}{R^2} \qquad 3.1$$

And Q_1 Q_2 can be positive or negative charge quantities, R is the distance separation and K is constant of proportional. R is in meters, and F in newton for SI system.

$k = \frac{1}{4\pi\epsilon_0}$, Where ϵ_0 is called the permittivity of free space;

$$\epsilon_o = 8.854 \times 10^{-12} = \frac{1}{36\pi}10^{-9}F/m\ (C^2/N.m^2)$$

$$\Longrightarrow \qquad F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \qquad 3.2$$

He said force acts on the line joining the two charges: Like charges repulse, while unlike charges attract each other.

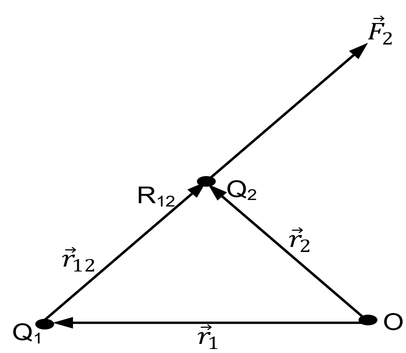


Figure 3.1 Two Like Charge $Q_1\ and\ Q_2$ at a Point Source

In Fig. 3.1 Q_1 *and* Q_2 charges are like, hence the direction of the arrow depicting $\vec{F}_2$. The vector $\vec{r}_1$ locates Q_1 *and* $r_2$locates Q_2.$\vec{r}_2 - \vec{r}_1 = \vec{R}_{12}$ is the directed line segment from Q_1 *to* Q_2.

$$\vec{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}\hat{a}_{12} \qquad 3.3$$

And $\hat{a}_{12} = \frac{\vec{R}_{12}}{|R_{12}|} = \frac{\vec{R}_{12}}{\vec{R}_{12}} = \frac{r_2 - r_1}{|\vec{r}_2 - \vec{r}_1|}$ is the unit vector in $\vec{R}_{12}$ direction.

Example 3.1: Locate a charge of $Q_1 = 2 \times 10^{-4}$ C at $A\,(1, 2, 3)$ and a charge of $Q_2 = -10^{-4}$ at $B(2, 0, 5)$ in a vacuum. Determine the force exerted on Q_2 *by* Q_1

Proc: $\vec{R}_{12} = \vec{r}_2 - \vec{r}_1 = (2-1)\hat{a}_x + (0-2)\hat{a}_y + (5-3)\hat{a}_z$

$$= \hat{a}_x - 2\hat{a}_y + 2\hat{a}_z \Longrightarrow \left|\vec{R}_{12}\right| = \sqrt{1^2 + 2^2 + 2^2} = 3$$

$$\Longrightarrow \quad \hat{a}_{12} = \frac{\hat{a}_x - 2\hat{a}_y + 2\hat{a}_z}{3}$$

$$\Longrightarrow \quad \vec{F}_2 = \frac{(2 \times 10^{-4}) \times 10^{-4}}{4\pi\left({}^{1}\!/_{36}\,\pi\right)10^{-9} \times 3^2}\left(\frac{\hat{a}_x - 2\hat{a}_y + 2\hat{a}_z}{3}\right)$$

$$= -20\left(\frac{\hat{a}_x - 2\hat{a}_y + 2\hat{a}_z}{3}\right)N,$$

Where 20N is the modulus (magnitude) of the force.

$$\vec{F}_2 = -\frac{20}{3}\hat{a}_x - \frac{40}{3}\hat{a}_y + \frac{40}{3}\hat{a}_z$$

$$components - \frac{20}{3}N, \frac{-40}{3}N \; and \; \frac{40}{3}N$$

In x, y, z directions respectively.

Similarly, since the charge repulse,

$$\vec{F}_1 = -\vec{F}_2 = -\frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}\hat{a}_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}\hat{a}_{21}$$

Increasing Q_1, results in increasing F_2 by the same proportion because coulomb's law obeys the law of linearity. So, also when several charges act on a charge, the result is the sum of the different, charges acting individually.

Example 3.2: Three-point charges, $q_1 = -4nC, q_2 = 5nC,\ q_3 = 3nC$ are faced as shown

If $r_1 = 0.5m\ \ and\ r_3 = 0.8m$, find force on q_2 due to other two charges.

Solution: we first find force on q_2 due to q_1

$$F_1 = k \times \frac{q_1 \times q_2}{r_1^2} = \frac{9 \times 10^9 \times 4 \times 10^9 \times 5 \times 10^{-9}}{(0.5)^2} = 7.2 \times 10^{-7} N$$

Which is directed to left. The force on q_2 due to q_3 is found as:

$$F_3 = \frac{kq_2\, q_3}{r_3^2} = \frac{9 \times 10^9 \times 3 \times 10^9 \times 5 \times 10^{-9}}{(0.8)^2} = 2.11 \times 10^{-7} N$$

Which is also directed to left. Thus, the total force on q_2 is given by

$$7.2 \times 10^{-7} + 2.11 \times 10^{-7} = 9.31 \times 10^{-7} N$$

Thus, the total force q_2 is $9.31 \times 10^{-7} N$ directed to left.

Example 3.3: A charge $Q_1 = 3 \times 10^{-4}$ is located at $A(1, 2, 3)$ and a charge $Q_2 = -2 \times 10^{-4}$ is located at $B(2, 0, 5)$ in vacuum. Determine the vector force exerted on Q_2 by Q_1.

Solution: Given $A(1, 2, 3)$ and $B(2, 0, 5)$.

$$\vec{R}_{12} = \vec{r}_2 - \vec{r}_1 = (2 - 1)\hat{a}_x + (0 - 2)\hat{a}_y + (5 - 3)\hat{a}_z$$

$$= \hat{a}_x + 2\hat{a}_z - 2\hat{a}_y$$

$$|\hat{R}_{12}| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$$

The unit vector $\hat{a}_{12} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|} = \frac{\hat{a}_x - 2\hat{a}_y + 2\hat{a}_z}{3}$

$\therefore$ $$F_{21} = \frac{3\times10^{-4}\times(-2\times10^{-4})}{4\pi\times\frac{1}{36\pi}\times10^{-9}\times3^2}\times\left(\frac{\hat{a}_x - 2\hat{a}_y + 2\hat{a}_z}{3}\right)$$

$$= -60\left(\frac{\hat{a}_x - 2\hat{a}_y + 2\hat{a}_z}{3}\right)N$$

$\Longrightarrow$ $$\vec{F}_{21} = (-20\hat{a}_x + 40\hat{a}_y - 40\hat{a}_z)\,N$$

The force expressed by Coulomb's law is a mutual force, for each of the two charges experiences a force of same magnitude, although of opposite direction, we can write.

$$\vec{F}_1 = -\vec{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}}\,\hat{a}_{21} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}}\,\hat{a}_{12} \qquad 3.4$$

Example 3.4(a): What happens when one of the charges is negative?

Solution: Look at coulomb's Law! If one charge is positive and other is negative, then the product $Q_1\, Q_2$ is negative. The resulting force vectors are therefore negative they point in the opposite direction of the previous (i.e., both positive) case in Fig. 3.2

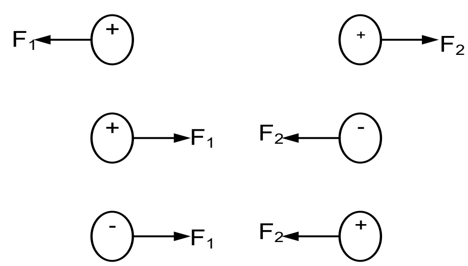


Figure 3.2 Coulombs Law

Example 3.4(b): The charges 0f $0.25\mu C$ are placed at vertices of an equilateral triangle whose side is 100mm. determine the magnitude and direction of result force on one charge due to other charges.

Solution:

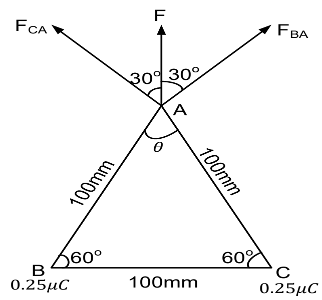


Figure 3.3

$$F_{BA} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}} = \frac{9\times 10^9 \times (0.25)^2 \times (10^{-12})}{(100)^2 \times 10^{-6}}$$

$$= 56.25 \times 10^{-3} N$$

$$= \frac{5.625 \times 10^{-4} \times 10^6}{(100)^2} = \frac{5.625 \times 100}{100 \times 100}$$

$$= 5.625 \times 10^{-2} N$$

$$F_{CA} = 5.625 \times 10^{-2} N$$

$$\vec{F} = \vec{F}_{BA} + \vec{F}_{CA}$$

$$= 2 \times 5.625 \times 10^{-2} \cos 30^o = 2 \times 5.625 \times 10^{-2} \times \frac{\sqrt{3}}{2}$$

$$\vec{F} = 9.7425 \times 10^{-2} N \;\; \textit{in direction perpendicular to BC}$$

3.1 Electric Field Intensity

For a text charge Q_1, there exist a force field associated with charge Q_1, and a force on the test charge given by

$$\vec{F}_t = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}\hat{a}_{it} \qquad 3.5$$

On per unit charge basis, this force delivers electric field intensity given by

$$\vec{E}_1 = \frac{\vec{F}_t}{Q_t} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}\hat{a}_{it} \qquad 3.6$$

Called the vector force arising from charge Q_1 acting on a unit positive test charge. For the general defining expression:

$\vec{E} = \frac{\vec{F}_t}{Q_t}\frac{N}{C}$ the vector $\vec{E}$ is the electric field intensity evaluated at the test charge location arising from all other charges in the vicinity-with the exception of the test charge itself. The unit of $\vec{E}$, Newton per coulomb, can be converted to a more practical expression involving the volt (Newton-meters per coulomb), by diving by the meter:

N/C = (N-m /C)/m = volts/meters (V/m)

For a Q_1 arbitrary located at the counter (origin) of a spherical coordinate system, the unit vector then becomes coincidental with radial unit vector $\hat{a}_r, R$ becomes r, giving rise to $\vec{E} = \frac{Q_1}{4\pi\epsilon_0 r^2}\hat{a}_r$as a general expression.

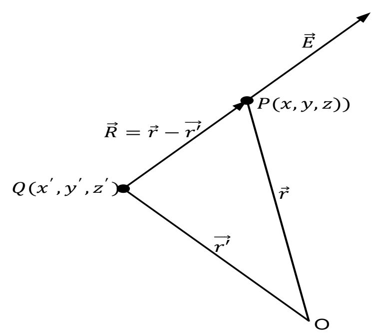


Figure 3.4

For a general charge not at the origin, no spherical symmetry comes to the rescue and we are forced to resort to rectangular system. Q is located at (x, y, z), that is,

$$\vec{E}(\bar{r}) = \frac{Q}{4\pi\epsilon_0|\vec{R}|^2}\frac{\vec{R}}{|\vec{R}|} = \frac{Q(\hat{r} - \widehat{r'})}{4\pi\epsilon_0|r - r'|^3}$$

$$= \frac{Q\left[(x - x')\hat{a}_x + (y - y')\hat{a}_y + (z - z')\hat{a}_z\right]}{4\pi\epsilon_0[x - x')^2 + (y - y')^2 + (z - z')^2]\frac{3}{2}}$$

Linearity principle, again, holds with respect to electric field intensity:

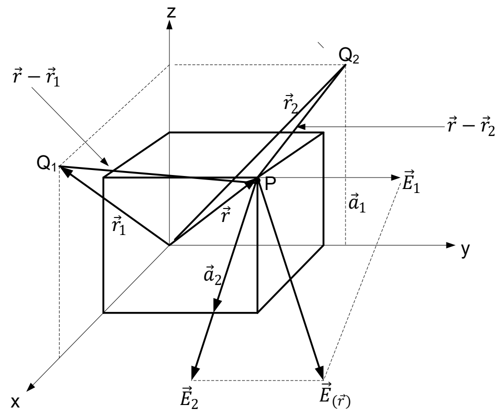


Figure 3.5 Rectangular Coordinate System

For the diagram on the proceeding page, charges Q_1 and Q_2 are located $\hat{r}_1$ and $\hat{r}_2$ from the rectangular origin respectively originating from the point P at with the electric field intensity is to be evaluated.

$$\hat{r}-\hat{r}_1-"R_1", \hat{r}-\hat{r}_2="R_2"$$

Total electric field intensity resulting from the two-point charges Q_1 and Q_2 is the sum of the charge acting individually:

$$\vec{E}(\bar{r})=\frac{Q_1}{4\pi\epsilon_0|\hat{r}-\hat{r}_1|^2}\hat{a}_1+\frac{Q_2}{4\pi\epsilon_0|\hat{r}-\hat{r}_2|^2}\hat{a}_2 \qquad 3.7$$

For n, point charges

$$\vec{E}(\bar{r})=\sum_{m=1}^{n}\frac{Q_m}{4\pi\epsilon_0|\hat{r}-\hat{r}_m|^2}\hat{a}_m$$

Example 3.4: Determine $\vec{E}\ at\ P(1,1,1)$ caused by 4 identical 2-nC charges located at P_1 (1, 1, 0), P_2 (-1, 1, 0), P_3(-1, -1, 0), P_4 (1, -1, 0) see Fig. 3.6

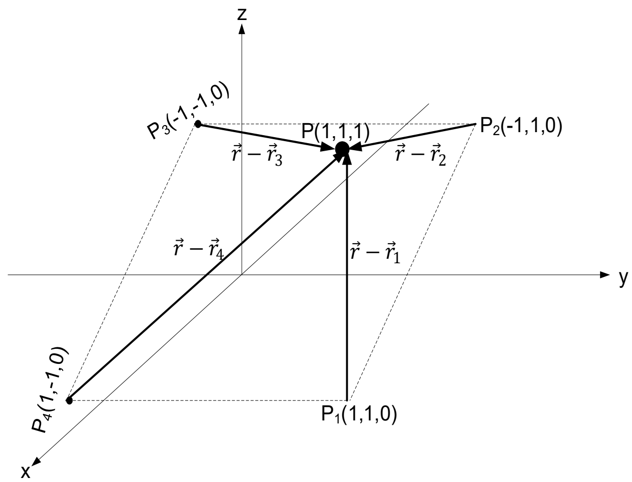


Figure 3.6

(This is a rough "Pyramid" Egyptian or not 1 cannot tell! However, it lacks symmetry as P does not lie on z-axis)

$$\vec{r} = \hat{a}_x + \hat{a}_y + \hat{a}_z, \hat{r}_1 = \hat{a}_x + \hat{a}_y, \hat{r}_2 = -\hat{a}_x + \hat{a}_y$$

$$\overrightarrow{r_3} = -\hat{a}_x - \hat{a}_y, \hat{r}_4 = \hat{a}_x - \hat{a}_y$$

$$|\vec{r} - \vec{r}_1| = |\hat{a}_z| = |; |\hat{r} - \hat{r}_2| = |2\hat{a}_x + \hat{a}_z| = \sqrt{5}$$

$$|\vec{r} - \vec{r}_3| = \left|2\hat{a}_x + 2\hat{a}_y + \hat{a}_z\right| = 3; |\hat{r} - \hat{r}_4| = \left|2\hat{a}_y + \hat{a}_z\right| = \sqrt{5}$$

$$\frac{Q}{4\pi\epsilon_0} = \frac{2 \times 10^{-9}}{4\pi \times 10^{-12}} = 17.97 \text{ V. m}$$

$$\Rightarrow \quad \vec{E} = 17.97 \left[\frac{\hat{a}_z}{1}\frac{1}{1^2} + \frac{2\hat{a}_x + \hat{a}_z}{\sqrt{5}}\frac{1}{\sqrt{5^2}} + \frac{2\hat{a}_x + \hat{a}_y + \hat{a}_z}{3}\frac{1}{3^2} + \frac{2\hat{a}_y + \hat{a}_z}{\sqrt{5}}\frac{1}{\sqrt{5^2}}\right]$$
$$= 4.55\hat{a}_x + 4.55\hat{a}_y + 21.87\hat{a}_z \text{ V/m}$$

3.1.1 Field from a Continuous Volume Charge Distribution

For a continuous distribution of charges, disregarding irregularities or ripples in the field owing to electron-to-electron idiosyncrasies, we take a macroscopic view of things and ignore the internal, microscopic phenomena.

For volume charge distribution ρr,with the units of coulombs per cubic meter (C/m^3), for a small amount of charge.

$\Delta Q = P_v \nabla_v$, where Δ_N stands for a small volume.

Mathematically defined,

$$\rho_r = \lim_{\Delta N \to 0} \frac{\Delta Q}{\Delta N}$$

Total charge within a finite volume

$$Q = \int_{Vol} \rho v \; dN.Vol \qquad \textit{indicating a } \text{Triple integration.}$$

Example 3.5: Determine the total charge contained in a 4.cm length of an electron in Fig. 3.7.

$\rho_v = -5e^{-10z\rho} \, C/_{m^3}$

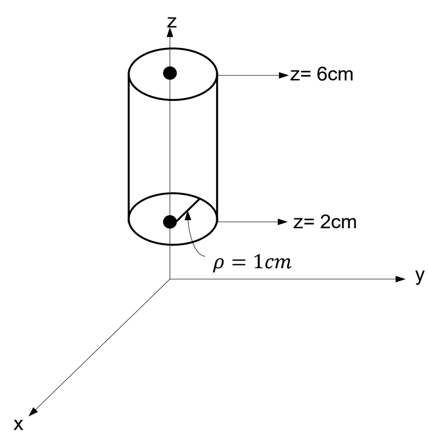


Figure 3.7

Charge density is given as $\Longrightarrow \rho_v = -5e^{-10z\rho}\mu \, \mathrm{C}/_{\mathrm{m}^3} = -5 \times 10^{-6} \, e^{-10z\rho} \, \mathrm{C}/_{\mathrm{m}^3}$

$$Q = \int_{0.02}^{0.06} \int_{0}^{2\pi} \int_{0}^{0.01} -5 \times 10^{-6} e^{-10z\rho} \rho d\rho d\emptyset dz$$

With respect to ∅,

$$(w.r.t.z)\quad Q = \int_{0.02}^{0.06}\int_{0.5}^{0.01} -10^{-5}\pi e^{-10ez}\,\rho d\rho dz$$

$$Q = \int_{0}^{0.01} \frac{-10^{-5}\pi e^{-10}}{-10^{5}e}\,\rho d\rho\Bigg|_{z=0.02}^{z=0.06}$$

Note that you have to multiply the above integration equation by $\frac{1}{\rho}$

$$\int_{0}^{0.01} -10^{-10}\pi\,(e^{-2000\rho} - e^{-6000\rho})d\rho$$

$$(w.r.t.e)\quad : Q = -10^{-10}\pi\left(\frac{e^{-2000\rho}}{-2000} - \frac{e^{-6000\rho}}{-6000}\right)_{0}^{0.01}$$

$$= -10^{-10}\pi\left(\frac{1}{2000} - \frac{1}{6000}\right)$$

$$= -10^{-10}\pi\left(\frac{2}{6000}\right)$$

$$= -10^{-12}\left(\frac{\pi}{30}\right) = -0.105\,p\text{C}\;(picocoulombs)$$

For an incremental charge $\Delta Q\ at\ \vec{r}$, the incremental contribution to the electric field intensity produced is:

$$\Delta\vec{E}(\bar{r}) = \frac{\Delta Q}{4\pi\epsilon_0|\hat{r} - \hat{r}_1|^2}\frac{\vec{r} - \vec{r}}{|\vec{r} - \vec{r}|}$$

$$= \frac{eV\Delta v}{4\pi\epsilon_0|\hat{r} - \hat{r}_1|^2}\frac{\vec{r} - \vec{r}}{|\vec{r} - \vec{r}|}$$

In the limiting edge,

$$\vec{E}(\bar{r}) = \int_{vol} \frac{eV(\bar{r})dv}{4\pi\epsilon_0|\hat{r} - \hat{r}_1|^2} \frac{\vec{r} - \vec{r}}{|\vec{r} - \vec{r}|} \qquad 3.8$$

Which is a triple integral, with vector $\bar{r}$ from the origin locating the field point where $\vec{E}$ is being determined, while vector $\bar{r}$, extends from the origin to the source point where $ev(\bar{r})dv$ is located. $(\bar{r} - \bar{r})/|\bar{r} - \bar{r}|$ is a unit vector. In the direction from source point to field point, with the variables of integration being "X" Y, Z of rectangular coordinates.

Example 3.6: Calculate electric field E at a point N (3, -4, 2) in free space caused by

(a) A charge $Q_1 = 2\mu C \ at \ P_1 \ (0,0,0)$
(b) A charge $Q_2 = 3\mu C \ at \ P_2 \ (-1,2,3)$
(c) Both charges $Q_1 \ and \ Q_2$

Solution: the vector from point P_1 to point N is

$$\vec{r}_1 = (3-0)\hat{a}_x + (-4-0)\hat{a}_y + (2-0)\hat{a}_y = 3\hat{a}_x - 4\hat{a}_y + 2\hat{a}_x$$

$$|\vec{r}_1| = \sqrt{29}$$

Unit vector from P_1 to N

$$\hat{a}_{r1} = \frac{3\hat{a}_x - 4\hat{a}_y + 2\hat{a}_z}{\sqrt{29}}$$

Field at N due to charge at P_1

$$\vec{E}_{N1} = \frac{Q_1}{4\pi\varepsilon_0|\vec{r}_1|^2}\hat{a}_{r1} = \frac{9\times10^9\times2\times10^{-6}}{29}\times\frac{3\hat{a}_x - 4\hat{a}_y + 2\hat{a}_z}{\sqrt{29}}$$

$$= 345\hat{a}_x + 460\hat{a}_y + 230\hat{a}_z \ \ V/m$$

(b) the vector from point $P_2 \ to \ N$ is

$$\vec{r}_2 = (3+1)\hat{a}_x + (-4-2)\hat{a}_y + (2-3)\hat{a}_z = 4\hat{a}_x - 6\hat{a}_y - \hat{a}_z$$

$$|\vec{r}_2| = \sqrt{53}$$

$$\hat{a}_{r2} = \frac{4\hat{a}_x - 6\hat{a}_y - \hat{a}_z}{\sqrt{53}}$$

Field at N due to charge at P_2

$$\vec{E}_{N2} = \frac{Q_2}{4\pi\varepsilon_0\, |\vec{r}_2|^2}\hat{a}_{r2} = \frac{9\times10^9\times3\times10^{-6}}{53}\cdot\frac{4\hat{a}_x - 6\hat{a}_y + \hat{a}_z}{\sqrt{53}}$$

$$= 280\hat{a}_x - 419\hat{a}_y - 69.9\hat{a}_z \;\; V/m$$

(c) the total field at point N

$$\vec{E}_N = \vec{E}_{N1} + \vec{E}_{N2}$$

$$= \left(345\hat{a}_x - 460\hat{a}_y + 230\hat{a}_z\right) + \left(280\hat{a}_x - 419\hat{a}_y - 69.9\hat{a}_z\right)$$

$$625\hat{a}_x - 879\hat{a}_y + 160.11\hat{a}_z \;\; \text{V/m}$$

3.2 Field of a Line Charge

For a cylindrical charged conductor of very small radius, it is convenient to treat the charge as a live charge with charge density $e_L C/m$.

Assuming a straight-line charge extending along the z-axis in a cylindrical coordinate system (an obvious choice) from $-\infty\ to\ \infty$, we desire the electric field intensity $\vec{E}$ at any and every point resulting from a uniform line charge density e_L see Fig. 3.8

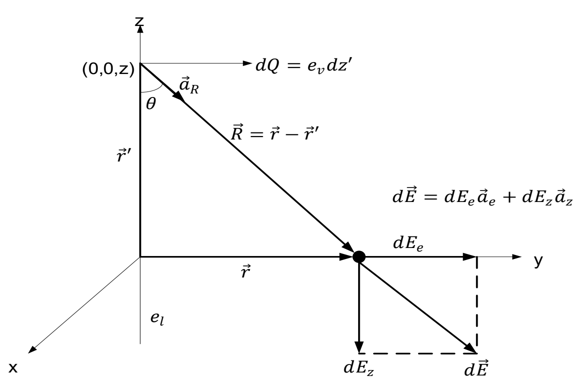


Figure 3.8

Keeping e and z constant, and varying Ø, the line charge does not change, indicating arithmethal symmetry. Also, axial symmetry allows the line charge to not charge with charges in z. as e charges (increases), however, the line charge also charges (becomes weaker) so the field is only a function of e elements of charges produce no $E\emptyset$ components (above and below), leaving only E_e that depends on e. For a point $P(0, y, 0)$ (on the y axis same direction as e,) to find the incremental field due to the incremental charge $dQ = e_L\, dz$,

$$d\vec{E} = \frac{e_L\, dz}{4\pi\epsilon_0 |\hat{r} - \hat{r}_1|^2} \frac{\vec{r} - \vec{r}}{|\vec{r} - \vec{r}|}$$

With $\hat{r} = y\hat{a}_y = e\hat{a}_e$

$$r = z\hat{a}_z = \hat{r} - \hat{r} = e\hat{a}_e - z\hat{a}_z$$

$$\Rightarrow \quad d\vec{E} = \frac{e_L\, dz(p\hat{a}_e - z\hat{a}_z)}{4\pi\epsilon_0 (e^2 z^2)^{3/2}} \Rightarrow \frac{e_L e\, dz}{4\pi\epsilon_0 (e^2 z^2)^{3/2}} = Ee$$

As only the $\vec{E}e$ component is present

$$\Rightarrow \qquad \vec{E} = \frac{e_L}{2\pi\epsilon_0 p}\hat{a}_e \qquad 3.9$$

So, for a line charge, the intensity varies only inversely with the distance, as opposed to a point charge where it varies inversely with the square of the distance.

Discarding symmetry, however, let's look at a case where an infinite line charge is parallel to z axis at x = y, y = 6.

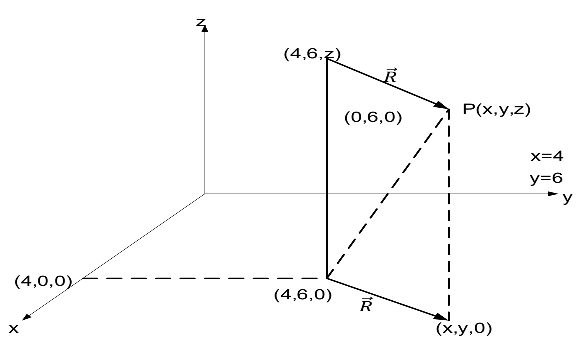


Figure 3.9

To find $\vec{E}$ at the general field point $P(x, y, z)$, e ⟹radial distance between the line charge and point,

$$P, R = \sqrt{(x-4)^2 + (y-6)^2 \hat{a}_e = \hat{a}_R}$$

$$\Longrightarrow \quad \vec{E} = \frac{e_L}{2\pi\epsilon_0\sqrt{(x-4)^2 + (y-6)^2}}\hat{a}_r$$

$$\hat{a}_R = \frac{\vec{R}}{|R|} = \frac{(x-4)\hat{a}_x + (y-6)\hat{a}_y}{\sqrt{(x-4)^2 + (y-6)^2}}$$

$$\Longrightarrow \quad \vec{E} = \frac{eL}{e\pi\epsilon_0} - \frac{(x-4)\hat{a}_x + (y-6)\hat{a}_y}{\sqrt{(x-4)^2 + (y-6)^2}} \; independent\ of\ z$$

3.3 Field of a Sheet of Charge

For an infinite sheet of charge with a uniform density of $e_s \, {}^{c}/_{m^2}$ (s indicating surface), such a charge distribution may often be used to approximate that found on the conductors of a strip transmission line or a parallel plate capacitor (static charges exist on the surfaces of conductor and not in the interiors.

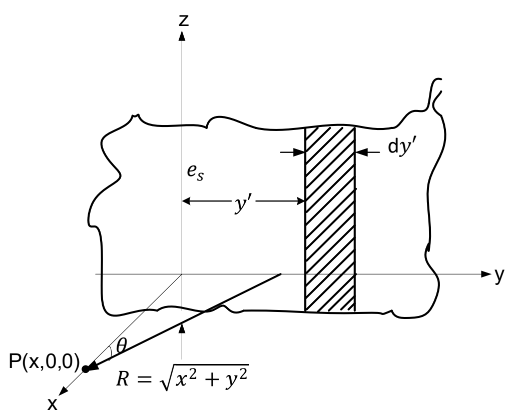


Figure 3.10

In Fig. 3.10, a sheet of charge is placed in the y-z plane. The field is independent of changes in y and/or z, and the y and z components from differential elements of charge symmetrically located with respect to the point at which the field is evaluated will cancel. So, the only component present is Ex and this depends on x along.

$From\ \vec{E} = \frac{eL}{2\pi\varepsilon_0\ e}\hat{a}_e$, Used in made of the field of infinite line charge by dividing the infinite sheet into differential width strips. Shown at the end of the proceeding page. The density of the line charge (charge per unit length) is $e_L = e_s dy'$, and the distance from this line charge to a general point P on the x axis is $R = \sqrt{x^2 + y^2}$. From the diferential width strip the contribution to Ex is:

$$dE_x = \frac{e_s d_y}{2\pi\epsilon_0\sqrt{x^2 + y^2}}\cos\theta = \frac{e_s}{2\pi\epsilon_0}\frac{xdy}{x^2 + y^2}$$

For the entire range of strips,

$$E_x = \frac{es}{2\pi e}\int_{-\infty}^{\infty}\left[\frac{xdy}{x^2 + y^2} = \frac{e_s}{2\pi\epsilon_0}\ tan^{-1}\frac{y}{x}\right]_{-\infty}^{\infty} = \frac{e_s}{ze_0}$$

And for point P on the negative x axis, $E_x = -\frac{e_s}{2\epsilon 0}$ because the field is always directed away from the positive charge.

For a unit vector $\hat{a}_N$,normal to the theet and directed outward (away) from it,

$$\vec{E} = \frac{e_s}{2\epsilon_o}\hat{a}_N \qquad 3.10$$

This shows that the field is constant in magnitude and direction, that is a constant vector!

For a second infinite sheet of charge with a negative charge density e_s located in the plane $x = a$

The total field is found by adding the contribution of each sheet. In the region $x > a$.

$$\vec{E}_1 = \frac{e_s}{2\epsilon_0}\hat{a}_x, E = -\frac{e_s}{2\epsilon_0}\hat{a}_x \Longrightarrow \vec{E} = \vec{E}_1 + \vec{E} = 0$$

For $x < 0$, $\vec{E}_1 = \frac{e_s}{2\epsilon_0}\hat{a}_x, E = -\frac{e_s}{2\epsilon_0}\hat{a}_x \Longrightarrow \vec{E} = \vec{E}_1 + \vec{E} = 0$

For $0 < x < a$,

$$\vec{E} = \frac{P_s}{2\epsilon_0}\hat{a}_x, \vec{E} = \frac{e_x}{2\epsilon_0}\hat{a}_x$$

3.4 Streamlines and Sketches of Fields

Field about a line charge, $\vec{E} = \frac{e_L}{2\pi\epsilon_0 e}\hat{a}_e$

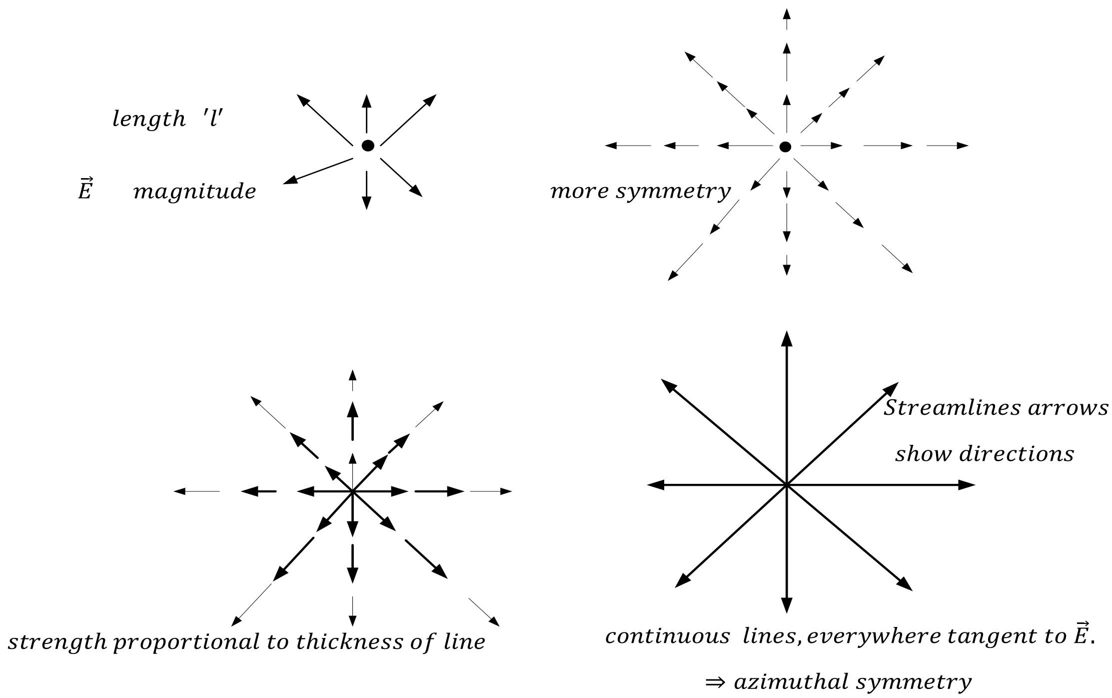


Figure 3.11

For streamline sketch, the magnitude of the field is inversely proportional to the spacing of the streamlines for special cases. Therefore, the closer together the streamlines, the stronger is the field. Bet it noted that for a point charge, the field does also vary into and away from this book, and so sketching is limited to two dimensional fields.

For $E_z = 0$, the streamlines are limited to lanes for which z is constant, and the sketch is the same for any such plane.

For the several streamlines indicated below.

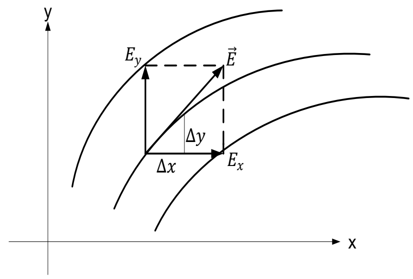


Figure 3.12

With E_x and E_y indicated at a general point $\frac{Ey}{Ex} = \frac{dy}{dx}$, solved by a form of the functions E_x and E_y.

Example 3.6: For the field of a uniform lime charge with $e_L = 2\pi\epsilon o$, $\vec{E} = \frac{1}{e}\hat{a}_e$

In rectangular coordinates,

$$\vec{E} = \frac{x}{x^2 + y^2}\hat{a}_x + \frac{y}{x^2 + y^2}\hat{a}_y$$

$$\Rightarrow \frac{dy}{dx} = \frac{Ey}{Ex} = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

$$\Rightarrow \; Equation\ of\ streamlines\ y = Cx \qquad\qquad **$$

To find the Equ ** of a streamline passing through $P(3, -5, 12)$, we substitute the coordinates of that point into our equation and then evaluate C.

$$-5 = C(3) \Rightarrow C = -{}^{5}/_{3} = -1.667 \Rightarrow y = -1.667x$$

For every streamline there's an associate value of C, and he radial lines of the last diagram on the top right corner of the proceeding page are obtained with $C = 0, 1, -1\ and \frac{1}{c} = o$, meaning for the last one that C increases without bound, C just being the gradient of the straight line passing through the origin in x- y coordinates.

3.5 Electric Flux Density

Electric flux $\psi(psi)$ and the total charge on an inner sphere, enduced by charged outer sphere, is given in Faraday's experiment by $\psi = Q$coulombs

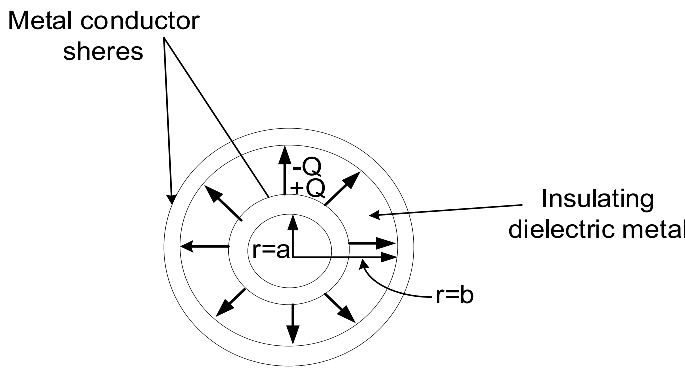


Figure 3.12

On the diagram of the preceding, an inner sphere has a radius of a, and the outer sphere has a radius of b, will charge of +Q and –Q respectively. Electric flux ψ extend from the inner sphere to the outer sphere and the paths are indicated by the symmetrically distributed streamlines drawn radially from the inner sphere to the outer sphere.

The inner sphere has surface area of $4\pi a^2 m^2$, and charge of $Q = \psi c$ is distributed uniformly over it, giving rise to flux density of $\frac{\psi}{4\pi a^2} = \frac{Q}{4\pi a^2} {}^{c}/_{m^3} = D$, a vector field. $\vec{D}$ has a direction at a point which is the same as that of the flux lines, at that point and the magnitude is given by the number of flux lines crossing a surface normal to the lines divided by the surface area. In the Fig. 3.12, $\vec{D}$ is in the radial direction, with a value of

$$\vec{D}|_{r=a} = \frac{Q}{4\pi a^2}\hat{a}_r \ \ for\ inner\ sphere$$

$$\vec{D}|_{r=b} = \frac{Q}{4\pi b^2}\hat{a}_r \ \ for\ outer\ sphere$$

In the limit as the inner sphere becomes smaller and smaller while retaining the charge of Q, it results in a point charge y but the electric flux density at a point **r** meter from the point charge is still

$$\vec{D}|_{r=a} = \frac{Q}{4\pi r^2}\hat{a}_r$$

Became Q lines of flux are symmetrically directed outward from that point and pass through an imaginary spherical surface of area $4\pi r^2$.

From $\vec{E} = \frac{Q}{4\pi\epsilon p a^2}\hat{a}_r$, derived earlier then $\frac{\vec{D}}{\epsilon_0} = \vec{E} \Longrightarrow \vec{D} = \epsilon_0\vec{E}$ for only free space. For a general charge distribution in free space,

$$\vec{E} = \int_{vol} \frac{e_v dv}{4\pi\epsilon_0 R^2}\hat{a}_R, for\ only\ free\ space.$$

$$Similarly, \vec{D} = \int_{vol} \frac{e_v dv}{4\pi R^2}\hat{a}_R,$$

Be it noted that, for a point charge embedded in an infinite ideal dielectric medium, the equation (expression at the bottom of the proceeding page still holds, and therefore the volume integral expression for $\vec{D}$ above still holds also. The expression before it relating $\vec{E}$ is only however, for free space. In free space, $\vec{D}$ is directly proportional to $\vec{E}$, with the constant of proportionality being the permittivity of free space ϵo.

3.6 The Electric Force

Say a charge Q is locate at some point in space, a point denoted by position vector, r. Likewise, there exists everywhere in space an electric field (we neither know nor care how this electric field was created).

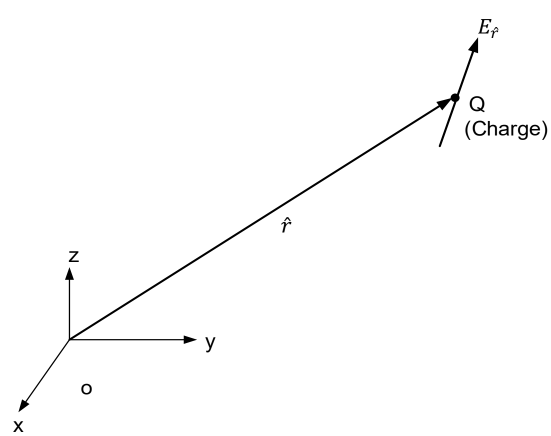


Figure 3.13 Vector at point r

The value (both magnitude and direction) of the electric field vector at point r is $E(\bar{r})$ as in Fig. 3.13.

Example 3.7(a): Our "Field theory" of electromagnetic says that the electric field will apply a force on the charge. Precisely what is this force (i.e, its magnitude and direction)?

Solution: fortunately, the answer is rather simple! The force Fe on charge Q is the product of the charge (a scalar) and the value of the electric field (a vector) at point where the charge is located:

$$F_e = QE(\bar{r}) \ [N] \qquad 3.11$$

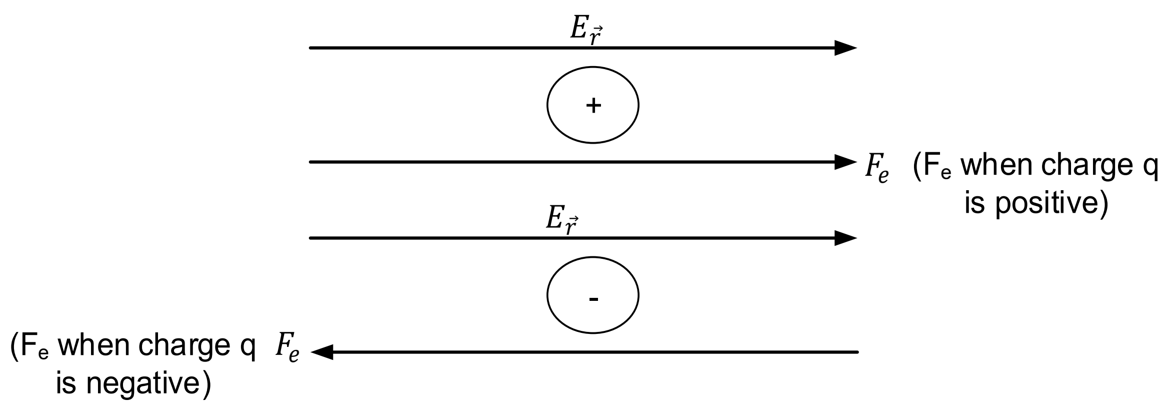


Figure 3.13

Note therefore, that the force vector will be parallel (or antiparallel) to the electric field!

Q > O (charge is negative) so F_e points in the opposite direction as the electric field as in Fig. 3.14

Note the magnitude of the electric force will increase proportionally with an increase in charge and/or increase in the electric field magnitude.

Example 3.7(b) Calculate the force of attraction between $Q_1 = 3 \times 10^{-8}$ and $Q_2 = 2 \times 10^{-5}$ and spaced by 10cm apart in vacuum. What is force if it is placed in kerosene whose $\varepsilon_r = 2$?

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2} = \frac{2 \times 4 \times 10^{-13}}{4\pi \times 8.854 \times 10^{-12} \times (10 \times 10^{-2})^2}$$

$$\frac{1}{4\pi\varepsilon_0 r^2} = 9 \times 10^9$$

∴ $\vec{F} = 0.72$

In kerosene $\vec{F} = \frac{0.72}{2} = 0.36\text{ N}$ ***Ans***

3.7 Coulomb's Law of Force

Consider two-point charges, $Q_1 \; and \; Q_2$ located at positions r_1 and r_2 respectively.

We will find that each charge has a force F (with magnitude and direction) exerted on it.

This force is dependent on both the sign (+ or -) and the magnitude of charges $Q_1 \; and \; Q_2$ as well as the distance R between the charges.

Charles Coulomb determined this relationship in the 18th century! We call his result Coulomb's law:

$$F_1 = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R^2}\hat{a}_{21}[N] \quad 3.12$$

This force F1 is the force exerted on charge Q_1. Likewise, the force exerted on charge $Q_2$is equal to:

$$F_2 = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R^2}\hat{a}_{21}\;[N] \quad 3.13$$

In these formulae, the value ε_0 is a constant that describes the permittivity of free space (i.e vacuum).

$$\varepsilon_0 = permittivity\ of\ free\ space = 8.854 \times 10^{-12}\left[\frac{C^2}{Nm^2} = \frac{farads}{m}\right]$$

Note the only difference between the equations for force F_1 and F_2 are the unit vectors $\hat{a}_{21} \; and \; \hat{a}_{12}$.

Unit vector $\hat{a}_{12}$ points from the location of $Q_2(i.e., \bar{r}_2)$ to the location of charge $Q_1(i.e., \bar{r}_1)$.

$$F_1 = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R^2}\hat{a}_{21}$$

Likewise, unit vector $\hat{a}_{21}$ points from the location of $Q_1(i.e., \bar{r}_1)$ to the location of charge $Q_2(i.e., \bar{r}_1)$.

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R^2}(\hat{a}_{21})$$

$$= \left(\frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R^2}\hat{a}_{21}\right) = -F_2 \qquad 3.14$$

Note therefore, that these unit vectors point

in opposite directions, a result we express

mathematically as:

$$\hat{a}_{21} = -\hat{a}_{12}$$

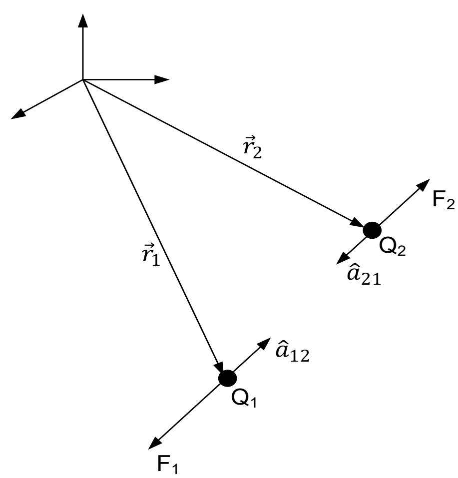


Figure 3.14 $\vec{F}_1$ and $\vec{F}_2$ are Equal and opposite

Look! For F_1 and F_2 have equal magnitude, but point in opposite directions!

Example 3.8: what happens when one of the charges is negative?

Solution: Look at coulomb's Law! If one charge is positive and other is negative, then the product $Q_1\, Q_2$ is negative. The resulting force vectors are therefore negative they point in the opposite directions of the previous (i.e., both positive) case in Fig.3.5.

3.8 Electric Field and Electric Field Strength (Intensity) $\vec{E}$

Electric field strength is defined as force per unit test charge.

$$\therefore \qquad \vec{E} = \hat{R}\frac{q}{4\pi\varepsilon_0 R^2}\ (V/m)$$

Where R is the distance between the charge and the observation point, $\hat{R}$ is the radial unit vector pointing away from the charge.

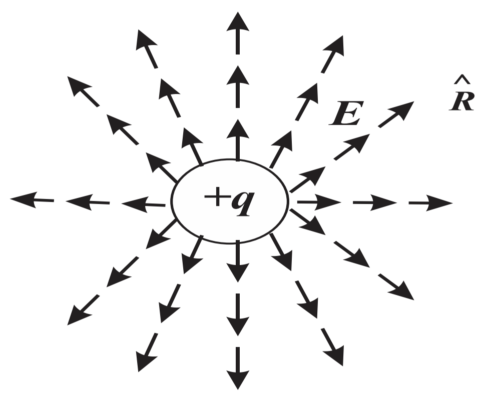


Figure 3.15 Force per unit Test

Electric field due to charge distribution:

The electric potential of a point at distance R from a point charge 'Q' referred to that at infinity is given as.

$$V = -\int_{\infty}^{R} \frac{q}{4\pi\varepsilon_0 R^2}\, dR$$

Which gives

$$V = \frac{q}{4\pi\varepsilon_0 R}$$

Above is a scalar quantity and depends only on distance R and charge ‘q’. Thus, potential difference between any two points P_2 and P_1 at distance R_2 and R_1 respectively from ‘q’ is

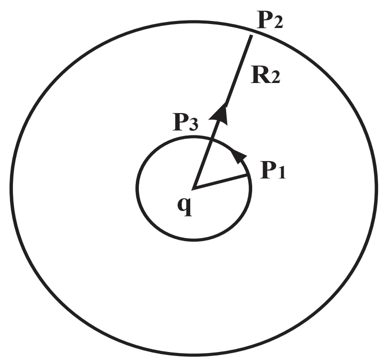


Figure 3.16 Path of Integration

$$V_{21} = V_{p2} - V_{p1}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R_2} - \frac{1}{R_1}\right]$$

Surprisingly, shown in Fig.3.16 above results is true even is P_2 and P_1may not lie on the same radial line through ‘q’.

Here $\quad V_{p2} - V_{p1} = V_{p1}$

You will conclude that choosing path of integration from P_1 to P_3 and them from P_3 to P_2 will give the result. as no work is done in moving a charge on equipotential surface, thus from P_1 *to* P_3 $\vec{E}.\overrightarrow{dl} = 0$.

The electric potential at R due to system of n distance point charges $q_1, q_2 \dots.. q_n$ located at $R_1, R_2 \dots. R_n$is by superposition, the sum of potential due to individual charges.

$$V = \frac{1}{4\pi\varepsilon_0} \sum_{R=1}^{n} \frac{q_R}{|\vec{R}\, R_k|}$$

Hereby, V is a scalar sum, so $\vec{E} = -\nabla V$ than vector sum

The electric potential due to continuous distribution of charge confined in a given region is obtained by integration the contribution of an element of charge over the charged region.

VOLUME CHARGE DISTRIBUTION $$V = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_v}{R}\, dv \ \ (V)$$

SURFACE CHARGE DISTRIBUTION $$V = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\rho_s}{\vec{R}}\, ds' \ \ (V)$$

LINE CHARGE DISTRIBUTION $$V = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\rho_L}{R}\, dl' \ \ (V)$$

3.8.1 Example: Electric field due to two-point charges in Fig 3.8

Two points charges $q_1 = 2 \times 10^{-5} C$ and $q_2 = 2 \times 10^{-5} C$ are located in free space at (1, 3, -1) and (3, 1, -2), respectively, in a Cartesian coordinates system.

$$E = E_1 + E_2 = \frac{1}{3\pi \cdot \varepsilon_0}\left[\frac{q_1\,(R - R_1)}{|R - R_1|^3} + \frac{q_2\,(R - R_2)}{|R - R_2|^3}\right]$$

If a small probe charge "Δq" is located at any point near a second fixed charge "q" the probe charge experience a force,

$$\Delta F = q\Delta q / 4\pi\ \varepsilon r^2$$

$$\Longrightarrow \qquad E = \frac{\Delta F}{\Delta q} = \frac{q}{4\pi\ \varepsilon r^2} \qquad 3.15$$

Now, let us calculate $\vec{E}$ for different charge distribution

3.9 Electric Potential (V)

In order to bring two charges near each other work must be done. In order to separate two opposite charges, work must be done.

$$V = \frac{W}{q_{moved}} \ \ (V) \qquad 3.16$$

Work or energy can be measured in Joules and charge is measured in Coulombs so the electrical can be measured in Joules per Coulomb, which has been defined as a volt.

The differential electric potential energy dW per unit charge is called the differential electric potential (or differential voltage) dV.

That is:

$$dV = \frac{dW}{q}$$

The potential difference between any two points P_1, P_2 is obtained by integrating.

$$V_{21} = V_2 - V_1 = -\int_{P1}^{P2} \bar{E} \cdot \bar{dl} \qquad 3.17$$

Along any path between them.

Where dl is the vector differential distance.

Example 3.9: An electric field is expressed in rectangular coordinates by $\vec{E} = 6x^2\hat{a}_x + 6y\,\hat{a}_y + 4\hat{a}_z V/m$. Find:

(a) V_{MN} if points M and N are specified by M (2, 6, -1) and N (-3, -3, 2)
(b) V_m if V = 0 at Q (4, -2, -35).
(c) V_N if V = 2 at P (1, 2, -4).

Solution:

(a) – 139 V
(b) – 120 V
(c) 19 V

Example 3.10: A nC (point charge is at origin in free space. Calculate V_1 if point P_1 is located at P_1 (-2, 3, -1) and (a) V = 0 at (6, 5, 4)

(b) V = 0 at infinity

(c) V =5V at (2, 0, 4).

Solution:

(a) 20.67 V
(b) 36 V
(c) 10.89 V

Example 3.11: An infinite charge sheet with surface charge density $\sigma_s c/m^2$ has a circular hole of radius "a". the sheet is placed in xy plane with its centre at origin. Using Coulomb's law or otherwise. Find the potential V and electrical field density $\vec{E}$ at any point distance 'z' away from the origin and along the positive z-direction.

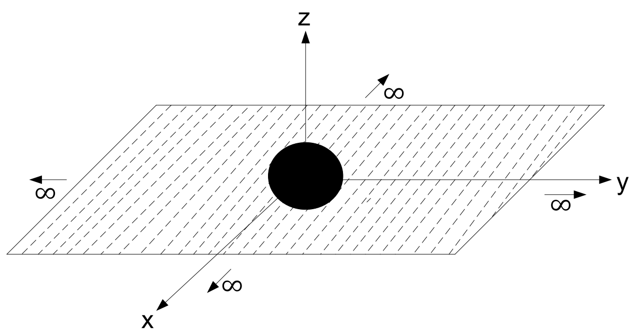


Figure 3.17

Solution:

for cases where we have a hole in conducting sheet having surface charge density σ_s we can imagine the hole having surface charge density of $-\sigma_s$ and solve the problem by two parts.

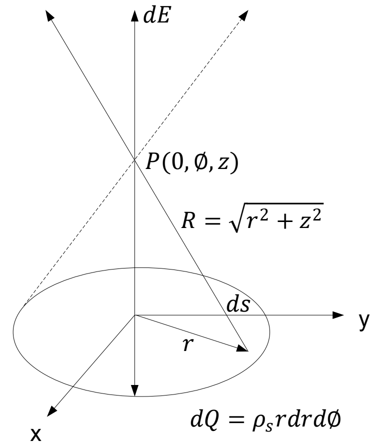


Figure 3.18

I. Electric field due to infinite conducting sheet with surface charge density $+\sigma_s$ c/m^2.

II. Electric field due to hole on charge sheet with surface charge density $-\sigma_s$ c/m^2 which can be separately considered as a circular sheet of radius "a" centered at origin with charge density $-\sigma_s$ c/m^2.

So, from text, I. $E_1=\frac{\sigma_s}{2\varepsilon_0}\hat{z}$

II. $$\int dE_2=\int_{\theta=0}^{2\pi}\int_{r=0}^{a}\frac{-\sigma_s\, r\, dr\, d\emptyset\, z\hat{a}_z}{4\pi\varepsilon_0(r^2+z^2)^{\frac{3}{2}}}$$

$$\vec{E}_2=\frac{-\sigma_s\, z}{4\pi\varepsilon_0}\int_{\theta=0}^{2\pi}\int_{r=0}^{a}\frac{r\, dr\, d\emptyset}{(r^2+z^2)^{3/2}}\ \hat{a}_z$$

$$=\frac{-\sigma_s\, z}{4\pi\varepsilon_0}\int_{r=0}^{a}\frac{r\, dr}{(r^2+z^2)^{\frac{3}{2}}}\ [\phi]_0^{2\pi}\hat{a}_z$$

$$\vec{E}_2=\frac{-\sigma_s\, z.2\pi}{4\pi\varepsilon_0}\int_{r=0}^{a}\frac{r\, dr}{(r^2+z^2)^{3/2}}\ \hat{a}_z$$

$$=\frac{-\sigma_s\, z}{2\pi\varepsilon_0}\int_{r=0}^{a}\frac{r\, dr}{(r^2+z^2)^{\frac{3}{2}}}\ \hat{a}_z$$

$$But \quad 2^2+z^2=t^2$$

$$| \; or\ 2r\ dr = 2t\ dt$$

$$| or\ r\ dr = t\ dt$$

$$\Longrightarrow \qquad \vec{E}_2 = \frac{-\sigma_s\ z}{2\ \varepsilon_0}\int_{r=0}^{a}\frac{t\ dt}{(t^2)^{3/2}}\ \vec{a}_z = \frac{+\sigma_s\ z}{2\ \varepsilon_0}\left[t^{-1}\right]_{r=0}^{a}\hat{a}_z$$

$$= \frac{+\sigma_s\ z}{2\ \varepsilon_0}\left[\frac{1}{\sqrt{r^2+z^2}}\right]_{r=0}^{a}$$

$$\Longrightarrow \qquad \vec{E}_2 = \frac{+\sigma_s\ z}{2\ \varepsilon_0}\left[\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{z^2}\right]\hat{a}_z$$

$$\vec{E}_2 = \frac{-\sigma_s}{2\ \varepsilon_0}\left[1 - \frac{z}{\sqrt{a^2+z^2}}\right]\hat{a}_z\ \mathrm{V/m}$$

So, net $\qquad E_{net} = E_1 + E_2$

$$\vec{E} = \frac{-\sigma_s z}{2\ \varepsilon_0\sqrt{a^2+z^2}}a\hat{z}\ \ \mathrm{V/m}$$

$$V = -\int_{init}^{final}\vec{E}\cdot\vec{dl} = -\int_0^z \frac{\sigma_s z}{2\ \varepsilon_0\sqrt{a^2+z^2}}dz$$

$$= \frac{-\sigma_s}{2\ \varepsilon_0}\int_0^z \frac{\sigma_s z}{\sqrt{a^2+z^2}}dz \qquad (let\ a^2+z^2 = t^2\ 2z\ dz = 2t\ dt)$$

$$= \frac{-\sigma_s}{2\ \varepsilon_0}\int_a^{\sqrt{a^2+z^2}}\frac{t\ dt}{\sqrt{t^2}} = \frac{-\sigma_s}{2\ \varepsilon_0}\int_a^{\sqrt{a^2+z^2}} dt = \frac{-\sigma_s}{2\ \varepsilon_0}\left[\sqrt{a^2+z^2} - a\right]$$

$$V = \frac{-\sigma_s}{2\ \varepsilon_0}\left[\sqrt{a^2+z^2} - a\right]\ \ \mathrm{V}$$

Example 3.12: We wish to find $\vec{D}$ in region about uniform line charge of 4 $n\mathrm{C/m}$ lying along z-axis in free space.

Solution: the $\vec{E}$ field

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}a_\rho = \frac{4\times10^{-9}}{2\pi\ (8.85\times10^{-12})\rho} = a_\rho = \frac{71.9}{\rho}a_\rho\ \ V/m$$

At $\rho = 3m, \vec{E} = 23.97 a_\rho$ V/m

Associated with E field we find

$$\vec{D} = \frac{\rho_L}{2\pi\rho} a_\rho = \frac{4 \times 10^{-9}}{2\pi\ \rho} a_\rho = \frac{0.6365 \times 10^{-9}}{\rho} a_\rho C/m^2$$

The value at $\rho = 3m\ is\ \vec{D} = 0.212\ a_p\ n\mathrm{C/m}$

3.10 Lines of Force

Lines of force is a curve drawn so that at every point it has direction of electric field

The number of lines per unit area is made proportional to magnitude of electric field strength 'E'

3.11 Lines of Flux

Lines of flux is a curve drawn so that at every point it has direction of electric flux density or displacement density, the number of flux lines per unit area is used to indicate the magnitude of displacement density, 'D'

3.12 Gauss Law

Mr. Gauss was perhaps, history's, greatest mathematician, inventing the summation theory by the age of ten!).

The electric flux passing through any imaginary spherical surface lying between two conducting spheres in equal to the charge enclosed within that imaginary surface. The enclosed charge is either distributed on the surface of the inner sphere, or it might be concentrated as a point charge at the center of the imaginary sphere. Because of the relationship between electric flux and charge (equality) the actual geometrical configuration of the space occupied by the charges, is immaterial since the result is the same.

Gauss Law: the electric flux passing through any closed surface is equal to the total change enclosed by that surface.

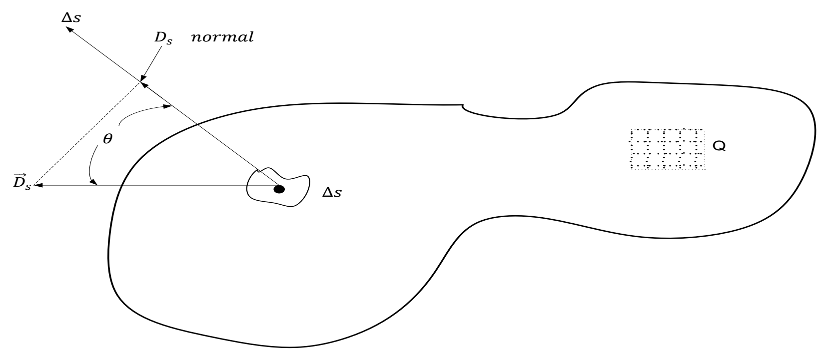


Figure 3.19 Charge in a Closed Surface of Arbitrary Shape

In the above diagram, total charge Q is enclosed within a closed surface of arbitrary shape. According to Gauss's law, then Q coulombs of electric flux will pass through then closed surface. The electric flux density vector $\vec{D}$ will have some value $\vec{D_s}$ in general varies from point to point on this surface, in magnitude and direction.

We have an incremental area ΔS that approximates a position of a plane surface, and is a vector quantity having magnitude and direction pointing normal away from the said incremental "plane" a plane tangent to the surface at that "point".

Taking an incremental element of surface ΔS at any point P, and letting $\vec{D}_s$make an angle $\theta\ with\ \Delta S$, the flux crossing ΔS is then the product of the normal component of $\vec{D}_s$ and $\Delta S, \Delta\psi = Flux\ crossing\ \Delta S = D_s,\ norm\ \Delta S = D_s \cos\theta\ \Delta S = \vec{D}_s.\Delta S$.

Total flux passing through the closed surface, is the sum of all each of which cross ΔS,

$$\psi = \int d\psi = \oint_{closed\ surface} \vec{D}_s.ds, \qquad 3.18$$

Which is a closed surface integral, a double integral being the integral of element involving an area which has two dimensions.

Gauss's law: $Q = \oint_S \vec{D}_s.\overrightarrow{ds} =$ charge enclosed $= Q$

$$Q = \int_{e_L} dL\ \ for\ a\ line\ charges, or$$

$$Q = \int_s es\, ds\ for\ a\ geenral\ surface\ charge, or$$

$$Q = \int_{ve} ev\ dv\ \ for\ a\ volume\ charge\ distribution$$

$$\Longrightarrow \quad Gauss's\ Law: \oint_s \vec{D}_s . \overrightarrow{ds} = \int_{vol} ev\ dv$$

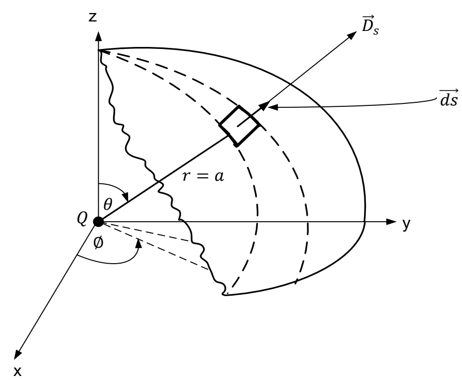


Figure 3.20 A Point Charge on a Spherical Coordinate System

In the preceding page, a point charge is at the origin of a spherical coordinate system and by choosing a closed surface of radius a.

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{a}_r$$

At the surface of the sphere,

$$\overrightarrow{D_s} = \frac{Q}{4\pi a^2}\hat{a}_r$$

$$ds = r^2 \sin\theta\ d\theta\ d\emptyset = a^2 \sin\theta\ d\theta\ d\emptyset \Longrightarrow \overrightarrow{ds} = a^2 \sin\theta\ d\theta\ d\emptyset\ \hat{a}_r$$

$$\vec{D}_s . ds = \frac{Q}{4\pi a^2}\ a^2 \sin\theta d\ \theta d\ \emptyset \hat{a}_r . \hat{a}_r = \frac{Q}{4\pi}\sin\theta\ d\theta\ d\emptyset$$

$$\Longrightarrow \int_{\emptyset=0}^{\emptyset=2\pi} \int_{\theta=\emptyset}^{\theta=\pi} \frac{Q}{4\pi}\sin\theta\ d\theta\ d\emptyset$$

$$\int_0^{2\pi} \frac{Q}{4\pi}(-\cos\theta)_0^{\pi}\, d\emptyset = \int_0^{2\pi} \frac{Q}{2\pi}\, d\emptyset = Q, consistent\ with\ Gauss's\ law$$

3.13 Electric Flux Density $(\vec{D})$

In addition to the electric field intensity E, we will often find it convenient to also used a related quantity called the electric flux density D, given by

$$\bar{D} = \varepsilon\bar{E} \quad (C/m^2) \qquad 3.19$$

In vacuum $D = \varepsilon_0 E$

$E = f$(Magnitude, position of charge q and dielectric constant of medium)

$D \neq f$ (medium)

3.13.1 "Faraday's Experiment" (with concentric spheres)

Electric displacement from charge on inner sphere through the medium to the outer sphere as in Fig. 3.21.

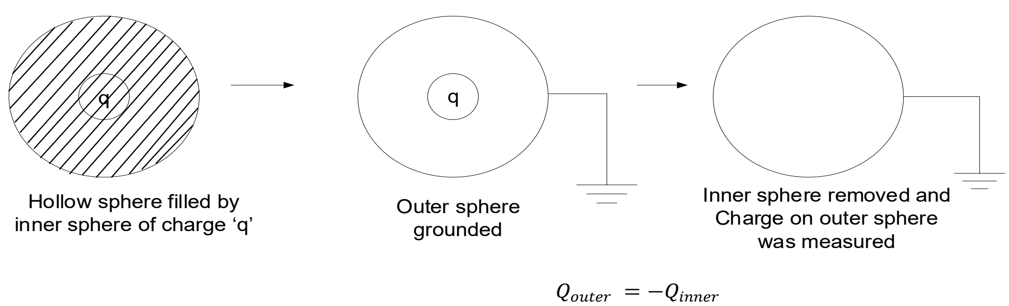


Figure 3.21. Faraday's experiment

The amount of this displacement (Ψ) depends only upon the magnitude of charge Q.

$$\Rightarrow \qquad \Psi = Q \qquad 3.20$$

For the case of an isolated point charge 'q' remote from outer bodies the outer sphere is assumed to have infinite radius.

The electric displacement per unit area at any point on a spherical surface of radius 'r' centered at isolated charge q will.

$$D = \frac{\Psi}{4\pi r^2} = \frac{q}{4\pi r^2}\,\hat{r} \quad \frac{Coulomb}{sqm}$$

Vector Quantity. Its direction being taken as that direction of normal to the surface element which makes the displacement through the element of area of maximum.

Above is true for isotropic media

For anisotropic dielectric

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \qquad 3.21$$

3.14 Alternative Statement of Gauss's Law

Gauss's law: $\oint_S \vec{D} \cdot \overrightarrow{da} = \int_v \rho \, dV$

Applying divergence theorem

$$\oint_V \nabla \cdot D \, dv = \int_V \rho \, dV \qquad 3.22$$

$$\nabla \cdot D = \rho_v \qquad 3.23$$

(As volume considered is reduced to an element volume)

At every point in a medium the divergence of electric displacement density is equal to the charge density

3.15 The Use of Gauss's Law

For a known charge distribution, given $Q = \oint_s \overrightarrow{Ds}. ds$, $\vec{D}_s$ can be determined even though it's the quantity appearing in the integral! But the, therein lies the advantage that the law of Gauss provides. To proceed a closed surface has to be chosen so that:

1. $\vec{D}_S.\vec{d}_S$ is everywhere either normal so that it becomes simply $D_s$ds or its tangential so that it is zero;
2. Where $\vec{D}_S.\vec{d}_S$ is not zero, Ds = constant, so that only ds can then vary.

$$\Rightarrow \quad Q = \int_S D_S.\overrightarrow{ds} = \oint_S Ds\ ds = D_S \oint_S ds \qquad 3.24$$

For normal arousing:

For a point charge, spherical coordinates are conveniently chosen so that the two conditions above would be satisfied since $\vec{D}_S$ has the same value throughout the entire surface (symmetry is obvious).

Going further, therefore from the Eq (3.24) we have that:

$$Q = D_S \oint_S ds = D_S \int_{\emptyset=0}^{\theta=2\pi} r^2 \sin\theta\ d\theta\ d\emptyset = 4\pi r^2 D_S$$

And we have "performed" a seemingly intractable integral without breaking a sweat.

So,

$$D_S = \frac{Q}{4\pi r^2}$$

Since $0 \leq r < \infty$, and because of the outwardly radial nature of $\vec{D}_S$, then:

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{a}_r \ , \qquad \vec{E} = \frac{Q}{2\pi\epsilon_0 r^2}\hat{a}_r \qquad 3.25$$

As earlier shown. Limitations, however are towards symmetry.

For a line charge, the only component present is the radial, obviously for uniform line charge.

$$\Rightarrow \quad \vec{D} = D_\rho\ \hat{a}_\rho = f(\rho)\hat{a}_\rho$$

Showing that D_ρ (a scalar) depends only on ρ.

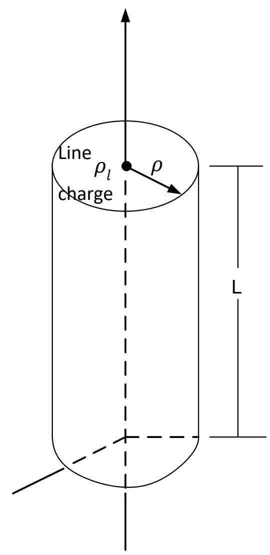


Figure 3.22 A close right circular cylinder of radius 'ρ' from z = 0 to z = L, and closed line surfaces normal to z-axis

$$Q = \oint_s \vec{D}_s . \vec{ds} = D_s \int_{sides} ds + 0 \int_{top} ds + 0 \int_{bottom} ds$$

$$= D_s \int_{z=0}^{L} \int_{\emptyset=0}^{2\pi} e d\emptyset \; dz = D_s \; 2\pi\rho L$$

$$\Longrightarrow \qquad D_\rho = D_\rho = \frac{Q}{2\pi\rho L} \qquad 3.26$$

But then, Q is equally $\rho L \, \frac{coulomb}{unit \; lenght} \times lenghts$

$$= \rho_L L \Longrightarrow D = \frac{\rho_L L}{2\pi\rho L}$$

So, $\qquad D_\rho = \frac{\rho L}{2\pi\rho} \Longrightarrow Ep = \frac{\rho L}{2\pi\epsilon_0\rho}$

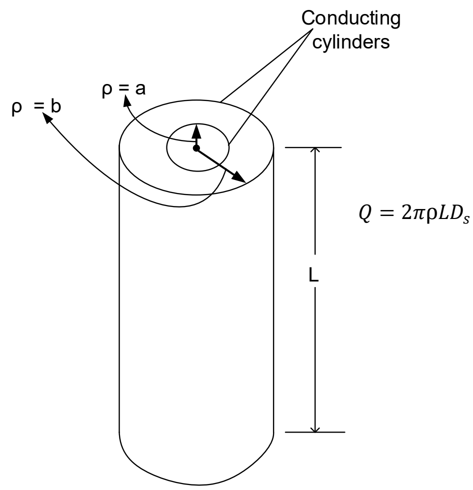


Figure 3.23

Linear conductor

$$Q = \int_{z=0}^{L}\int_{\emptyset=0}^{2\pi} \rho_s a \; d\emptyset \; dz = 2\pi a L \rho s$$

$$\Longrightarrow \quad D_s 2\pi\rho_L = 2\pi a L\rho s = D_s = \frac{\hat{a}_{\rho s}}{p} = \vec{D} = \frac{a\rho s}{\rho}\hat{a}_\rho$$

For inner conductor, charge per unit length

$$\frac{Q}{L} = \frac{2\pi a\rho s L}{L} = 2\pi a\rho s = \rho L$$

$$\Longrightarrow D_s = \frac{Q}{2\pi\rho L} = \frac{2\pi a L(\rho s)}{2\pi\rho L} = \frac{a}{\rho}(\rho s) = \frac{a}{\rho}\left(\frac{\rho_L}{2\pi a}\right) = \frac{\rho_L}{2\pi\rho}$$

$$\vec{D} = \frac{\rho_L}{2\pi\rho}\hat{a}_\rho$$

3.16 Total Charge on Outer Cylinder

For a Gaussian surface of cylinder with radius $\rho > b$, the total charge would be zero since there are opposite and equal charges on the inner and outer conducting cylinders.

So, $\quad D_s 2\pi\rho L = 0 = D_s = 0 \; for \; \rho > b$

Also, for ρ > a, same result, logically, so that charges are enclosed only between the two conductors for an infinite length of coaxial cable. Even for a finite length, the result still holds as long as the length is many times greater than the radius B, such that the non-symmetry at the ends do not appreciably affect the result.

Example 3.12: To determine the charge densities and $\vec{E}\ and\ \vec{D}$ of the cylinders of a coaxial cable with an inner radius of 1mm and outer of 5mm, with the space in between assumed to be filled with air. The total charge on the inner conductor is 40nC and the length of the cable is 80cm.

Solution:

$$\rho_s\ inner\ cyl = \frac{Q\ inner\ cyl}{2\pi aL} = \frac{40 \times 10^{-9}}{2\pi(10^{-3})(0.8)} = 7.96\mu c/m^2$$

$$\rho_s, outer\ cyl = -\ \frac{40 \times 10^{-9}}{2\pi(5 \times 10^{-3})(0.8)} = -\frac{1.59\mu C}{m^2}$$

(Internal fields):

$$D_s = \frac{a_{\rho s}}{\rho} = \frac{10^{-3}(7.9 \times 10^6)}{\rho} = 7.96\mu\, {}^{c}/_{m^2}$$

$$\Longrightarrow \quad E_\rho = \frac{D_\rho}{\epsilon_0} = \frac{7.96 \times 10^{-9}}{8.854 \times 10^{-12}\rho} = \frac{\frac{899}{\rho}V}{m} 1 < \rho < 5mm$$

3.17 Differential Volume Element

"Special" surface as a cube or a cylinder not having the length appreciably greater than the radius (and therefore, the diameter), lack symmetry, and therefore the application of the Gaussian principle cannot hold. To circumvent this difficulty, however, a very small size of these can be chosen, so that in the limiting case, symmetry is almost achieved. Applying Taylor's-series expansion for $\vec{D}$ using the first two terms a nearly correct result is achieved as the volume inside the Gaussian surface decreases.

In anticipation of one of Maxwell's four equations (basic to all electromagnetic theory), let's consider a point P located rectangularly inside a cube (its reluctant to say "middle of" since the cubes dimensions $\ggg zero$.

$$\vec{D} = \vec{D}_0 = D_{x_0}\vec{a}_x + D_{y_0}\vec{a}_y + D_{z_0}\vec{a}_z$$

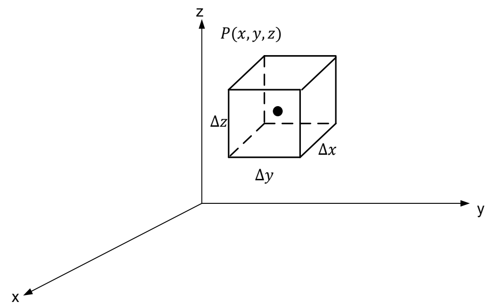


Figure 3.24 A differential sized gaussian of a 'P' surface used to determine the space rate of change of D in P neighborhood

For the above differential sized cube, $\vec{D}$ at point $P(x, y, z)$ may be expressed as:

$$\vec{D}_o = D_{0_r}\,\hat{a}_r + D_{o_y}\hat{a}_y + D_{o_z}\hat{a}_z \qquad 3.27$$

Gauss's law: $\oint_S \vec{D}.\overrightarrow{ds} = Q$

$$\oint_s \vec{D}.\overrightarrow{ds} = \int_{front} + \int_{back} + \int_{left} + \int_{right} + \int_{top} + \int_{bottom} +$$

$$\int_{front} = \vec{D}_{front}\Delta S_{front} = \vec{D}_{front}.\Delta y\,\Delta z\,\hat{a}_x = \vec{D}_{x\,front}\Delta y\,\Delta z$$

With D_x approximated at the front face, which lies a distance of $\frac{\Delta x}{2} fromP$.

$$\Longrightarrow \quad D_x, front = D_{x_0} + \frac{\Delta x}{2} \times rate\ of\ change\ of\ D_x\ with\ x$$

$$D_{x\,front} = D_{x_0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}, with\ D_{xo}\ being\ the\ value\ of\ D_x\ at\ P,$$

And partial derivatives are employed twice D_x also generally varies with y and z.

$$\Longrightarrow \qquad \int_{front} = \left(D_{x_0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y\,\Delta z$$

$$\Rightarrow \qquad \int_{back} = D_{back}.\Delta S_{back} = D_{back}.(-\Delta y\ \Delta z\ \hat{a}_x) = -D_{x,back}\ \Delta y\ \Delta z$$

$$D_{xback}\ = D_{x_0} - \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}$$

$$= \int_{back} = \left(-D_{x_0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y\ \Delta z$$

$$\Rightarrow\ \int_{front} + \int_{back} = \frac{\partial D_x}{\partial x}\ \Delta x\ \Delta y\ \Delta z$$

Similarly,

$$\int_{right} + \int_{left} = \frac{\partial D_y}{\partial y}\ \Delta x\ \Delta y\ \Delta z$$

$$\int_{top} + \int_{bottom} = \frac{\partial D_z}{\partial z}\ \Delta x\ \Delta y\ \Delta z$$

Collectively,

$$\oint_s \vec{D}.\overrightarrow{ds} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta x\ \Delta y\ \Delta z$$

$$= Q\ = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta_N \qquad 3.28$$

So, “almighty” result:

$$charge\ enclosed\ in\ volume\ \Delta N = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)x$$

Example 3.15: To find the total charge enclosed in an incremental volume of 10 m^3 located at the origin of $\vec{D} = e^x \sin y\ \hat{a}_x - e^{-x}\cos y\ \hat{a}_y + z\hat{a}_z\ {}^{c}/_{m^2}$

Proc:

$$\frac{\partial D_x}{\partial x} = -2e^{-2x}\sin y$$

$$\frac{\partial D_y}{\partial y} = e^{-x}\sin y$$

$$\frac{\partial D_z}{\partial z} = 1$$

$$At\ P(0,0,0)\ \ (origin), \qquad \frac{\partial D_x}{\partial x} = \frac{\partial D_y}{\partial y} = 0, \qquad \frac{\partial D_z}{\partial z} = 1$$

$$\Longrightarrow \quad \oint_c \vec{D}.\overrightarrow{ds} \approx 1\,\Delta v = 1\times 10^{-8} = 10nC$$

3.17.1 Divergence and Maxwell's First Equation

In the limit on ΔN shrinles to zero

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta N\to 0}\oint_s \frac{\vec{D}.\overrightarrow{ds}}{\Delta N} = \lim_{\Delta N} Q = \rho_v\ (charge\ density) \qquad 3.29$$

For any vector $\vec{A}$ equally,

$$\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) = \lim_{\Delta N\to 0}\oint_s \frac{\vec{A}.\overrightarrow{ds}}{\Delta N}$$

$$Divergence\ (\ \ "\ \ )\ of\ \vec{A} = div\ \vec{A} = \lim_{\Delta N\to 0}\oint_s \frac{\vec{A}.\overrightarrow{ds}}{\Delta N} \qquad 3.29.1$$

To wit, "the divergence of the vector flux density $\vec{A}$ is the outflow of flux from a small closed surface per unit volumes the volume shrinks to zero.

A divergence > 0 (positive) for any vector quantity indicates that, that point is a source, whereas a sink is indicated for divergence < 0 (negative).

$$div\ D = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\ rectangular$$

$$div\ D = \frac{1}{\rho}(\rho D\rho) + \frac{1}{\rho}\frac{\partial D\emptyset}{\partial\emptyset} + \frac{\partial Dz}{\partial z}\ (cylindrical)$$

$$div\ D = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 Dr) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ D\theta) + \frac{1}{r\sin\theta}\frac{\partial D\emptyset}{\partial\emptyset}\ (spherical) \qquad 3.29.2$$

Be it noted that, despite the appearance of three components in the expression for divergence, it is strictly still a scalar and the above three expressions appear without any associative, directional unit vectors. It merely tells how much flux is leaving a small volume on a per-unit volume basis.

So, the divergence div $\vec{D}$ of the example in the preceding page is

$$div\ \vec{D} = \frac{\partial Dx}{\partial x} + \frac{\partial Dy}{\partial y} + \frac{\partial Dz}{\partial z} = -2e^{-x}\sin y + e^{-x}\sin y + 1 = 1\ at\ origin\ \left(10n\ {}^{C}/_{m^3}\right)$$

Maxwell's first equation: $div\ \vec{D} = \rho_v$, applicable to electrostatic and steady magnetic fields. "Electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there. "Called the "point form of Gauss's law," relating the flux leaving any closed surface to the enclosed charge, similar to Maxwell's first equation that makes the same statement on a per-unit volume basis for a vanishingly small volume or at a point.

Maxwell's (differential equation) = Gauss's integral for $div\ \vec{D}$ in the region about a point charge Q at the origin.

$$\vec{D} = \frac{Q}{4\pi\, r^2}\ \hat{a}_r$$

$$\Longrightarrow\ div\ \vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\ (r^2 D_r) + \frac{1}{r\sin\theta}\ \frac{\partial}{\partial\theta}\ (1\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial D\emptyset}{\partial\emptyset}$$

$$D_\theta = D_\emptyset = 0 \Longrightarrow div\ \vec{D} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{Q}{4\pi r^2}\right) = 0, \qquad for\ r \neq 0$$

$$for\ r = 0\ (origin), div\ \vec{D} = \frac{1}{r^2}\frac{d}{dr}\left(\frac{Q}{4\pi}\right) = \infty,$$

The expression how being a full derivative of a constant term (Q).

The vector operator ∇ and the divergence theorem

Something. $\vec{D} = \frac{\partial Dx}{\partial x} + \frac{\partial Dy}{\partial y} + \frac{\partial Dz}{\partial z}$, to investigate what this "something" is

It has to be a dot operation not dot product, leading to an operation involving a vector

$$\vec{\nabla} = \frac{\partial}{\partial y}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z \text{ Defined as dot operator}$$

$$\vec{\nabla}.\vec{D} = \left(\frac{\partial}{\partial y}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right).\left(Dx\hat{a}_x + Dy\hat{a}_y + Dz\hat{a}_x\right)$$

$$= \frac{\partial Dx}{\partial x} + \frac{\partial Dy}{\partial y} + \frac{\partial Dz}{\partial z} = div\left(\vec{D}\right) = \vec{\nabla}.\vec{D}$$

Used with (against) any scalar field μ.

$$\nabla\mu = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)\mu = \frac{\partial\mu}{\partial x}\hat{a}_x + \frac{\partial\mu}{\partial y}\hat{a}_y + \frac{\partial\mu}{\partial z}\hat{a}_z$$

And the result is a vector for cylindrical coordinates

$$\vec{\nabla}.\vec{D} = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D\rho) + \frac{1}{\rho}\frac{\partial D\emptyset}{\partial\emptyset} + \frac{\partial Dz}{\partial z}\right)$$

$$\int_s \vec{\mathrm{d}}.\overrightarrow{ds} = Q$$

$$But \qquad Q = \int_{vol} e_v dv = \int_{vol} \vec{\nabla}.\vec{D}\, dv$$

$$\int_s \vec{\mathrm{D}}.\overrightarrow{ds} = \int_{vol} \vec{\nabla}.\vec{D} dv \qquad the\ Divergence\ theorem: \qquad 3.30$$

The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.

Stated another way, the divergence theorem says,

The total flux crossing the closed surface is equal to the integral of the divergence of the flux density throughout the enclosed volume.

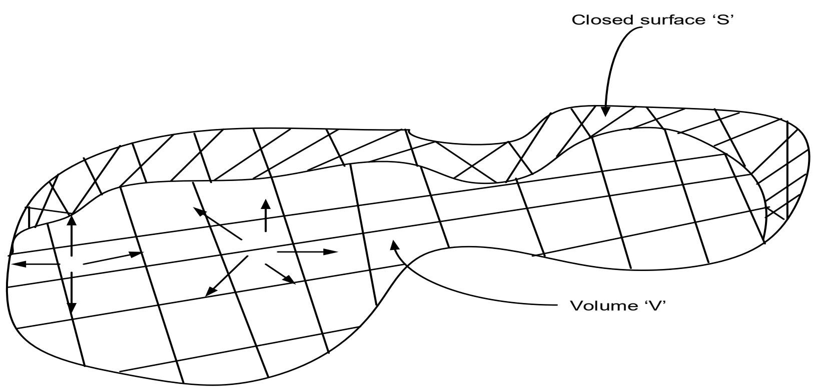


Figure 3.25

As can be discerned from the above diagram, a volume v shown on cross section for each differential-sized volume, the flux that diverge from it enters the adjacent cell, unless the cell contains a portion of the outer space, so that the divergence of the flux density throughout a volume leads to the same result as determining the net flux crossing the enclosed surface.

Example 3.16: For the field $\vec{D} = 3xy\hat{a}_x + 2x^2\,\hat{a}_y\ {}^{c}/_{m^2}$ and the rectangular paralldepiped formed by the planes $x = 0\ and\ 1, y = 0\ and\ 2, and\ z = 0\ and\ 3$, show. The validity of the divergence theorem (by evaluating both sides of the equation).

Solution:

$\vec{D}$ is parallel to the surfaces at $z = 0\ \ and\ 3\ (since\ \vec{D}$ has no solution when plugging into it those values of z). $\Longrightarrow\ \vec{D}.\overrightarrow{ds} = 0$ there. For the remaining four surfaces we have

$$\Longrightarrow\ \oint_s \vec{D}.\overrightarrow{ds} = \int_0^3\int_0^2 (\vec{D})_{x=0} \bullet (-dy\,dz\,\hat{a}_x) + \int_0^3\int_0^2 (\vec{D})_{x=1} \bullet (dy\,dz\,\hat{a}_x)$$

$$+\int_0^3\int_0^1 (\vec{D})_{y=0} \bullet (-dx\,dz\,\hat{a}_y) + \int_0^3\int_0^1 (\vec{D})_{y=2} \bullet (dx\,dz\,\hat{a}_y)$$

$$= -\int_0^3\int_0^2 (D)_{x=0}(dy\,dz\,) + \int_0^3\int_0^2 (D)_{x=1}(dy\,dz\,)$$

$$-\int_0^3\int_0^1 (D_y)_{y=0} dx\,dz + \int_0^3\int_0^1 (D_y)_{y=2} dxdz$$

But

$$(D_x)_{x=0} = 0, 3xy| = 0, \qquad (D_y)_{y=0} = 2x^2|_{y=0} = (D_y)_{y=2} = 2x^2|_{y=2} = 2x^2$$

$$D = \frac{\partial}{\partial x}(3xy) + \frac{\partial}{\partial y}(2x) = 3y\ and\ d\mathbf{S} = \mathrm{dydz}$$

$$\Rightarrow \oint_s \vec{D}.\overrightarrow{ds} = \int_0^3\int_0^2 (D_x)_{x=1} dy\,dz = \int_0^3\int_0^2 3y\,dydz$$

$$= \int_0^3 \frac{3y^2}{2}\bigg|_0^2 dz = \int_0^3 6\,dz = 18$$

But

$$\Rightarrow \oint_{vol} \vec{\nabla}.\vec{D}\,dv = \oint_s \vec{D}.\overrightarrow{ds} = \int_0^3\int_0^2\int_0^1 \left[\frac{\partial}{\partial x}(3xy) + \frac{\partial}{\partial y}(2x^2)\right] dx\,dy\,dz$$

$$= \int_0^3\int_0^2\int_0^1 (3y)\,dx\,dy\,dz = \int_0^3\int_0^2 3ydydz = 3\int_0^3 \frac{y^2}{2}\bigg|_0^2 dz = 6\int_0^3 dz = 18$$

3.18 Vector Analysis of Gauss Divergence Theorem

In vector analysis, of the theorem is,

$$\oint_S \vec{E}\cdot\hat{n}\,da = \int_{volume} \nabla\cdot\,\vec{E}\,d\tau \qquad 3.31$$

Known as Gauss's Divergence theorem

$$\text{Now} \qquad Q_{enclosed} = \oint_{volume} p d\tau$$

ρ: charge density and $d\tau$: volume element

$$\therefore \qquad \oint \vec{E} \cdot \hat{n}\, da = \frac{Q_{enclosed}}{\varepsilon_0}$$

Becomes

$$\int_{volume} \nabla \cdot \vec{E}\, d\tau = \frac{1}{\varepsilon_0} \int_{volume} p\, d\, \tau$$

$$\therefore \qquad \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad 3.32$$

Or

$$\varepsilon_0 \oint_S \vec{E}\, \hat{n}\, da = \int_{Volume} \rho_v\, dv$$

$$\Longrightarrow \qquad \oint_S \varepsilon_0 \vec{E} \cdot \hat{n}\, da = \int_{Volume} \rho_v\, dv$$

$$\Longrightarrow \qquad \oint_{.S} D da = \int_{Volume} \rho_v\, dv \qquad 3.33$$

The net outward displacement through a closed surface is equal to charge contained in the volume enclosed by the surface.

Example 3.16 Given the field $\vec{D} = 2xy\, \hat{a}_x + x^2\, \hat{a}_y\, C/m^2$ and the rectangular parallelepiped formed by planes x = 0 and 1, y = 0 and 2 and z = 0 and 3. Evaluate left hand side of Gauss divergence theorem.

Solution: we know that $\vec{D}$ ia parallel to surface at z = 0 and z = 0, so $\vec{D}.\overrightarrow{ds} = 0$ there. For remaining four surfaces.

We have

However, $[D_x]_{x=0} = 0$ and $[D_y]_{y=0} = [D_y]_{y=2}$ which leaves only

$$\oint \vec{D}\,.\,\overrightarrow{dS} = \int_0^3 \int_0^2 [D_x]_{x=1} dydz + \int_0^3 \int_0^2 2y\;dydz$$

$$= \int_0^3 4\;dz = 12$$

$$\nabla \bullet D = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial x}(x^2) = 2y$$

The volume integral becomes

$$\oint_{vol} \nabla \bullet D\;dv = \int_0^3 \int_0^2 \int_0^1 2ydxdydz = \int_0^3 \int_0^2 2y\;dydz = \int_0^3 4\;dz = 12$$

3.19 Conditions for Applications of Gauss Law

Gauss law is always true, but it is not always useful only 3 kinds of symmetry work:

(a) Spherical symmetry: Make your Gaussian surface a concentric sphere
(b) Cylindrical symmetry: make your Gaussian surface a coaxial cylinder
(c) Plane symmetry: Use a Gaussian “Pillbox” which straddles the surface

Example 3.16. Let us select a 100cm length of coaxial cable having an inner radius of 1mm and an outer radius of 4mm. the space between the conductor is assumed to be filled with air. The total charge on the inner conductor is 30 nC. We wish to know charge density on each conductor, and the $\vec{E}$ and $\vec{D}$ fields.

Solution: we begin by finding the surface charge density n inner cylinder,

$$\rho_s \text{inner cylinder} = \frac{Q_{inner}}{2\pi a L} = \frac{30 \times 10^{-9}}{2\pi \times 10^{-3} \times 1} = 4.775\;\mu C/m^2$$

The negative charge density on inner surface of outer cylinder is

$$\rho_s \text{ outer cylinder} = \frac{Q_{outer}}{2\pi b L} = \frac{-30 \times 10^{-9}}{2\pi \times (4 \times 10^{-3}) \times 1} = -1.195\;\mu C/m^2$$

The internal field may therefore be calculated easily:

$$D_p = \frac{a_{ps}}{\rho} = \frac{10^{-3} \times 4.775 \times 10^{-6}}{\rho} = \frac{4.775}{\rho} \;\; nC/m^2$$

And

$$\vec{E}_p = \frac{D_p}{\varepsilon_0} = \frac{4.775 \times 10^{-9}}{8.854 \; \times 10^{-12}} = \frac{539.5}{\rho} \;\; \mathrm{V/m}$$

Both of these expressions apply to region where $1 < \rho < 4\mathrm{mm}$. for $\rho < 1\mathrm{mm}$ or $\rho > 4\mathrm{mm}$. $\vec{E}$ and $\vec{D}$ are zero.

3.20 Coulomb's Law for Charge Density

Consider the case where there are multiple point charges present. What is the resulting electrostatic field? (Fig. 3.26).

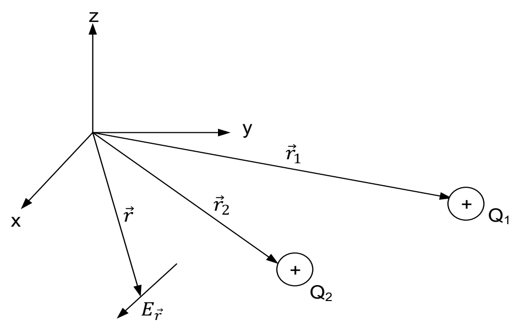


Figure 3.26 Coulomb's Law

The electric field produced by the charges is simply the vector sum of the electric field produced by each (i.e., superposition)

$$E(\bar{r}) = \frac{Q_1}{4\pi\varepsilon_0} - \frac{\bar{r} - \bar{r}_1}{|\bar{r} - \bar{r}_1|^3} + \frac{Q_2}{4\pi\varepsilon_0} - \frac{\bar{r} - \bar{r}_2}{|\bar{r} - \bar{r}_2|^3} \qquad 3.34$$

Or more generally, for N point charges.

Consider now a volume V that is filled with a "cloud" of charge, described by volume charge density $\rho_v\ (\bar{r})$. (Fig 3.27)

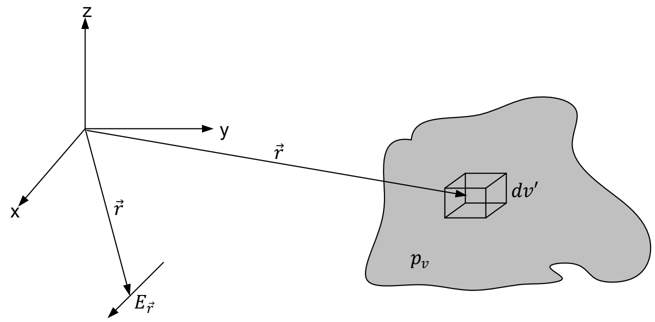


Figure 3.27 Cloud of Charge

A very small differential volume dv, located at point $\bar{r}$, will thus contain charge $dQ = \rho_v\ (\bar{r})dv$.

This differential charge produces an electric field at point $\bar{r}$ equal to:

$$dE(\bar{r}) = \frac{\rho_v\ (\bar{r})\ dv\ \bar{r} - \bar{r}'}{4\pi\varepsilon_0\ |\bar{r} - \bar{r}|^3} \qquad 3.35$$

The total electric field at $\bar{r}$ (i.e., $E\ (\bar{r})$ is the summation (i.e integration) of all the electric field vectors produced by all the little differential charges dQ that make up the charge cloud.

$$E(\bar{r}) = \iiint\limits_v \frac{\rho_v\ (\bar{r})\ \ \bar{r} - \bar{r}'}{4\pi\varepsilon_0\ |\bar{r} - \bar{r}|^3}\ dv \qquad 3.36$$

Note: The variables of integration are the primed coordinates, representing the locations of the charges (i.e sources).

Similarly, we can show that for surface charge:

$$E(\bar{r}) = \iint_S \frac{\rho_s\ (\bar{r})\ dv\ \bar{r} - \bar{r}'}{4\pi\varepsilon_0\ |\bar{r} - \bar{r}|^3}\ ds \qquad 3.37$$

And for line charge:

$$E(\bar{r}) = \int_C \frac{\rho_s\ \ dv\ \bar{r} - \bar{r}'}{4\pi\varepsilon_0\ |\bar{r} - \bar{r}|^3}\ dl \qquad 3.38$$

Point to remember

The potential at a point ‘p’ due to number of charges is obtained as a simple algebraic addition or superposition of the potential at a point by each of the charge acting above.

If $q_1, q_2, q_3 \dots q_n$ are charges located at distance $R_1, R_2, R_3 \dots . R_n$ respectively, from point p, the potential at p is given by

$$V = \frac{1}{4\pi\varepsilon}\left(\frac{q_1}{R_1} + \frac{q_2}{R_2} + \cdots \frac{q_n}{R_n}\right) = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{n}\frac{q_i}{R_i} \qquad 3.39$$

If charge is distributed continuously throughout a region, rather than being located at a discrete number of points, the region can be divided into elements of volume ΔV each containing charge $+\Delta V$

$$V = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{n}\frac{\rho_i\,\Delta V_i}{R_i} \qquad 3.40$$

Where R_i is distance to p from the ith volume element.

As the size of volume element chosen is allowed to become small.

$$V = \frac{1}{4\pi\varepsilon}\int_V \frac{\rho dV}{R} \qquad 3.41$$

Its often written in form

$$V = \int_v \rho\, G\, dV \qquad 3.42$$

In which

$$G = \frac{1}{4\pi\varepsilon R}$$

The function G is the potential of a unit charge and is often referred to as electrostatic Green’s function for an unbounded homogeneous region

Example 3.17: A Charge configuration in cylindrical coordinates is given by $\rho = 15re^{-2r}\ \mathrm{C/m^3}$. Use Gauss's law to find $\vec{D}$.

Solution: since r is not a function of Ø or z, the flux Q is completely radial.

Using Gauss's law

$$Q_{\text{inc}} = \int \vec{D} \cdot \overrightarrow{ds}$$

$$\int_0^L \int_0^{2\pi} \int_0^r 15re^{-2r}\ r\, dr d\emptyset dz = D(2\pi rL)$$

$$\Longrightarrow \qquad 15\ \pi L\left[e^{-2r}\left(r^2 - r - \frac{1}{2}\right) + \frac{1}{2}\right] = D\ (2\pi rL)$$

Hence, $$\vec{D} = \frac{7.5}{r}\left[\frac{1}{2} - e^{-2r}\left(r^2 + r + \frac{1}{2}\right)\right]\hat{a}_r \quad (C/m^2)$$

Example 3.18 For $\vec{E} = \frac{2xy}{z^2+1}\hat{a}_x + \frac{3x^2}{z^2+1}\hat{a}_y + \frac{2x^2\ yz}{z^2+1}\hat{a}_z\ mV/m$ calculate the total charge enclosed in a tiny sphere of radius $1\mu m$ that is centered at a point with coordinates (5, 8, 1).

Solution: Given that

$$\vec{E} = \frac{2xy}{z^2+1}\hat{a}_x + \frac{3x^2}{z^2+1}\hat{a}_y + \frac{2x^2\ yz}{z^2+1}\hat{a}_z$$

From Gauss's law we can write as

$$Q_{enclosed} = \oint \vec{D} \cdot \overrightarrow{ds}$$

Using $\vec{D} = \varepsilon_0\,\vec{E}$ (for only free space)

$$\Longrightarrow \qquad Q_{enclosed} = \varepsilon_0 \oint \vec{E} \cdot \overrightarrow{ds} = \varepsilon_0 \oint E\ ds = \varepsilon_0 E\ 4\pi r^2$$

$$\left(as \oint ds = 4\pi r^2\right)$$

Taking magnitude of E, substituting $(x, y, z) = (5, 8, 1)$

$$|E| = \sqrt{\left(\frac{2 \times 5 \times 8}{1+1}\right)^2 + \left(\frac{3 \times 25}{1+1}\right)^2 + \left(\frac{2 \times 25 \times 8 \times 1}{1+1}\right)^2}$$

$$= \sqrt{1600 + 1406.25 + 40000} = \sqrt{43006.25}$$

$$|E| = 207.38 \text{ V/m}$$

Now putting |E| is $Q_{enclosed} = \varepsilon_0 E \,(4\pi r^2)$

$$Q = 8.85 \times 10^{-12} \times 207.38 \times 10^6 \times 4 \times 3.14 \times (10^{-6})^2$$

$$= 23061.95 \times 10^{-18}$$

$$Q = 0.0231.\rho C$$

Example 3.19 Determine an expression for volume charge density associated with each $\vec{D}$ field following:

(a) $\vec{D} = \frac{4xy}{z}\hat{a}_x + \frac{2x^2}{z}\hat{a}_x - \frac{x^2 y}{z^2}\,\hat{a}_x$
(b) $\vec{D} = 2z\sin\emptyset\,\hat{a}_\rho + z\cos\phi\,\hat{a}_\phi + r\sin\phi\,\hat{a}_z$
(c) $\vec{D} = \sin\theta\sin\phi\,\hat{a}_\rho + 2\cos\theta\sin\phi\,\hat{a}_\theta + \cos\phi\hat{a}_\phi$

Solution: (a) Gauss divergence theorem $\nabla.\vec{D} = \rho_v$

$$\frac{d\left(\frac{4xy}{z}\right)}{dx} + \frac{d\left(\frac{2x^2}{z}\right)}{dy} + \frac{d\left(\frac{-x^2 y}{z^2}\right)}{dz} = \rho_v$$

$$\frac{4y}{z} + 0 + \frac{2x^2}{z^3} = \rho_v$$

$$\rho_v = \frac{4y}{z^3}(z^2 + x^2)$$

(b) $$\nabla.\,\vec{D} = \frac{1}{r} \times \frac{d}{dr}(2r\,z\sin\emptyset) + \frac{1}{r} \times \frac{d}{d\emptyset}(z\cos\emptyset) + \frac{d}{dz}(r\sin\emptyset)$$

$$= \frac{1}{r}(2z\sin\emptyset) - \frac{1}{r}(z\sin\emptyset) + 0$$

$$\vec{\nabla}.\vec{D} = \rho_v = \frac{z\sin\emptyset}{r}.\ Ans$$

(c) $\vec{\nabla}\ .\vec{D} = \frac{1}{r^2}\frac{d}{dr}(r^2\sin\theta\sin\emptyset) + \frac{1}{r\sin\theta}\frac{d}{d\theta}(2\sin\theta\cos\theta\sin\emptyset) + \frac{1}{r\sin\theta}\frac{d}{d\emptyset}(\cos\emptyset)$

$$= \frac{1}{r^2}2r\sin\theta\sin\emptyset + \frac{2}{r\sin\theta}\frac{\sin\emptyset}{2}\frac{d}{d\theta}(\sin 2\theta)\ \frac{-\sin\emptyset}{r\sin\theta}$$

$$= \frac{2\sin\theta\sin\emptyset}{r} + \frac{\sin\emptyset}{2r\sin\theta}(2\cos 2\theta) - \frac{\sin\emptyset}{r\sin\theta}$$

$$= \frac{2\sin\theta\sin\emptyset}{r} + \frac{\sin\emptyset}{r\sin\theta}(-4\,cos^2\theta + 2) - \frac{\sin\emptyset}{r\sin\theta}$$

$$= \frac{2\sin\theta\sin\emptyset}{r} + \frac{2\ \sin\emptyset}{r\sin\theta} - \frac{\sin\emptyset}{r\sin\theta} - \frac{4\sin\theta\sin\emptyset}{r}$$

$$\vec{\nabla}\ .\ \vec{D} = \frac{\sin\emptyset}{r\sin\theta} - \frac{2\sin\theta\sin\emptyset}{r} = \rho_v.\ Ans$$

Example 3.20 A charge distribution with spherical symmetry has volume charge density

$$\rho_v = \{\rho_0\quad 0 \le r \le a$$

$$\{0\ \ r > a$$

Calculate:

(a) The electric field intensity at all points
(b) Potential at all points
(c) Total energy stored in electrostatic field

Solution: considering Fig 3.20 and apply Gauss's law for $r < a$, we have

$$\int_S \vec{D}\cdot\overrightarrow{ds} = \int_V \rho v\cdot\overrightarrow{dV}$$

$$D \cdot 4\pi r^2 = \frac{4}{3}\,\pi r^2\,\rho_0$$

$$D = \frac{r\rho 0}{3}$$

$$E = \frac{r\rho 0}{3\ \epsilon_0}\,\hat{a}_r \quad V/m \quad 0 \le r \le a$$

$r \ge a$, Gauss's law

$$D.4\pi r^2 = \frac{4}{3}\,\pi a^3\,\rho_0$$

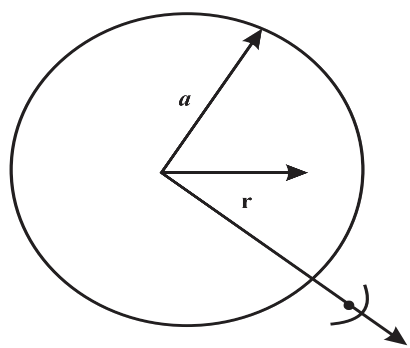


Figure 3.28

$$D = \frac{a^3\,\rho 0}{3r^2}$$

$$E = \frac{\rho_0\,a^3}{3\ \epsilon_0\ r^2}\hat{a}_r\,V/m \quad r \ge a$$

At r = a

$$E = \frac{\rho_0\,a}{3\ \epsilon_0}\hat{a}_r \qquad V/m \quad r = a$$

(b) The potential at any point can be obtained from

$$V = -\int \vec{E}\,.\overrightarrow{dl}$$

$$for\ \ r\ \geq a \qquad\qquad V = \int \frac{\rho_0\ a^3}{3 \in r^2} dr = \int \frac{\rho_0\ a^3}{3 \in r} + A$$

At $r = \infty$, $\ Let\ V = 0,\ So, A = 0$

$$V = \frac{\rho_0\ a^3}{3 \in r^2} \quad at\ r \geq a$$

$$At\ \ r = a \qquad V = \frac{\rho_0\ a^2}{3\ \epsilon_0}$$

$$For \qquad r \leq a \qquad\qquad V = -\int \vec{E}.\overrightarrow{dl} = -\int \frac{\rho_0\ r}{3\ \epsilon_0} dr = -\frac{\rho_0\ a^2}{6\ \epsilon_0} + B$$

At $r = a$

$$V = \frac{\rho_0\ a^2}{3\ \epsilon_0}$$

$$\frac{\rho_0\ a^2}{3\ \epsilon_0} = -\frac{\rho_0\ r^2}{6\ \epsilon_0} + B$$

$$V = \frac{\rho_0}{6\ \epsilon_0}\ (3a^2 - r^2)$$

(c) The energy stored is

$$W_E = \frac{1}{2} \int_V \epsilon_0\ E^2 dV$$

Considering both region E,

$$= \frac{\rho_0^2}{18\ \epsilon_0} \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\emptyset=0}^{2\pi} r^2 \cdot r^2 \sin\theta\ dr\ d\theta\ d\emptyset$$

$$+ \frac{\rho_0\ a^6}{18\ \epsilon_0} \int_{r=a}^{\infty} \int_{\theta=0}^{\pi} \int_{\emptyset=0}^{2\pi} \frac{1}{r^4} \cdot\ r^2 \sin\theta\ dr\ d\theta\ d\emptyset$$

$$= \frac{\rho_0\ a^5}{18\ \epsilon_0\ 5} 4\pi + \frac{\rho_0^2\ a^6}{18\ \epsilon_0} \left(-\frac{1}{r}\right)\ 4\pi$$

$$W_E = \frac{4\pi\rho_0^2 a^5}{15\varepsilon_0} J$$

Example 3.21: A dielectric sphere of radius 'b' and permittivity $' \in '$ is situated in vacuum. It is charged throughout its volume by a charge density $\rho_V = \frac{5b}{r}$, r being the distance from the center of sphere. Determine the electrostatic energy of system.

Solution: For $\vec{D}$ outside sphere

Using Gauss's law

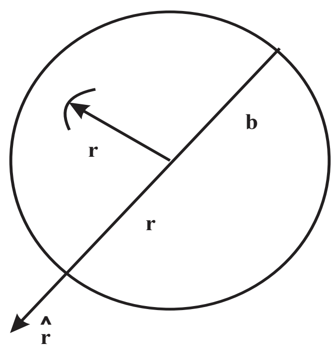


Figure 3.28

$$\int \vec{D} \cdot \overrightarrow{ds} = \int \rho_V \cdot dV$$

$$D.4\pi r^2 = \int_{r=0}^{b} \int_{\theta=0}^{\pi} \int_{\emptyset=0}^{2\pi} \frac{5b}{r} r^2 \sin\theta \, dr \, d\theta \, d\emptyset$$

$$= 5b^3 \times 2\pi = 10\pi b^3$$

$$\vec{D} = \frac{10\pi b^3}{4\pi r^2} = \frac{5}{2}\frac{b^3}{r^2}\hat{a}_r \quad b < r < \infty$$

$\vec{D}$ Inside sphere by Gauss's law at any radius r

$$D.4\pi r^2 = \int_{0}^{r} \int_{\theta=0}^{\pi} \int_{\emptyset=0}^{2\pi} \frac{5b}{r} r^2 \sin\theta \, dr \, d\theta \, d\emptyset$$

$$5b.\frac{r^2}{2} \cdot 4\pi = 10\pi \, br^2$$

$$\vec{D} = \frac{5}{2}\, b\hat{a}_r \;\; 0 < r < b$$

Total energy of system

$$W_E = \int_V \frac{1}{2}\frac{D^2}{\epsilon_0} dV$$

$$= \frac{1}{3\varepsilon_0}\left[\int_{r=b}^{\infty}\int_{\theta=0}^{\pi}\int_{\emptyset=0}^{2\pi}\frac{25\, b^6}{4r^4} r^2 \sin\theta \; dr\; d\theta\; d\emptyset \; + \int_0^b\int_{\theta=0}^{\pi}\int_{\emptyset=0}^{2\pi}\frac{25}{4} b^2 r^2 \sin\theta\; dr\; d\theta\; d\emptyset\right]$$

$$= \frac{1}{3\varepsilon_0}\left[\frac{-25}{4} b^6 \cdot\; 4\pi \frac{1}{r}\right| + \frac{24}{4}\frac{b^2 r^3}{3}\Big|_0^6 4\pi\right] = \frac{50\,\pi}{3\varepsilon_0}$$

$$W_E = 5.914 \times 10^{12}\; b^5\, J$$

3.21 Introduction to Electric Potential (V)

Recall that a point charge Q, located at the origin ($\bar{r} = 0$) produces a static electric field:

$$E(\bar{r}) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r \qquad 3.43$$

Now, we know that this field is the gradient of some scalar field:

$$E(\bar{r}) = -\nabla V(\bar{r}) \qquad 3.44$$

Example 3.22: What is the electric potential function $V\,(\bar{r})$ generated by a point charge Q, located at the origin?

Solution: We find that it is:

$$V\,(\bar{r}) = \frac{Q}{4\pi\varepsilon_0 r} \qquad 3.45$$

Example 3.23: Where did this come from? How do we know that this is the correct solution?

Solution: We can show it is the correct solution by direct substitution

$$E(\bar{r}) = -\nabla V(\bar{r}) = -\nabla\left(\frac{Q}{4\pi\varepsilon_0 r}\right) = -\frac{\partial}{\partial r}\left(\frac{Q}{4\pi\varepsilon_0\, r}\right)\hat{a}_r + 0 = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r$$

The correct result.

Example 3.24: What if the charge is not located at the origin?

Solution: Substitute r with $|\bar{r} - \bar{r}|$ and we get:

$$V(\bar{r}) = \frac{Q}{4\pi\varepsilon_0|\bar{r} - \bar{r}|} \quad 3.46$$

Where, as before the position vector $\vec{r}$ denotes the location of the charge Q, and the position vector $\bar{r}$ denotes the location in space where the electric potential function is evaluated.

Example 3.25: Given the field $\vec{E} = 40\,xy\,\hat{x} + 20x^2\,\hat{y} + 15\hat{z}$, calculate V_{PQ} given $(1, -1, 0)$ and $Q(2, 1, 3)$.

Solution: Given vector is

$$\vec{E} = 40\,xy\hat{x} + 20x^2\hat{y} + 15\hat{z}$$

In Cartesian coordinates, we can write V_{PQ} as

$$V_{PQ} = -\int_Q^P \vec{E}\,\cdot\overrightarrow{dl}$$

$$\vec{E}\,\bullet\overrightarrow{dl} = \left[40\,xy\,\hat{a}_x + 20x^2\,\hat{a}_y + 15\hat{a}_z\right]\bullet\left[dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z\right]$$

$$= -\left[\int_2^1 40\,xy\,dx + \int_1^{-1} 20x^2\,dy + \int_3^0 15\,dz\right]$$

$$= -\left[40\times\frac{x^2}{2}\times y\Big|_{x=2}^{1} + 20x^2y|_{y=1}^{-1} + 15(-3)\right]$$

$$= -[20\,x^2y|_{x=2}^{1} + 20\,x^2y|_{y=1}^{-1} - 45]$$

$$= -[-60y - 40x^2 - 45]$$

$for \quad Q(2,1,3)$

$$= -[-60 - 160 - 45]$$

$$V_{PQ} = -[-265]V$$

Example 3.26: If we take the zero reference for potential at infinity, find the potential at $(0,0,2)$ caused by this charge configuration in free space.

(a) $12\ nC/m$ on the line $p = 2.5m, z = 0$
(b) Point charge of $18nCat\ (1,2,-1)$
(c) $12n\ C/m$ on line $y = 2.5, z = 0$

Solution

(a) 529 V
(b) 43.2 V
(c) 67.4 V

3.22 Electric Potential Function for Charge Densities

Let us look at a review on superposition principles. Recall the total static electric field produced by 2 different charges (or charge densities) is just the vector sum of the fields produced by each:

$$E(\bar{r}) = E_1(\bar{r}) + E_2\ (\bar{r}) \qquad 3.47$$

Since the fields are conservatives, we can write this as:

$$E(\bar{r}) = E_1(\bar{r}) + E_2\ (\bar{r})$$

$$-\nabla V(\bar{r}) = -\nabla V_1(\bar{r}) - \nabla V_2\ (\bar{r})$$

$$-\nabla V(\bar{r}) = -\nabla (V_1(\bar{r}) + V_2\ (\bar{r}))$$

Therefore, we find

$$V(\bar{r}) = V_1(\bar{r}) + V_2\ (\bar{r}) \qquad 3.48$$

In other words, superposition also holds for the electric potential function. The total electric potential field produced by a collection of charge is simply the sum of the electric potential produced by each.

3.22.1. Sign for work done

If a body acted upon by force is moved from one point to another, work will be done on/by body.

If there is no mechanism by which energy represented by this work can be dissipated, then field is said to be conservative.

Energy must be stored in either potential/kinetic form. If some point is taken as a reference/zero point, the field of force can be described by the work that must be done in moving the body from the reference point up to any point in the field.

A reference point that is commonly used is a point at infinity.

If a small body charge 'q' and a second body with small test charge $'\Delta q'$ is moved from infinity along a radius line to a point $'p'$ at a distance R from the charge $'q'$, then work done on the system in moving the test charge against the force F will be.

$$\text{work} = \int_{-\infty}^{R} \mathrm{F}_r \, dr \qquad 3.49$$

Work done on test charge

$$= \frac{-q \, \Delta q}{4\pi\varepsilon} \int_{-\infty}^{R} \frac{1}{r^2} \, dr = \frac{q \, \Delta q}{4\pi\varepsilon \, R}$$

Work done on test charge per unit charge is

$$\mathrm{V} = \frac{\mathrm{q}}{4\pi\epsilon\mathrm{R}} \qquad 3.50$$

V is potential is only magnitude and on dir' → **Scalar potential**

3.22.2. Conversation field

When work done in moving from one point to another is independent of path then field is called as conservative field (fig 3.22)

There is no mechanism for dissipating energy corresponding to positive work done and no source from which energy could be absorbed, if work were negative.

A $$dW = dV = -\vec{E} \cdot ds$$

Or $$\left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz\right) = -\vec{E} \cdot ds$$

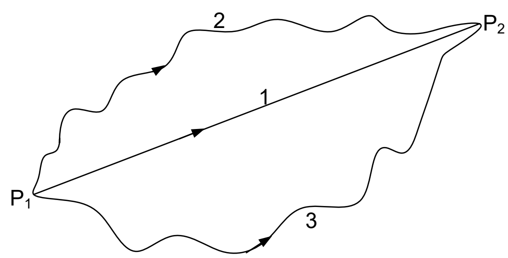


Figure 3.29 Conservative Field

Or $$\left(\hat{x}\frac{\partial V}{\partial x} + \hat{y}\frac{\partial V}{\partial y} + \hat{z}\frac{\partial V}{\partial z} dz\right) \cdot (\hat{x}dx + \hat{y}dy + \hat{z}dz) = -\vec{E} \cdot ds$$

Or $$\nabla V . ds = -\vec{\varepsilon} \cdot ds$$

$$\Longrightarrow \quad \vec{E} = -\nabla V \qquad 3.51$$

Points to remember are that:

1. Electric field strength at any point is just the negative of the potential gradient at that point.
2. The direction of electric field is the direction in which the gradient is greatest or in which the potential charges most rapidly.

Once we find the electric potential function $V\,(\bar{r})$. We can then determine the total electric field by taking the gradient.

$$E(\bar{r}) = -\nabla V(\bar{r})$$

Thus, we now have three (!) potential methods for determining the electric field produced by some charge distribution $\rho_v\ (\bar{r})$.

1. Determine $E(\bar{r})$ from coulomb's law.
2. If $\rho_v\ (\bar{r})$ is symmetric, determine $E\ (\bar{r})$ from Gauss law.
3. Determine the electric potential function $V\ (\bar{r})$ and then determine the electric field as $E(\bar{r}) = -\nabla V(\bar{r})$.
   Which of the three should be use?

To a certain extent it does not matter! All three will provide the same result (although $\rho_s\ (\bar{r})$ must be symmetric to use method 2!)

However, if the charge density is symmetric, we will find that using Gauss's law (method 2) will typically result in much less work!

Otherwise (i.e., for non-symmetric $\rho_v\ (\bar{r})$), we find that sometime method 1 is easiest but in other cases method 3 is a bit less stressful (i.e you decide).

Example 3.27 Given the potential field, $V = 2x^2y - 4z$ and point $P(-4, 3, 6)$, we wish to find several numerical values at point P, the potential V, the electric field intensity E, the direction of E, the electric flux density D, the volume charge density ρ_v.

Solution: The potential at $P\ (-4, 3, 6)$ is

$$V_p = 2(-4)^2 \times 3 - 4 \times 6 = 72\ \text{V}$$

We may use gradient operation to obtain the electric field intensity

$$E = -\nabla\, V = -4xy\ \hat{a}_x - 2x^2\ \hat{a}_y + 4\hat{a}_z$$

The value of E at point P is

$$\vec{E}_p = 48\hat{a}_x - 32\hat{a}_y + 4\hat{a}_z\ \ \text{V/m}$$

And $$\left|E_p\right| = \sqrt{48^2 + 32^2 + 4^2} = 57.83\ \text{V/m}$$

The directions of $\vec{E}$ at P is given by unit vector

$$\vec{a}_{E.P} = \frac{48\hat{a}_x - 32\hat{a}_y + 4\hat{a}_z}{57.83}$$

$$\vec{a}_{E\,.P} = 0.83\hat{a}_x - 55\hat{a}_y + 0.069\hat{a}_z$$

If we assume these fields exist in free space, then

$$\vec{D} = \varepsilon_0\vec{E} = -35.4xy\ \hat{a}_x - 17.71x^2\ \hat{a}_y + 44.3\hat{a}_z\ pC/m^3$$

Finally, we may use divergence relationship to find volume charge density that is the source of the given potential field.

$$\rho_v = \nabla.D = -35.4\ y\ pC/m^3$$

At $\qquad\rho_v = -106.2\ pC/m^3$

Example 3.26 A spherical charge distribution is given by

$$\rho = \rho_0\ \left(1 - \frac{r}{a}\right)\ \ When\ \ r \le a$$

$$\rho = 0\ \ \ r > a$$

Calculate

i. The electric field intensity inside and outside the charge distribution
ii. The value of r for which the field is maximum
iii. The electrostatic potential at the center

Solution: The volume density of a spherical charge at a distance $R < a$ from the center is given by

$$\rho = \rho_0\left(1 - \frac{R}{a}\right) \qquad 3.52$$

i. Consider a spherical shell of radius R and thickness dR inside the charge distribution. The electric field strength at a point distant r< a from the center due to the small charge dq on shell is given as.

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2} \qquad 3.53$$

And $$dq = (4\pi\, R^2\, dR)\rho$$

The electric field intensity due to the whole charge distribution at any point inside the sphere distant r from the center is obtained by integrating Eq (3.53)

ii. Between the limit 0 to r.

$$E_r = \frac{1}{4\pi\varepsilon_0}\int_0^r \frac{dq}{r^2} = \frac{1}{4\pi\varepsilon_0}\int_0^r \frac{(4\pi\, R^2\, dR)\rho}{r^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int_0^r (4\pi R^2\, dR)\cdot\ \rho_0\left(1-\frac{R}{a}\right) \quad [from\ equation\ (3.52)]$$

$$= \frac{\rho_0}{\varepsilon_0 r^2}\int_0^r \left(R^2 - \frac{R^3}{a}\right)dR = \frac{\rho_0}{\varepsilon_0 r^2}\left[\frac{R^3}{3} - \frac{R^4}{4a}\right]_0^r$$

$$= \frac{\rho_0}{\varepsilon_0 r^2}\left[\frac{r^3}{3} - \frac{r^4}{4a}\right] = \frac{\rho_0}{\varepsilon_0}\left[\frac{r}{3} - \frac{r^2}{4a}\right]$$

The electric field intensity at any point distance R > a from the center of spherical charge distribution is given as

$$E_0 = \frac{1}{4\pi\varepsilon_0}\int_0^a \frac{(4\pi\, R^2\, dR)\rho}{r^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\int_0^a \frac{(4\pi\, R^2\, dR)\rho\ \left(1-\frac{R}{a}\right)}{r^2}$$

$$= \frac{\rho_0}{r^2\varepsilon_0}\int_0^a \left(R^2 - \frac{R^3}{a}\right)dR = \frac{\rho_0}{\varepsilon_0 r^2}\left[\frac{R^3}{3} - \frac{R^4}{4a}\right]_0^a$$

$$= \frac{\rho_0}{r^2\varepsilon_0}\left(\frac{R^2}{3} - \frac{a^3}{4}\right) = \frac{\rho_0 a^3}{12\varepsilon_0 r^2}$$

ii. the electric field intensity at any point distant r from the center is

$$E_i = \frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3} - \frac{r^2}{4a}\right)$$

The field will be maximum for the value of r, for which $\frac{dE}{dr} = 0$

$$\frac{d}{dr}\left[\frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3} - \frac{r^2}{4a}\right)\right] = 0$$

$$\frac{\rho_0}{\varepsilon_0}\left(\frac{1}{3} - \frac{2r}{4a}\right) = 0$$

Or $$r = \frac{2a}{3}$$

iii. The electric field intensity at any point distance r from the center is

$$E_i = \frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3} - \frac{r^2}{4a}\right)$$

The potential V at a point distant r from the center is given by

$$V = -\int \vec{E}_i \cdot \overrightarrow{dr} = -\int E_i dr = \int \frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3} - \frac{r^2}{4a}\right) dr = -\frac{\rho_0}{\varepsilon_0}\left(\frac{r^2}{6} - \frac{r^3}{12a}\right)$$

At the center of the spherical charge $r = 0$

∴ The potential at the center of the spherical distribution is zero

Example 3.29 A charge q coulomb is distributed uniformly throughout a non-conducting spherical volume of radius R meter. Show that the potential at a distance r from the center where $r < R$, is given by

$$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q(3R^2 - r^2)}{2R^3}$$

Solution: Let a charge q coulomb is distributed uniformly throughout a non-conducting sphere of radius R meter. The electric intensity at any point distant $r < R$ from the center of the spherical charge is

$$\vec{E}_0 = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}\vec{V}_r.$$

The electric potential at the surface of spherical charge is

$$V_s = -\int_{\infty}^{R} \vec{E}_0 \cdot \overrightarrow{dr} = -\int_{\infty}^{R} \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\vec{v}_r.\vec{d}_r$$

$$= -\frac{1}{4\pi\varepsilon_0} \cdot \int_{\infty}^{R} \frac{q}{r^2}\, dr = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R} \qquad 3.54$$

The intensity at any point distant r (CLR) from the center is

$$\vec{E}_0 = \frac{1}{4\pi\varepsilon_0} \cdot \frac{qr}{R^2}\, \vec{v}.$$

∴ The potential at a point distant $r\ (< R)$ from the center is

$$V = -\int \vec{E}_i\, \overrightarrow{dr} = -\int \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}\vec{v}_r \cdot \vec{d}_r = -\frac{1}{4\pi\varepsilon_0}\int \frac{qr}{R^3}\, dr$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{qr^2}{2R^3} + C \qquad 3.55$$

Where C is an integration constant.

But Equ (3.54) given the electric potential at the surface of the sphere.

$$\therefore \qquad \frac{1}{4\pi\varepsilon_0}\frac{q}{R} = -\frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{2R} + C$$

$$C = \frac{1}{4\pi\varepsilon_0}\frac{3q}{2R}$$

Putting this value in Equ (3.55), we get

$$V = \frac{1}{4\pi\varepsilon_0}\frac{qr^2}{2R^3} + \frac{1}{4\pi\varepsilon_0}\frac{3q}{2R} = \frac{1}{4\pi\varepsilon_0}\left[\frac{3q}{2R} - \frac{qr^2}{2R^3}\right]$$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q(3R^2 - r^2)}{2R^3}\ Proved$$

3.23 Electric Field Due to Surface Charge

Let surface charge density be $'\rho_s'$ coulombs per square with enclosed an element of surface in a volume of "pillpox" shape with its flat surfaces parallel to the conductor surface, depth $d \ll diameter$ in as in the case of a conducting pathway of an Electric circuit.

Electric displacement through its edge surface ≪ Electric displacement through its flat surface.

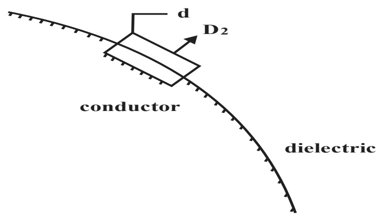


Figure 3.30 (a) boundary between conductor and dielectric

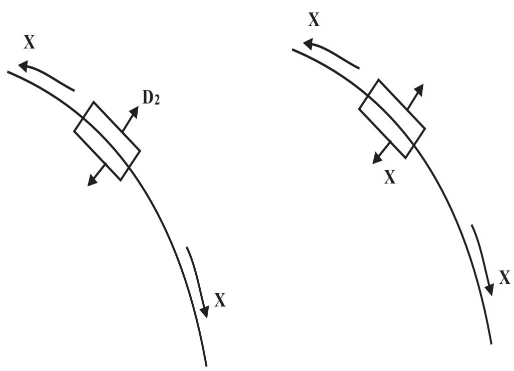


Figure 3.30 (b) conditions at Boundary

Electric displacement through top and bottom surface gets cancelled as they are equal and opposite

There can be displacement through the left-hand surface submerged in conductor sec in Fig 3.30.

$$\Longrightarrow \qquad D_n\, da = \rho_s\, da \qquad 3.56$$

$$\Longrightarrow \qquad D_n = \rho_s \text{or } E_n = \frac{\rho_s}{\varepsilon}$$

The electric displacement density at the surface of conductor is normal to surface and equal in magnitude to surface charge density.

OR

The electric field strength is normal to surface and is equal to surface charge density divided by dielectric constant.

3.24 Method of Electrostatic Images

The method of image involves the conversion of an electrostatic field into another equivalent field.

The applications of above method reflex to following distributions:

a. Combination of concentrated charges and planes: electrons in space close to essentially plane conductors is its known example as seen in Fig 3.30
b. Replacing the conductor by one or more-point charges: it will be seen that it is possible to replace the conductors by one or more-point charges in such a way that the conductor surfaces are replaced by equipotential surfaces at the same potential.
c. Combination of several cylindrical conductors: This combination of several cylindrical conductors is taken with reference to earth as large conducting plane. An open wire transmission line is a very typical example of above.

Note: Therefore, the application of method of electrostatic images would require a knowledge of the surface charge density distribution on large conduction plane.

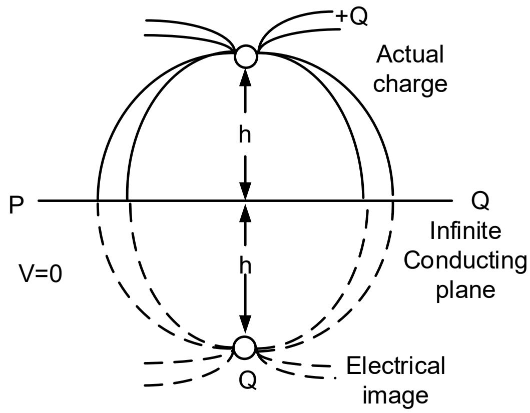


Figure 3.31 Electrical Image Method

Conclusion: Image methods permits

1. The determination of the electric field behavior in dielectric without knowing the actual charge distribution on the plane.
2. The determination of electric field behavior in the vicinity of concentrated or line charge close to a plane boundary between two different dielectric media.
3. The determination of relationship involved between two external spheres by an extension of this method as a series of images.

With reference to the above Fig. 3.31, the fictitious charge $-Q$ shown below the conducting plane is usually referred to as electrostatic or electric image similar to optical image that is why the name is method of images.

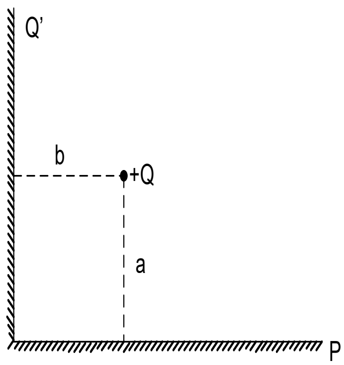
This method has the following steps.

1. Firstly, the mirror image of the conductor in ground plane is taken
2. The image conductor is assumed to carry a negative of the charge
3. The ground plane is then removed

Obviously, then the electric field of the two conductors obeys the right boundary conditions at the actual conductor and by symmetry has an equipotential surface where the ground plane was.

Hence, this field is the field in the region between the actual conductors and the ground plane.

If $-Q$ charge is placed "$2h$" unit below actual charge $+Q$ the plane PQ has potential of zero. Thus, the upper portions of two fields are identical and equivalent configuration can be used to determine all field quantities in the upper half space.



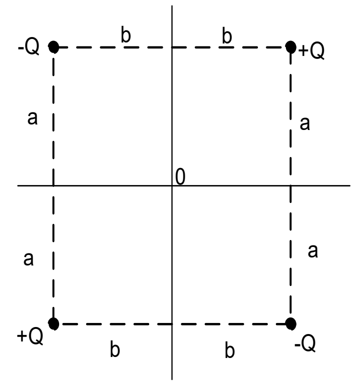


Figure 3.32 **(Left)** rights angle conducting plane charge system **(Right)** equivalent image configurations

With reference to Fig 3.32, let a point charge $+Q$ be brought near a right-angle corner between two conducting plane boundaries at zero potential in Fig. 3.33 (a). by adding a single image charge directly below the original charge will make the potential of plane OP zero but will not satisfy zero potential condition in plane OQ. However, a system of three image charges in Fig 3.33 (b) will satisfy the condition on both planes OP and OQ.

Example 3.31. Point charge near an infinite grounded conducting plane.

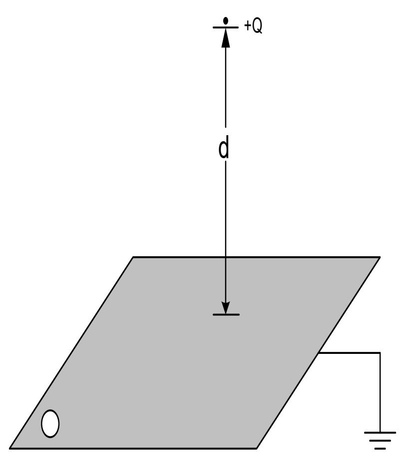


Fig 3.33 (a) point charge near a grounded conducting plane

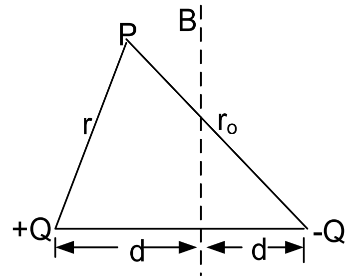


Fig 3.33 (b) the conducting plane replaced image charge $-Q$

Solution: Follow the steps as:

1. The conducting plate is of infinite extent, its potential must be constant at ll points on it. Assume this potential to be zero, whatever be the charges induced on it.
2. Earth has a large capacitance of $4\pi\varepsilon_0\, R \approx 600\mu F$ and hence addition or substraction of even large amounts of charge has a negligible effect in its potential.
3. Remove the rounded conductor and replace it by a charge $-Q$, at a distance $'d'$ behind the plane, then every point of the plane will be equidistant from $+Q$ and from $-Q$ and hence be at zero potential as in Fig 3.33 (b).
4. The charge $-Q$ is said to be image charge $+Q$ in the plane

The potential V at point P whose coordinates are r, Q is given by

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{1} - \frac{Q}{1}\right) \qquad 3.57$$

Where,

$$r_0^2 = r^2 + (2d)^2 + 2\,.2\; dr\cos\theta$$

$$r_0 = \sqrt{r^2 + 4d^2 + 4\; dr\cos\theta}$$

Now, the components of electric field intensity at ρ are component of ∇V

$$\vec{E}_r = \frac{-dV}{dr}\,\vec{a}_r \qquad 3.58$$

$$E_\theta = -\frac{1}{r}\frac{dV}{d\theta}\vec{a}_\theta \qquad 3.59$$

Solving above Equs (3.58 and 3.59)

$$E_r = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r^2} - \frac{(r+2d\cos\theta)}{(r^2+4d^2+4dr\cos\theta)^{\frac{3}{2}}}\right] \qquad 3.60$$

$$E_\theta = \frac{2dQ\sin\theta}{4\pi\varepsilon_0(r^2+4d^2+4dr\cos\theta)^{\frac{3}{2}}} \qquad 3.61$$

Let us draw the lines of force as in Fig 3.34

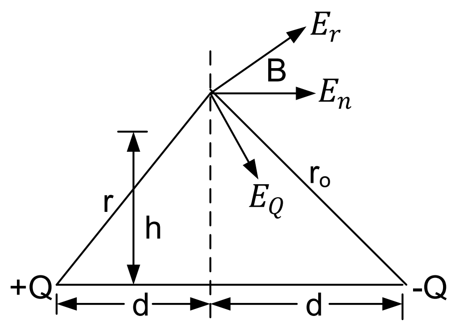


Figure 3.34 Lines of Force

The electric field intensity $\vec{E}_n$ at surface of grounded conducting plane is calculated from fields of $+Q\ and - Q$.

$\vec{E}_n$ is vector sum of $\vec{E}_r$ and $\vec{E}$ is normal to surface

$$E_n = E_r\cos\theta - E_\theta\sin\theta \qquad 3.62$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\left(\frac{1}{r^2} - \frac{r+2d\cos\theta}{r_0^3}\right)\cos\theta - \left(\frac{2d\sin\theta}{r_0^3}\right)\sin\theta\right]$$

$$E_n = \frac{Q}{4\pi\varepsilon_0}\left[\frac{\cos\theta}{r^2} - \frac{r\cos\theta}{r^3} - \frac{2d}{r^3}\right] = \frac{-Q}{4\pi\varepsilon_0}\left(\frac{2d}{r^3}\right)$$

Or

$$E_n = \frac{-Qd}{4\pi\varepsilon_0 r^3} \qquad 3.63$$

By Gauss's law

$$E_n = \frac{+\rho_s}{\varepsilon_0}$$

$$\rho_s = \frac{-Qd}{2\pi r^3} \qquad 3.64$$

Conclusion: surface charge density varies inversely as cube of distance of the position on the plate from charge $+Q$. Hence, amount of charge induced on conducting plane is greatest near the foot of perpendicular O.

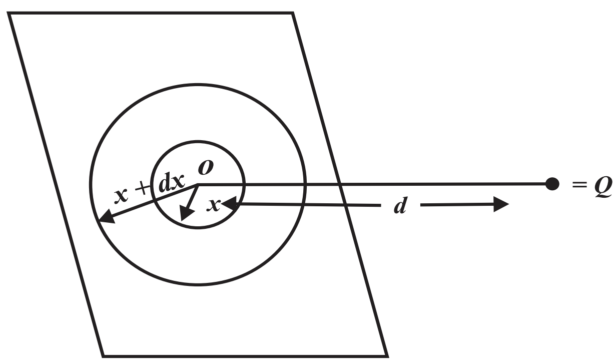


Figure 3.35 Annular Ring

Assume as in Fig 3.35 $'ds'$ area of infinite simally small element of surface which may be taken as annular ring between circles of radii $'x'$ and $'x + dx'$ on plane with O as centre.

$\therefore \qquad ds = \pi(x + dx^2) - \pi(x)^2$

$$= 2\pi x\, dx \qquad (\text{Neglecting } dx^2)$$

Thus, total induced charge is

$$Q = \int_0^\infty \rho_s\, ds = -\int_0^\infty \frac{Q\, d}{2\pi r^3} \cdot 2\pi x\, dx$$

$$= Qd \int_0^\infty \frac{x\,dx}{r^3} = -Qd \int_0^\infty \frac{dx}{(x^2 + d^2)^{\frac{3}{2}}}$$

$$= \frac{-Qd}{2} \int_0^\infty \frac{2x\,dx}{(x^2+d^2)^{3/2}}$$

$$Q' = -Q \qquad 3.65$$

This shows that total charge induced on plane is $' - Q'$.

Similarly, there are number of problems which can be solved by image method.

Example 3.32. Find surface density at $P(2,5,0)$on the conducting plane $z = 0$ if there is a line charge of $30nC/m$ located at $x = 0, z = 3$ as shown Fig 3.36.

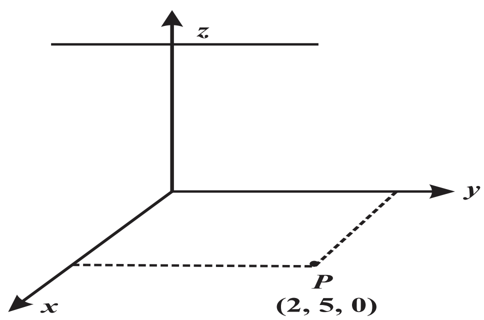


Figure 3.36

Solution: Remove the plane and install an image line charge of $-30nC/m$ at $x = 0, z = -3$, as illustrated in Fig. 3.37

The field at P may now be obtained by superposition of known field of line charge to P is $R_+ = 2\hat{a}_x - 3\hat{a}_z$ while $R_- = 2\hat{a}_x + 3\hat{a}_z$

Thus, the individual fields are

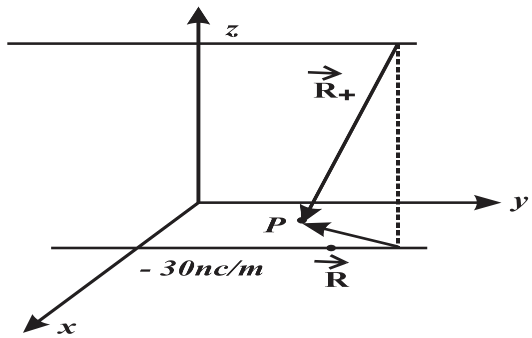


Figure 3.37

$$\vec{E}_+ = \frac{\rho_L}{2\pi\varepsilon_0 R_+}\hat{a}_{R+} = \frac{30\times10^{-9}}{2\pi\varepsilon_0\sqrt{13}}\times\frac{(2\hat{a}_x - 3\hat{a}_z}{\sqrt{13}}$$

$$\vec{E}_- = \frac{30\times10^{-9}}{2\pi\varepsilon_0\sqrt{13}}\times\frac{(2\hat{a}_x - 3\hat{a}_z}{\sqrt{13}}$$

Adding these results, we have

$$\vec{E} = \frac{-180\times10^{-9}\ \hat{a}_z}{2\pi\varepsilon_0(13)}\ \vec{E} = 248\hat{a}_z\ V/m$$

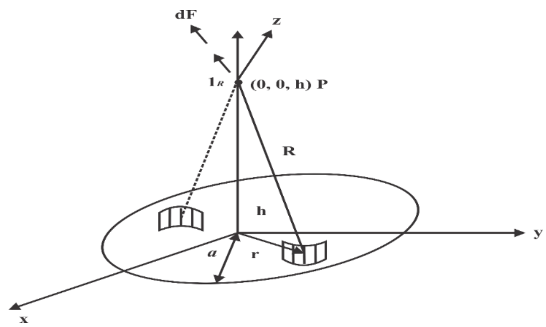
This then is the field at (or just above) P in both configuration

$$\vec{D} = \varepsilon_0\vec{E} = -2.20\hat{a}_z\ C/m^2 \quad \text{(Directed towards the conducting plane)}$$

$$\rho_S = -\frac{2.20nC}{m^2}\ \text{ at P}$$

3.25 Exercise

1. A point charge $Q_1 = 300\ \mu C$ located at $1,-1,-3)\ m$ experience a force $\vec{F}_1 = 8\hat{x} - 8\hat{y} + 4\hat{z}\ N$ due to a point charge $Q_2\ at\ (3,-3,-2)m$. Determine Q_2.
2. A point charge $Q_1 = 10\ \mu C$ is located at a point $P_1\ (1,2,3)$ in free space while $Q_2 = -5\mu C$ as at $P_2\ (1,2,10)$. Find: (a) Force extended on $Q_2\ by\ Q_1$. (b) the coordinates of a point at which a point charge Q_3 experiences no force.
3. Find the force on a point charge $'q'$ located at $(0,0,h)m$ due to charge of surface charge density $p_s\ c/m^2$ uniformly distributed over the circular disc $r \le a.z = 0m$.



Figure

4. (a) A circular disc of radius $'a'$ is situated in the xy plane at $z = 0$, with its center at the origin. Charge density on disc is $p_s = constant\ C/m^2$. Calculate the field at any point $(0, 0, h)$ in cylindrical coordinates system.
   (b) Extend the result of part (a) for calculating the field at any point due to infinite uniform plane sheet of charge density $p_1\ C/m$.
5. (a) Develop on expression for electric field intensity at a general point P due to an infinite straight-line charge with charge density $p_i\ C/m$
   (b) Find the field intensity at point if the line is having semi-infinite length.
6. Consider a line charge distribution with charge density p_i C/m along the line between $(0,0,-a)$ and $(0, 0, a)$ in cylindrical coordinates. Obtain expression for electric field intensity at $(r, \emptyset, 0)$.
7. Find the general equation of flux lines which represent the field
$$\vec{E} = \frac{\rho}{4\pi\varepsilon_0 r^3}\ (2\cos\theta\ \hat{r} + \sin\theta\ \hat{\theta})$$
   In spherical coordinate system, when $'p'$ dipole moment
   Formulas to be used"
   (a) $\frac{dx}{Ex} = \frac{dy}{Ey} = \frac{dz}{Ez}$ (b) $\frac{dr}{Er} = \frac{rd\emptyset}{E\emptyset} = \frac{dz}{Ez}$ (c) $\frac{dr}{Er} = \frac{rd\theta}{E\theta} = \frac{r\sin\theta\ d\emptyset}{E\emptyset}$
8. Three coaxial cylindrical sheets of charge are present in free space; $\rho_s = 5\ c/m^2$ at $r = -2m\ \rho_s = 2c/m^2$ at $r = 4$, and $\rho_s = -3\frac{c}{m}^3\ at\ r = 5m$. Find the displacement flux density $\vec{D}$ at (i) $r = 1m$ (ii) $r = 3m$ (iii) $r = 4.5m$ (iv) $r = 6m$
9. Determine the charge density due to each of following electric flux densities
   (a) $\vec{D} = 6xy\ \hat{x} + 4x^2\ \hat{y}$ (b) $\vec{D}\ r\sin\emptyset\ \hat{r} + 2r\cos\emptyset\ \hat{\emptyset} + 3z^2\ \hat{z}$
10. In a spherical coordinate system, the volume charge density is
$$\rho_v = \rho_0\left(\frac{r}{a}\right)^{3/2}\ c/m^3$$

i. How much charge lies in sphere of radius $'a'$?

ii. Find the electric flux density at $r = a$.

11. A volume charge distribution is represented as

$$\rho(r_1\emptyset_1 z) = 0; 0 < r < a$$
$$= \rho_0 r; a < r < b$$
$$= 0; b < r < \infty$$

Find electrical field intensity at all points using Gauss's law in integral form

12. Calculate the field due a line charge considering it a special Gaussian surface. (keep it along z-axis)

13. A potential field is expressed by $V = \left(\frac{50r^2 \cos\emptyset}{z+1}\right)V$. Given a point $A\ (4, 30, 2)$ in free space. Calculate:
    a. Potential at A
    b. $\vec{E}\ at\ A$
    c. Volume charge density at A
    d. Unit vector in the direction of potential gradient

14. Given a field $\vec{E} = \left(-\frac{6y}{x^2}\right)\hat{x} + \left(\frac{6}{x}\right)\hat{y} + 5\hat{z}\ \frac{V}{m}$. Calculate $'V_{AB}'$ given $A\ (-5, 2, 1)$ and $B\ (6, 1, 2)$.

15. A circular line charge of density $\rho_0 \frac{C}{m}$ of radius $'a'$ is lying in xy value with its centre at origin, calculate the potential at point $(0, 0, h)$.

16. Calculate potential at point $(0, 0, h)$ due to circular disc of radius is having a surface charge density of $P_s\ C/m^2$ with its centre at origin. Calculate the field at point $P\ (0, 0, z)$.

17. Find the potential that gives rise to $\vec{E} = 2xy\ \hat{x} + x^2\hat{y} - \hat{z}$.

18. Given cylindrical electric fields

$$\vec{E} = \frac{5}{r}\hat{r}\ V/m \quad 0 \le r \le 2m$$

$$\vec{E} = 2.5\ \hat{r}\ V/m\ \ r \ge 2m$$

Find the potential difference $V_{AB}\ for\ A(1, 0, 0) and\ B(4, 0, 0)$

19. (a) State Coulomb's law.
    (b) Point charges 1mC and -2mC, are located at (3,2, -1) and (-1, -1, 4) respectively (i) compute the electric force on a 10nC charge located at (0,3,1) (ii) determine the electric field intensity at that point.

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20. (a) State Gauss Law

(b) Given that D=$z\rho cos^2\emptyset a_z C/m^2$, determine the charge density at $(1,\pi/4, 3)$.

(c) (i) Use the point form of Gauss law to determine the total charge enclosed by the cylinder of radius 1m with -2 $\leq z \leq 2m$.

(ii) determine same using integral form of Gauss law (3pts)

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21. Discuss coulomb's law of force with diagrams:
22. Define electric field strength, how do you define electric force?
23. Discuss electric potential with reference to various charge distribution
24. What is electric flux density?
25. Dow does Faraday's experiment infer you to result of electric flux density?
26. Define electric line of flux and lines of flux
27. Why can't lines of force and flux cross each other over a single point?
28. What is green function?
29. State Gauss's law
30. Prove Gauss's law taking any closed surface in context.
31. Write the integral form of Gauss's law
32. Write Gauss divergence theorem
33. Give the condition for application of Gauss law
34. Give coulomb's law for charge density
35. Give electric field strength for following charge distributions.
    a. Volume charge
    b. Surface charge
    c. Line charge
36. What is Green's function?
37. Relate Green's function to electric potential
38. Define electric potential
39. Discuss superposition principle with reference to electric potential function
40. What do you mean by conservative field? Explain with example
41. Explain sign used for work done 'on' or 'by' the body

CHAPTER 4

CAPACITANCE OF CAPACITOR

4.0 Explanation of Term Capacitance

Consider two conductors with a potential difference of V volts as in Fig 4.1

- Since there is a potential difference between the conductor, there must be and electric potential field $V\ (\bar{r})$, and therefore and electric field $E\ (\bar{r})$ in the region between the conductors.
- Likewise, if there is an electric field, then we can specify an electric flux density $D\ (\bar{r})$, which we can use to determine the surface charge density $\rho_s\ (\bar{r})$ on each of the conductors.
- We find that if the total net charge on one conductor is Q then the charge on the other will be equal to – Q.

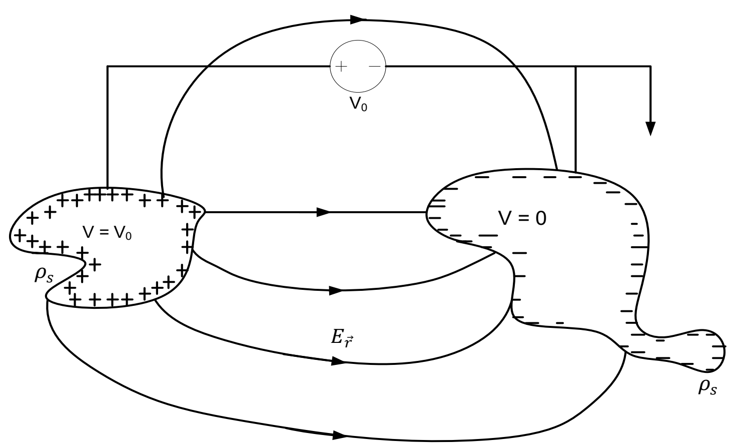


Figure 4.1 Capacitor

In other words, the total net charge on each conductor will be equal but opposite

Recall that the total charge on a conductor can be determined by integrating the surface charge density $\rho_s\ (\bar{r})$ across the entire surfaces S of a conductor:

$$Q = \oiint_S \rho_s + (\bar{r})\, ds = \oiint_S \rho_s - (\bar{r})\, ds \qquad 4.1$$

But recall also that the surfaces charge density on the surface of a conductor can be determined from the electric flux density $D\ (\bar{r})$.

$$\rho_s\ (\bar{r}) = D(\bar{r}) \cdot \hat{a}_n \qquad 4.2$$

Where $\hat{a}_n$ is a unit vector normal to the conductor

Note that this does not mean that the surface charge densities on each conductor are equal (i.e., $\rho_{s+}\ (\bar{r}) \neq \rho_{s-}(\bar{r})$). Rather, it means that:

$$\oiint_S \rho_s + (\bar{r})\, ds = -\oiint_S \rho_s - (\bar{r})\, ds = Q \qquad 4.3$$

Where surfaces S, is the surface surrounding the conductor with the positive charge (and the higher electric potential), while the surface S surrounds the conductor with the negative charge.

Example 4.1: How much free charge Q is there on each conductor, and how does this charge relate to the voltage V_0?

Solution: We can determine this from the mutual capacitance C of these conductors!

The mutual capacitance between two conductors is defined as:

$$C = \frac{Q}{V}\left[\frac{Coulombs}{Volt} = Farad\right] \qquad 4.4$$

Where Q is the total charge on each conductor, and V is the potential difference between each conductor (for example, $V=V_0$).

$$Q = \oiint_S D(\bar{r})\, \hat{a}_n ds = -\oiint_S D(\bar{r})\ \cdot\ \hat{a}_n\, ds$$

$$= \oiint_S D(\bar{r}) \cdot \overline{ds} = -\oiint_S D(\bar{r})\ \cdot \overline{ds}$$

Where we remember that $\overline{ds} = \hat{a}_n\, ds$.

Hey! This is no surprise! We already knew that:

$$Q = \oiint_S D(\bar{r}) \cdot \overline{ds} \qquad 4.5$$

Note since $D(\bar{r}) = \varepsilon E(\bar{r})$ we can also say:

$$Q = \oiint_S \varepsilon E(\bar{r}) \cdot \overline{ds}$$

The potential difference V between two conductors can likewise be determined as:

$$V = \int_C E(\bar{r}) \cdot \overline{dl}$$

Where C is any contour that leads from one conductor to the other.

Example 4.2: why any contour?

Solution: we can therefore determine the capacitance between two conductors as:

$$C = \frac{Q}{V} = \frac{\oiint_S \varepsilon E(\bar{r}) \cdot \overline{dl}}{\int_C E(\bar{r}) \cdot \overline{dl}} \quad [Faraday] \qquad 4.6$$

Where the contour C must start at some point on surface S, and end at some point on surface S.

Note this expression can be written as:

$$Q = CV \qquad 4.7$$

In other words, the charge stored by two conductors is equal to the product of their mutual capacitance and the potential difference between them.

Therefore, the greater capacitance, the greater the amount of charge that is stored.

By the way, try taking the time derivative of the above Equ 4.7.

$$\frac{dQ}{dt} = C\frac{dV}{dt}$$
$$I = C\frac{dV}{dt} \qquad 4.8$$

Look familiar?

By the way, the current I in this Equ 4.8 is displacement current.

Example 4.3 A 100 μF capacitor is charged to a potential of 100 V and the charging battery is then disconnected. this capacitor is then connected in parallel with second capacitor. If the potential difference drops to 50 V; calculate the capacitance of second capacitor.

Solution: The charge on first capacitor

$$Q = C_1 V \qquad 4.9$$

$$= (100 \times 10^{-6}\ F)(100\ \mathrm{V})$$

$$Q = 10^{-2}\ \mathrm{C}$$

When C_1 is connected is parallel to C_2 the capacitance of combination between $C_1 + C_2$. The total charge is still Q. Therefore, the potential difference across the combination would be $\frac{Q}{C_1+C_2}$

$$50 = \frac{Q}{C_1 + C_2}$$

$\Longrightarrow$ $$50 = \frac{10^{-2}}{(100\times10^{-6}) + C_2}$$

$\Longrightarrow$ $$5\ \times 10^{-3} + 50\ C_2 = 10^{-2}$$

$\Longrightarrow$ $$50C_2 = \frac{1}{100} - \frac{5}{1000} = \frac{5}{1000}$$

$\Longrightarrow$ $$C_2 = \frac{5}{1000} \times \frac{1}{50}$$

$\Longrightarrow$ $$C_2 = \frac{1}{10^4}$$

$\Longrightarrow \qquad C_2 = 10^{-4}F$

4.1 Capacitance of Various Distributions

Consider Capacitance of a Spherical Capacitor. Fig. 4.2 shows a spherical shell capacitor formed by two concentric spherical shells A and B of radii 'a' and 'b' charged with $+\theta$ and $-\theta$ respectively. Twise charge attract each other, and spread out uniformly on outer surface of inner shell and inner surface of outer shell. They produce an electric field $\vec{E}$ between two shells, radially outwards. Consider a Gaussian surface which is concentric spherical surface of radius 'r'. the electric flux through it is

$$\emptyset = \int_s \vec{E} \cdot \overrightarrow{ds} = \oint E\, ds\ (E\, ds \cos 0 = E\, ds)$$

$$\emptyset = E \oint ds = E \cdot\ 4\pi r^2$$

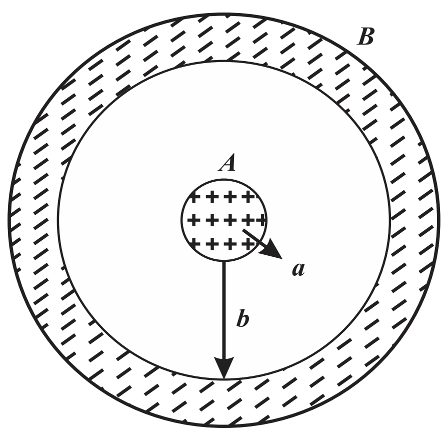


Figure 4.2

According to Gauss's law, the electric flux $\emptyset$ must be equal to $\frac{1}{\varepsilon_0}$ times the charge contained within surface.

$\therefore \qquad \emptyset = E(4\pi r^2) = \frac{Q}{\varepsilon_0}$

$\Longrightarrow \qquad E = \frac{Q}{4\pi\varepsilon_0\, r^2}$

$\vec{E}$ is radially outward and $\overrightarrow{dl}$ is inware, therefore $\vec{E}.\overrightarrow{dl} = E\, dl \cos 80^0 = -E\, dl$

$\Rightarrow$ $\vec{E} \cdot \overrightarrow{dl} = E\ dr$

Also, $V = -\int_b^a E\ dr$

$$V = -\frac{-Q}{4\pi\varepsilon_0}\left(\frac{-1}{r}\right)_b^a = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$V = \frac{-Q}{4\pi\varepsilon_0}\left(\frac{-1}{r}\right)_b^a = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{b-a}{ab}\right)$$

But capacitance $C = \frac{Q}{V}$

$\Rightarrow$ $C = 4\pi\varepsilon_0\ \frac{ab}{b-a}$

4.1.1 Capacitance of an Isolated Conducting Sphere

The above expression may be written as

$$C = 4\pi\varepsilon_0 \frac{ab}{b-a} = \frac{4\pi\varepsilon_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

If radius of outer sphere is infinite $(i.e, b = \infty)$, then we have an isolated conducting sphere

Putting $b = \infty$, we can get capacitance of an isolated sphere

$$C = \frac{4\pi\varepsilon_0}{\left(\frac{1}{a} - \frac{1}{b}\right)} = 4\pi\varepsilon_0\ a$$

4.1.2 Spherical Capacitor Appreciates to Parallel Plate Capacitor

Capacitance of a spherical capacitor is given by

$$C = 4\pi\varepsilon_0 \frac{ab}{b-a}$$

$$C = 4\pi\varepsilon_0 \frac{a^2}{b-a}$$

The surface area of sphere is $4\pi a^2 = A$ (say), then $C = \frac{\varepsilon_0 A}{b-a}$

the capacitance of parallel plate capacitor with plate separation $(b-a)$.

4.1.3. Capacitance of a Cylindrical Capacitor

Fig. 4.3 shows a cylindrical capacitor formed by two coaxial cylinders A and B of radius 'a' and 'b' respectively and each of length. They are charged with $+\theta$ and $-\theta$. These charges attract each other and spread out uniformly on outer surface of inner cylinder and inner surface of outer cylinder they produce an electric field $\vec{E}$ between two shells, which is radially outward. Consider a coaxial. The flux through the plane faces of ends of this surface is zero because $\vec{E}$ and $\overrightarrow{ds}$ are perpendicular.

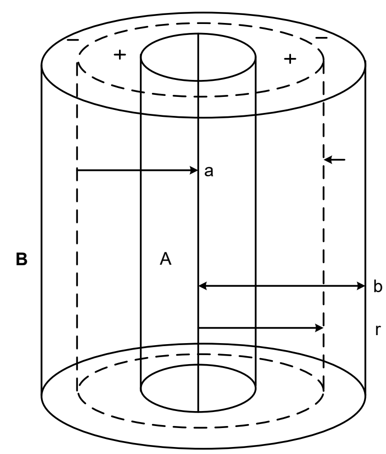


Figure 4.3

$$\emptyset = \int \vec{E} \cdot \overrightarrow{ds} = E \oint ds = E\,(2\pi\, rI)$$

$\Longrightarrow$ $$E\,(2\pi\, rl) = \frac{Q}{\varepsilon_0}$$

$\Longrightarrow$ $$E = \frac{Q}{2\pi\varepsilon_0\, rl}$$

$$V = -\int_b^a E \cdot dr = \frac{-Q}{2\pi\varepsilon_0\, l}\int_b^a \frac{dr}{r} = \frac{-Q}{2\pi\varepsilon_0 l}\,[\log_e r]_b^a = \frac{Q}{2\pi\varepsilon_0\, l}\,\log_e \frac{b}{a}$$

Capacitance $$C = \frac{Q}{V} = \frac{2\pi\varepsilon_0\, l}{\log_e\left(\frac{b}{a}\right)}$$

4.1.4 Parallel Plate Capacitor with Dielectric

Fig. 4.4 shows a parallel plate capacitor whose plates are charged with charges $+Q$ and $-Q$. Let E_0 be electric field in air between two plates. The weaker field within the dielectric is $\vec{E}$. Consider Gaussian surface $PQRS$ with wall QR in air. The electric flux through PS is zero (as field within the conductor is zero). The electric flux through PQ and SR is zero ($\vec{E}_0$ is $\perp$ to area vector of these surfaces).

Thu, electric flux through the entire Gaussian surface is flux through the surface QR only

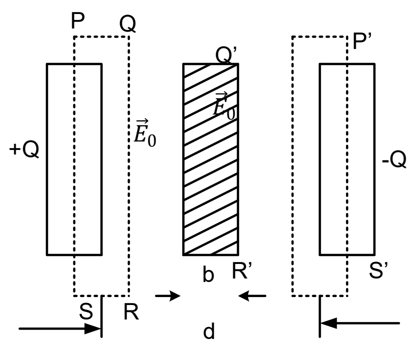


Figure 4.4

$\therefore$ $$\emptyset = \oint \vec{E}_0 \cdot \overrightarrow{ds} = E_0 A$$

Where A is area of plate

By Gauss's law $$\emptyset = \vec{E}_0 \cdot \overrightarrow{ds} = \vec{E}_0 \cdot A = \frac{Q}{\varepsilon_0}$$

Or $$E_0 = \frac{Q}{\varepsilon_0\, A}$$

Now consider $P'Q'R'S'$ with wall $Q'R'$ in dielectric where field $\vec{E}$ is weber.

Similarly, $\oint \varepsilon_0 \ \vec{E} \ \cdot \ \overrightarrow{ds} = \frac{Q}{\varepsilon_0}$

$\Longrightarrow$ $\varepsilon_0 E = \frac{Q}{\varepsilon_0}$

Or $E = \frac{Q}{\varepsilon_r \varepsilon_0 \ A}$

$\therefore$ The potential difference $'V'$ between plates is work done in density in carrying a unit charge from one plate to other in field to other length $(d - b)$ and in field E.

Over length b.

$\therefore$ $V = E_0 \ (d - b) + Eb$

$$= \frac{Q}{\varepsilon_{0 \ A}} \ (d - b) + \frac{Q}{\varepsilon_r \varepsilon_0 \ A} b$$

$$V = \frac{Q}{\varepsilon_0 \ A}\left(d - b + \frac{b}{\varepsilon_r}\right)$$

$$= \frac{Q}{V} = \frac{\varepsilon_0 A}{d - b + \frac{b}{\varepsilon_r}}$$

$$C = \frac{\varepsilon_r \varepsilon_0 A}{\varepsilon_r \ d - b \ (\varepsilon_r - 1)}$$

Special cases:

1. If no dielectric $(b = 0 \ or \ t_r = 1)$ is present
$$C = \frac{\varepsilon_0 A}{d}$$
2. If dielectric fills ensure space between plates $(b = d)$,
$$C = \frac{\varepsilon_r \varepsilon_0 A}{d}$$
3. It is clear from above equations that when plates of electric materials of thickness $'b'$ is placed between the plates of capacitor the thickness of air medium is reduced by $b\left\{1 - \frac{1}{\varepsilon_r}\right\}$, so capacitance increases.

4. Capacitance with copper slab: As electric field inside capacitor is zero, therefore for conductor (copper slab of thickness b) between plates of a capacitor with plate separation $'d'$, effective separation will be only $(d-b)$ in air.

$$C = \frac{\varepsilon_0 A}{d-b}$$

4.1.5 Spherical Capacitor with Two Dielectrics

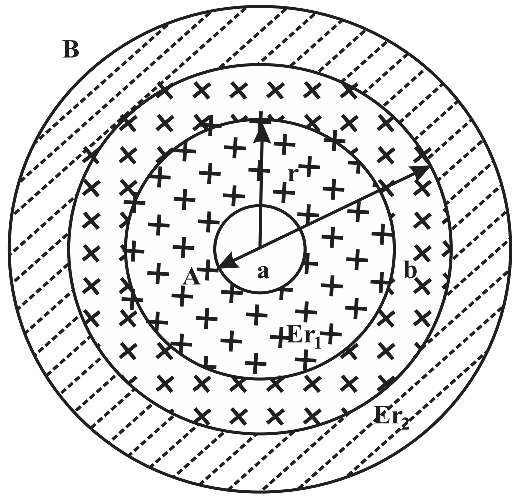


Figure 4.5

$$V_A - V_B = \frac{Q}{4\pi\varepsilon_0}\left[\int_a^r \frac{dr}{\varepsilon_{r_1} r^2} + \int_r^b \frac{dr}{\varepsilon_{r_2} r^2}\right]$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\left|\frac{-1}{r\varepsilon_{r_1}}\right|_a^r + \left|-\frac{1}{\varepsilon_{r_2} r}\right|_r^b\right]$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\varepsilon_{r_1}}\left(\frac{1}{a}-\frac{1}{r}\right) + \frac{1}{\varepsilon_{r_2}}\left(\frac{1}{r}-\frac{1}{b}\right)\right]$$

$$V_{AB} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r}\left(\frac{1}{\varepsilon_{r_2}}-\frac{1}{\varepsilon_{r_1}}\right) + \left(\frac{1}{\varepsilon_{r_1} a}-\frac{1}{\varepsilon_{r_2} b}\right)\right]$$

And $C = \frac{Q}{V_{AQ}}$

$$\Rightarrow \qquad C = 4\pi\ \left[\frac{1}{r}\left(\frac{1}{\varepsilon_{r_2}}-\frac{1}{\varepsilon_{r_1}}\right) + \left(\frac{1}{\varepsilon_{r_1} a}-\frac{1}{\varepsilon_{r_2} b}\right)\right]^{-1}$$

4.1.6 Surface Charge Distribution Capacitance Between Two Isolated Conductors

Let us first visualize two conductors merged in a homogenous dielectric medium with conductor C_1 carrying positive charge (q) and C_2 carrying equal and positive charge $(-q)$. This means the total charge of the system is zero and there are no other charges present.

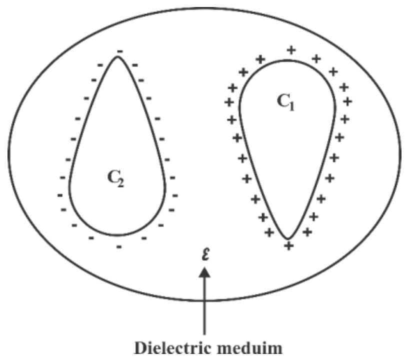


Figure 4.6 Dielectric medium with two oppositely charge conductor

Surface charge density is when the charge is carried on the surface. We know that electric field inside a conductor is zero and the electric field is normal to the conducting surface. Thus, we can say that each conductor acts moreover like an equipotential surface.

Let's now evaluate the direction of electric flux. We know that flux is directed from positive to negative charge i.e. from C_1 to C_2 as C_1 is at more positive potential. Thus, work must be done to carry a positive charge from C_2 to C_1. We say that the potential difference between C_1 and C_2 is V_0. Thus, capacitance of a two-conductor system is defined as ratio of the magnitude of total charge on either conductor to the magnitude of potential difference between conductors.

$$C = \frac{q}{V_0}$$

Or we can determine q by surface integral over positive conductor and we find V_0 by carrying a unit positive charge from negative to positive surface.

$$C = \frac{\oint_S \in \vec{E} \cdot \overrightarrow{ds}}{-\int_{-}^{+} \vec{E} \cdot dl}$$

Where numerator is defined from Gauss's law and denominator by relation that $\bar{E} = \frac{V}{l}$.

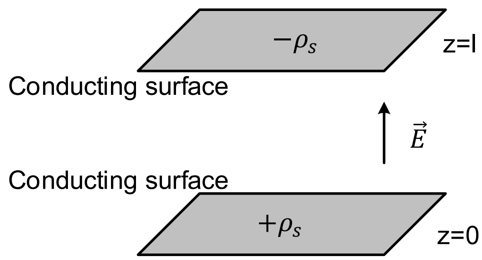


Figure 4.7 Parallel Plate capacitor

Thus, from the above equation one con infer that capacitance is sheerly a function of physical discussion of the system of conductors and of permittivity of the homogenous dielectric. It is independent of potential and total charge, i.e., if charge density is increased by some factor, then Gauss's law indicates that electric field intensity also increases by same factor, so does the potential difference.

Let us apply the above definition of capacitance to a simple two conductor system as shown in Fig 4.7.

We choose the lower conducting plane at $z = 0$, having uniform sheet of surface charge $+p_s$ and upper conducting plane at $z = l$, having uniform sheet of charge $-\rho_s$. This leads to uniform field.

$$\vec{E} = \frac{\rho_s}{\in} \hat{z}$$

This is from boundary condition where $\vec{D_n} = \rho_s\, \hat{n}$ *or* $D = \rho_s \hat{z}$

Where ∈ is permittivity of homogenous dielectric

∴ on lower plane

$$D_N = D_Z = \rho_s$$

(equal to surface charge density is negative of that on lower plane)

On upper plane

$D_N = -D_Z$ (The surface charge density is negative of that on lower plane)

Thus, potential difference between two planes is

$$V_0 = -\int_{upper}^{lower} \vec{E} \cdot \overrightarrow{dl} = -\int_{l}^{o} \frac{\rho_s}{\epsilon}\, dz = \frac{\rho_s}{\epsilon} l$$

$\therefore$ $$V_0 = \frac{\rho_s}{\epsilon} l$$

Example 4.3 Calculate capacitance of parallel-plane capacitor having mica dielectric, $\epsilon_r = 6$ and plate area $5m^2$ and separation of $0.02m$

Solution: $$S = 6 \times (0.0254)^2 = 3.23 \times 10^{-3}\, m^2$$

$$l = 0.02 \times (0.0254) = 5.08 \times 10^{-4}\, m^2$$

Where $S \rightarrow$ area of conducting plate

$l \rightarrow$seperation

$$\therefore C = \frac{\epsilon\, s}{l} = \frac{\epsilon_0 \epsilon_r\, S}{l} = \frac{8.854 \times 10^{-12} \times 6 \times 3.23 \times 10^{-3}}{5.08 \times 10^{-4}}$$

$$C = 3.377 \times 10^{-10} F$$

Or $$C = 0.34\, nF. \quad Ans$$

4.2 Energy Storage in Capacitors

Recall in a parallel plate capacitor, a surface charge distribution $\rho_{s+}(\bar{r})$ is created on one conductor, while charge distribution $\rho_{s-}(\bar{r})$ is created on the other, as in fig 4.8

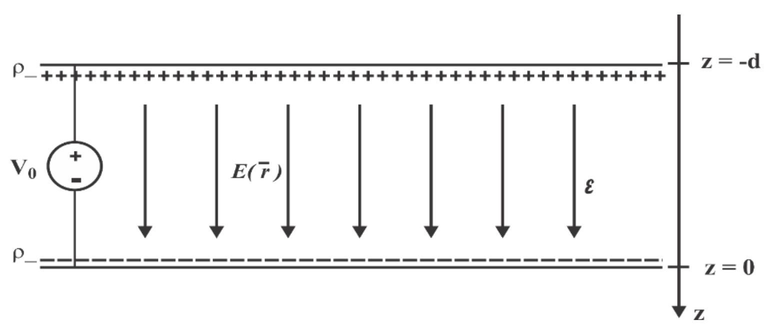


Figure 4.8 Parallel Plate capacitor

We learned that the energy stored by a charge distribution is

$$W_e = \frac{1}{2}\iiint_V \rho_V\ (\bar{r})\ V\ (\bar{r})dv$$

For the parallel plate capacitor, we must integrate over both plates:

$$W_e = \frac{1}{2}\oiint_S \rho_{s+}\ (\bar{r})V(\bar{r})ds + \frac{1}{2}\oiint_S \rho_{s-}\ (\bar{r})V(\bar{r})ds$$

But on the top plate (i.e., S_+) we know that:

$$V\ (z = -d) = V_0$$

While on the bottom (i.e., S_-):

$$V\ (z = 0) = 0$$

Therefore

$$W_e = \frac{V_0}{2}\oiint_{S_+} \rho_{s+}\ \ (\bar{r})ds + \frac{0}{2}\oiint_S \rho_{s-}\ \ (\bar{r})ds = \frac{V_0}{2}\oiint_S \rho_{s+}\ (\bar{r})ds$$

But the remaining surface integral we know to be charge Q.

$$Q = \oiint_S \rho_{s-}\ (\bar{r})ds$$

Therefore, we find,

$$W_e = \frac{1}{2}V_0\ Q$$

But recall that:

$$Q = CV$$

Where V is the potential difference between the two conductors (i.e., $V = V_0$).

Combining these two equations, we find:

$$W_e = \frac{1}{2}\, V_0\, Q = \frac{1}{2}\, V_0\, (CV) = \frac{1}{2}\, CV^2$$

The above equation shows that the energy stored within a capacitor is proportional to the product of its capacitance and the squared value of the voltage across the capacitor.

Recall that we also can determine the stored energy from the fields within the dielectric:

$$W_e = \frac{1}{2}\iiint_V D\ (\bar{r})\cdot\ E\ (\bar{r}) dv\ II$$

Since, the fields within the capacitor are approximately:

$$E(\bar{r}) = \frac{V}{d}\hat{a}_z\ D(\bar{r}) = \frac{\varepsilon V}{d}\hat{a}_z$$

We find:

$$W_e = \frac{1}{2}\iiint_V D(\bar{r})\ \ E(\bar{r}) dv = \frac{1}{2} W_e = \frac{1}{2}\iiint_V \frac{\varepsilon V^2}{d^2}\ dv$$

$$= \frac{1}{2}\frac{\varepsilon V^2}{d^2}\iiint_V dv = \frac{1}{2}\frac{\varepsilon V^2}{d^2}\ \ (Volume)$$

Where the volume of the dielectric is simply the plate surface area S time the dielectric thickness d:

$$\text{Volume} = \text{sd}$$

Resulting in the expression:

$$W_e = \frac{1}{2}\frac{\varepsilon V^2}{d^2}(Sd) = \frac{1}{2}\frac{\varepsilon S}{d}\ V^2$$

Recall, however, that the capacitance of a parallel plate capacitor is:

$$C = \frac{\varepsilon S}{d}$$

Therefore: $$W_e = \frac{1}{2}\frac{\varepsilon S}{d}V^2 = \frac{1}{2}CV^2$$

The same result as before!

$$\text{Workdone} = \text{Vdq} = \frac{q}{c}dq \qquad 4.10$$

$$\int_0^Q \frac{q}{c}\,dq = \frac{1}{2}\frac{Q^2}{c} = \frac{1}{2}\,V.Q = \frac{1}{2}CV^2$$

Work done in charging a capacitor to Q coulomb/total energy stored charged capacitor.

Example. 4.4 Calculate the energy stored in capacitor of capacitance 20 μF charged to potential of 90 volts.

Solution: the energy stored in capacitor is given by

$$U = \frac{1}{2}\,CV^2$$

$$= \frac{1}{2} \times 20 \times 10^{-6} \times (90)^2 = 81000 \times 10^{-6}$$

$$\varepsilon = 8.1 \times 10^{-2}\,J$$

Example 4.5 A parallel plate air capacitor of 2F capacity having a plate separation of 1 mm. Can this capacitor would have been constructed in laboratory?

Solution: Capacitance of a parallel plate capacitor is

$$C = \frac{\varepsilon_0 A}{d}$$

$$C = 2F, d = 1mm = 10^{-3}m$$

$$A = \frac{Cd}{\varepsilon_0} = \frac{2 \times 10^{-3}}{8.854 \times 10^{-12}} = 2 \times 10^8\,m^2$$

In laboratory, it is not possible to construct a parallel plate capacitor of such a large plate area.

4.3 Electrostatic Energy

Energy necessary to establish a given charge distribution surface is called electrostatic energy.

Suppose that all of space is initially field free and N point charges are brought in form infinity and located at specific point as in Fig 4.9

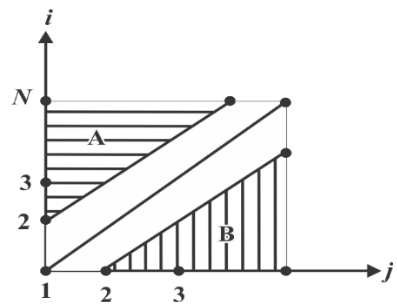


Figure 4.9 System of Charges

The energy expanded in locating the ith charge at point ri,

$$W_i = q_i v_i = \frac{q_i}{4\pi\varepsilon}\sum_{j=1}^{i=1}\frac{q_j}{R_{ij}}$$

$$R_{ij} = |r_i - r_j| \qquad 4.11$$

No energy is used up is locating the first charge

$$\text{Total energy} = \mathrm{W} = \sum_{i=2}^{N} W_i = \frac{1}{4\pi\varepsilon}\sum_{i=2}^{N}\sum_{\mathrm{J}=1}^{i-1}\frac{q_i q_J}{R_{ij}}$$

$$2W = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{q_i q_J}{R_{ij}} \; i \neq f \qquad 4.12$$

Or $$W = \frac{1}{8\pi\varepsilon}\int_V\int_{V'}\frac{\rho(r)dV\cdot\rho(r)'dV'}{R_i} \qquad (R = |r - r'|)$$

Or $$W = \frac{1}{2}\int_V\left[\frac{1}{4\pi\varepsilon}\int_V\left[\frac{\rho(r')\,\rho(r)\times dV\,dV'}{R_i}\right]\right]$$

As $$V\ (r) = \frac{1}{4\pi\varepsilon}\int_{V'}\frac{\rho(r')dV'}{R_i} \qquad 4.13$$

Identity $$\boldsymbol{\nabla}.(\boldsymbol{VD}) = \boldsymbol{VD}\cdot\ \boldsymbol{D} + \boldsymbol{D}\cdot\boldsymbol{\nabla V}$$

Or

$$W = \frac{1}{2}\int_V \rho V\ dV\ \Rightarrow \frac{1}{2}\int_V V\ (\nabla\cdot D)dV$$

$$= \frac{1}{2}\int_V [\nabla.(VD) - D.\cdot\nabla V]dV$$

$$= \frac{1}{2}\int_S VD\cdot\ \overrightarrow{da} + \frac{1}{2}\int_V D.\vec{E}\ dV$$

$$O''\left(if\ S\to\infty, \int\frac{1}{r^3}\to 0\right)$$

$\Rightarrow$ $$W = \frac{1}{2}\int_{all\ space}\varepsilon E^2\ dV \qquad 4.14$$

From these Equs (4.13 and 4.14), electrostatic, electrostatic energy is said to be

1. "Associated with electric charge"
2. "Associated with the electric field"

Example 4.6 A parallel plate capacitor has internal separation 'd' between plates. A dielectric slab with ε_r of thickness $'a'$ is placed on the lower plates of capacitor. Show that electric intensity in dielectric is

a. $E_1 = \frac{\emptyset}{\varepsilon_r\ d - a\ (\varepsilon_r - 1)}$ where $\emptyset =$ potential difference between the plates
b. Electric field intensity in the air space is $\varepsilon_0 = \varepsilon_r\ \varepsilon_1$
c. Capacitance of capacitor is

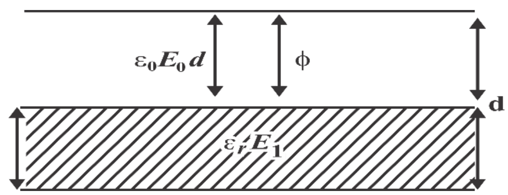


Figure 4.10

$$C_T = \frac{\varepsilon_0}{d}\left[\frac{\varepsilon_r}{\left(1-\frac{a}{d}\right)\varepsilon_r + \left(\frac{a}{d}\right)}\right]F$$

Solution: As the normal component of flu density is continuous. That is

$$\varepsilon_0 E_0 = \varepsilon_r E_1$$

$$E_0 = \frac{\varepsilon_r E_1}{\varepsilon_0} = \varepsilon_r E_1 \left(\varepsilon_r = \frac{\varepsilon_1}{\varepsilon_0}\right)$$

$$E = \frac{V}{d}$$

$$\emptyset' = \varepsilon_1 a \ \ and \ \emptyset_0 = E_0 (d-a)$$

$$V = V_0 + V_1 = E_0 (d-a) + E_1 a = \varepsilon_r E_1 (d-a) + E_1 a$$

$$= \varepsilon_r \varepsilon_1 d - \varepsilon_r E_1 a + E_1 a$$

$$\Longrightarrow \qquad E_1 = \frac{V}{\varepsilon_r (d-a) + a}$$

Hence $$E_1 = \frac{V}{\varepsilon_r d - a(\varepsilon_r + 1)}$$

Total capacitance

$C_T = C_0$ in series with C_1

$$C_T = \frac{C_0 C_1}{C_0 + C_1}$$

$$C_0 = \frac{\varepsilon_0 A}{d-a} \qquad \text{and} \qquad C_1 = \frac{\varepsilon_1 A}{a}$$

$$C_T = \frac{\varepsilon_0 \varepsilon_1\, A^2\ \left(\varepsilon_0 \dfrac{A}{d-a}\right) \times \left(\dfrac{\varepsilon_1 A}{d}\right)}{\dfrac{\varepsilon_1 A}{d-a} + \dfrac{\varepsilon_1 A}{a}} = \frac{\dfrac{\varepsilon_0 \varepsilon_1\, A^2}{(d-a)a}}{\dfrac{\varepsilon_0\, A_a + \varepsilon_1\ Ad - \varepsilon_1\ Ad}{(d-a)a}}$$

$$= \frac{\varepsilon_0 \varepsilon_1\, A}{a(\varepsilon_0 - \varepsilon_1) + \varepsilon_1 d} = \frac{\varepsilon_0 \varepsilon_1\, A}{\varepsilon_0 a + \varepsilon_1\,(d-a)}$$

$$C_T = \frac{\varepsilon_0\, A}{d}\left(\frac{\varepsilon_1}{\dfrac{\varepsilon_0 a}{d} + \varepsilon_1\,\left(1 - \dfrac{a}{d}\right)}\right) = \frac{\varepsilon_0\, A}{d}\left(\frac{\varepsilon_1}{\varepsilon_0\,\left(\dfrac{a}{d} + \dfrac{\varepsilon_1}{\varepsilon_0}\left(1 - \dfrac{a}{d}\right)\right)}\right)$$

$$C_T = \frac{\varepsilon_0\, A}{d}\left[\frac{\varepsilon_\rho}{\varepsilon_r\,\left(1 - \dfrac{a}{d}\right) + \dfrac{a}{d}}\right].\ Ans$$

4.4 Charge on Conducting Surface

Equation of work done relating charge on conducting surface is given by:

$$W = \frac{1}{2}\, V\ \int_V \rho_s\, ds \qquad 4.15$$

As charge is assembled on surface of conductor

$$\Longrightarrow \qquad W = \frac{1}{2} QV \qquad 4.16$$

Total charge on it potential of surface

4.5 Force on a Charged Conductor

Equation of work done relating force on a charged conductor is given by:

$$\Delta W = W \Delta s\ \Delta l \qquad 4.17$$

(if an elemental area ΔS on a charged conductor is depressed at distance Δl, the increase in stored energy is denoted by ΔW)

If depression must be carried out against force F, then,

$$\Delta W = F\Delta l = f\Delta S\ \Delta l \qquad 4.18$$

Where F is force per unit area = W (energy density)

$$\Rightarrow \qquad F = \frac{1}{2}\varepsilon E^2 = \frac{1}{2\varepsilon}\ D^2 = \frac{1}{2\varepsilon}\ \rho_s^2$$

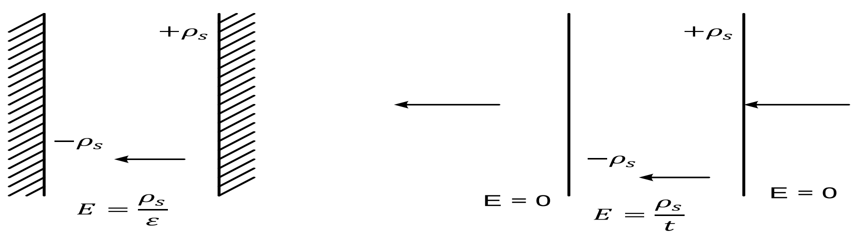


Figure 4.11

$$F = \frac{1}{2}\varepsilon E^2 = \frac{1}{2}\varepsilon \times \left(\frac{\rho_s}{\varepsilon}\right)^2 = \frac{1}{2\varepsilon}\ \rho_s^2$$

Example 4.7 We are given the non-uniform field $E = y\hat{a}_x + x\hat{a}_y + 2\hat{a}_z$. And we are asked to determine the work expended in carrying 2C from B (1, 0, 1) to A (0.8, 0.6, 1) along shorter arc circle.

$$x^2 + y^2 = 1; z = 1$$

Solution. We use $W = Q^A \int_B \vec{E} \cdot \overrightarrow{dl}$, where $\vec{E}$ is not necessarily constant. Working in rectangular coordinates, the differential path $\overrightarrow{dl}$ is $dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z$ and the integral becomes.

$$W = -Q^A \int_B \vec{E} \cdot \overrightarrow{dl}$$

$$= -2\int_{B}^{A} \left(y\hat{a}_x + x\hat{a}_y + 2\hat{a}_z\right)\cdot\left(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z\right)$$

$$= -2\int_{1}^{0.8} \mathrm{y\,dx} - 2\int_{0}^{0.6} \mathrm{x\,dy} - 4\int_{1}^{1} \mathrm{dz}$$

Where the limits on the integrals have been chosen to agree with the initial and final values of the appropriate variables of integration. Using the equation of circular path (and selecting the sign of the radical which is correct for quadrant involved), we have

$$W = -2\int_{1}^{0.8} \sqrt{1-x^2}\ dx - 2\int_{0}^{0.6} \sqrt{1-y^2}\ dy - 0$$

$$= -\left[x\sqrt{1-x^2} + \sin^{-1} x\right]_1^{0.8} - \left[y\sqrt{1-y^2} + \sin^{-1} y\right]_0^{0.6}$$

$$= -(0.48 + 0.927 - 0 - 1.571) - (0.48 + 0.644 - 0 - 0)$$

$$= -0.96\ \mathrm{J}$$

Example 4.8 Again, find the work required to carry 2C from B to A in the same field, but this time use the straight-line path from B to A.

Solution. We start by determining the equations of straight line. Any two of the following three equations for planes passing through the line are sufficient to define.

$$y - y_B = \frac{y_A - y_B}{x_A - x_B}\ (x - x_B)$$

$$z - z_B = \frac{z_A - z_B}{y_A - y_B}\ (y - y_B)$$

$$x - x_B = \frac{x_A - x_B}{z_A - z_B}\ (z - z_B)$$

For first equation, we have

$$y = -3(x-1)$$

And from the second equation we obtain, z=1

Thus,

$$W = -2\int_{1}^{0.8} y\,dx - 2\int_{0}^{0.6} x\,dy - 4\int_{1}^{1} dz$$

$$= 6\int_{1}^{0.8}(x-1)\,dx - 2\int_{0}^{0.6}\left(1-\frac{y}{3}\right)dy$$

$$W = -0.96\ \mathrm{J}$$

Example 4.9 Calculate the work done in moving a 4C charge from B (1, 0, 0) to A (0, 2, 0) along the path $y = 2 - 2x, Z = 0$ is field E =

a. $5\hat{a}_x \quad V/m$
b. $5x\hat{a}_x + 5y\hat{a}_y \qquad V/m$
c. $5x\hat{a}_x \quad \frac{V}{m}$

Solution:

a. $20\ \ \mathrm{J}$
b. $10\ \mathrm{J}$
c. $-30\ \mathrm{J}$

4.6 Poisson's and Laplace's Equation

We know that for the case of static fields, Maxwell's equations reduce to the electrostatic equation:

$$\nabla \times E(\bar{r}) = 0 \ \text{ and } \ \nabla \cdot E(\bar{r}) = \frac{\rho_v\,(\bar{r})}{\varepsilon_0} \qquad 4.19$$

We can alternatively write these Equ 4.19 in terms of the electric potential field, using the relationship

$$E(\bar{r}) = -\nabla V(\bar{r}) \colon 1$$

$$-\nabla \times \nabla V(\bar{r}) = 0 \quad and \ -\nabla \cdot \nabla V(\bar{r}) = \frac{\rho_v\,(\bar{r})}{\varepsilon_0}$$

Recall that this operation (second equation) is called the scalar Laplacian:

$$\nabla \cdot \nabla = \nabla^2 \qquad 4.20$$

Therefore, we can write the relationship between charge density and the electric potential field in terms of one in Equ 4.22a.

$$\nabla^2 V(\bar{r}) = -\frac{\rho_v(\bar{r})}{\varepsilon_0} \qquad 4.22a$$

Equ 4.22a is known as Poisson's equation and is essentially the "Maxwell's equation" of the electric potential field.

Note that for points where no charge exists, **Poisson's equation** becomes:

$$\nabla^2 V(\bar{r}) = 0 \qquad 4.22b$$

Equ 4.22b is known as Laplace's equation

$\hat{a}$ Although it looks very simply, most scalar functions will not satisfy Laplace's Equation. Only a special class of scalar fields, called analytic functions will satisfy Laplace's equation.

Laplace equation expanded in Cartesian coordinates

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad 4.23$$

This is a second order partial differential equations relating the rate of change of potential in the three component directions.

4.6.1 Procedure for solving Poisson's and Laplace's equation

Both the equations are subjected to "UNIQUENESS THEOREM" i.e, if a function V is found which is a solution of $\nabla^2 V = \frac{-\rho}{\epsilon_0}$ (or special case $\Delta^2 V = 0$) and if solutions also satisfy the boundary conditions, then it is the only solution.

- Solutions of Laplace's equation are known as harmonic functions. The several procedures of solving Laplace's equation is to construct a linear combination of harmonic functions so as to satisfy the boundary conditions of given procedures.

- For Poisson's equation, once we have any solution of the equation, then other solutions (including the one which obeys boundary conditions) can be obtain by adding to it solution to the corresponding Laplace's equation. The procedure for finding the correct solution to Poisson's equation is thus to obtain an initial solution to the equation which will most likely not satisfy boundary conditions. Next one adds to the solution corresponding Laplace equation until the final result does satisfy the boundary conditions. It will be clearer when you solve some numerical. Following steps should be performed, i.e.,

1. From given condition, interpret whether you have to use Laplace's or Poisson's equation.
2. From equation, find out the general solution of $'V'$ (potential) for given coordinate system.
3. Make this several solutions of $'V'$ particular by using boundary conditions. Thus, determine the final expression for $'V'$.
4. Calculate $\vec{E}$ from $\vec{E} = -\nabla V$, if V is known
5. Calculate $\vec{D}$ from $\vec{D} = \varepsilon\vec{E}$
6. Calculate $\vec{D}$ at either capacitor plates, depends on geometry given since $\vec{D} = \vec{D}_s = \vec{D}_n\, \hat{\imath}_n$
7. Recall $\rho_s = D_n$
8. Calculate Q by surface integration over capacitor plates $Q = \int \rho_s\, ds$
9. Finally, $C = \frac{Q}{v}$

$$\nabla . \vec{E} = 0 \qquad \text{[as } \vec{E} = -\nabla V] \qquad 4.24$$

In a homogenous charge free region, the number of lines of electric field strength emerging from a unit volume is zero (or in such a region). Lines of electric field strength are continuous.

Hence, we can generalize the procedure in following steps:

(1) Solve Laplace $'\emptyset\ (if\ \rho_v = 0)$ or poission $'\emptyset\ (if\ \rho_v \neq 0)$ equation either using direct integration (when V is a function of one variables) or separation of variables (if V is function of more than one variable) the solution is expressed at this point in terms of unknown integration constants and is not unique

(2) Hereby we find the unique solution for V by applying boundary condition

(3) After obtaining V, find $\vec{E}$ as $\vec{E} = -\nabla V, \vec{D} = \in \vec{E}$ and $\vec{J}$ from $\vec{J} = \in \ \vec{E}$.

(4) If required, find charge $'\theta'$ unduced on conductor using $\theta = \rho_s \, ds$, where $\rho_s = \vec{D}_n \ and \ \vec{D}_n$ is normal to conductor also we can find $C = \theta/v$, i.e., capacitance and $R = V/I$ where $I = \int \vec{J} \cdot \overrightarrow{ds}$ i.e., resistance.

4.6.2. Separation of variables

Determine the potential function for region inside the rectangular trough of infinite length whose cross-section is as shown in Fig 4.12.

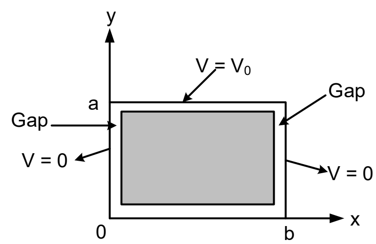


Figure 4.12 V(x,y)

Here the potential depends on x and y thus Laplace's equation becomes

$$\nabla^2 V = \frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} = 0$$

Subjected to boundary conditions

$$V\ (x = 0, 0 < y < a) = 0$$

$$V\ (x = b, 0 < y < a) = 0$$

$$V\ (0 < x < b, y = 0) = 0$$

$$V\ (0 < x < b, y = a) = V_0$$

We will seek a product solution of V by assuming

$$V(x, y) = X(x) Y\ (y)$$

So, Laplace's equation becomes

$$\frac{d^2(XY)}{dx^2} + \frac{d^2(XY)}{dy^2} = 0$$

$$Y\frac{d^2x}{dx^2} + \frac{Xd^2x}{dy^2} = 0$$

Dividing throughout by XY, we get

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

$$\Downarrow$$

$+\frac{A^2}{-B^2}$ $\quad (if\ B^2 > 0\ \&-A^2 <$

0

$$\therefore \qquad \frac{1}{X}\frac{d^2X}{dx^2} = A^2$$

$\Longrightarrow \qquad \frac{d^2X}{dx^2} - A^2X = 0$

$\Longrightarrow \qquad X = A_1e^{2x} + A_2e^{-\alpha x}$

Or $\qquad \cos h\ \alpha x = \frac{e^{\alpha x}+e^{-\alpha x}}{2}$

$$\sin h\ ax = \frac{e^{ax} - e^{-ax}}{2}$$

$\Longrightarrow \qquad e^{ax} = \cos h\ ax + \sin h\ ax$

$$e^{-ax} = \cos h\ ax - \sin h\ ax$$

$\therefore \qquad X\ (x) = \beta, \cos h\ ax + \beta_2 \sin h\ ax$

Where $\qquad \beta_1 = A_1 + A_2$

$$\beta_2 = A_1 - A_2$$

$\therefore \qquad$ Apply B. C

We get $B_1 = 0$

And $X(x = b) = \beta_2 \sin h\, ab = 0$

$\Longrightarrow$ $B_2 = 0\; as\; \alpha\; \neq 0\; \&\; b\; \neq 0$

$\therefore$ $X = 0$

Is a trivial solution we conclude that $-A^2$ can't be less than zero

$$\frac{1}{X}\frac{d^2X}{dx^2} = -B^2$$

$$\frac{d^2X}{dx^2} + B^2X = 0$$

$$X = C_0 e^{\alpha\beta x} + Ge^{-\alpha\beta x}$$

$$\text{y}e^{\alpha\beta x} = \cos\beta x + j\sin\beta x$$

$$e^{-\alpha\beta x} = \cos\beta x - jsin\,\beta x$$

$\Longrightarrow$ $X = g_0 \cos\beta x + g_1 \sin\beta x$

Where $g_0 = C_0 + C_1$

$g_1 = C_0 - \alpha C_1$

$\therefore$ Apply B .C we get

$g_0 = 0$

$$X\,(x = b) = 0 = g_1 \sin\beta\, b$$

$\Longrightarrow$ $\beta = \frac{n\pi}{b}$

Where $n = 1, 2, 3$

Note: Unlike $\sin hx$ which is zero only at $x = 0, \sin x = 0$ at infinite number of points

$\frac{1}{Y}\frac{d^2Y}{dy^2} = M^2Y$ $\Longrightarrow X_n(x) = g_n \sin\frac{n\pi x}{b}$

Which is similar to solution obtained (in case 1 of X)

i.e. $Y(y) = m_0 \cosh\, My + m_1 \sinh My$

apply B.C

$$Y(y=0) = 0 \implies m_0 = 0$$

Hence, $y_n(y) = m_n \sinh \frac{n\pi y}{b}$

Combining both we get

$$V_n(x,y) = g_n\, m_n \sin\frac{n\pi y}{b} \sinh \frac{n\pi y}{b}$$

This shows that there are many possible solutions V_1, V_2, V_3, V_4 and so or for $n = 1, 2, 3, 4 \ldots.$

4.7 Field Inside Parallel Plate Capacitor

Field inside a parallel plate capacitor can be calculated using Laplace's equation.

$$\ln 1 - \mathrm{D} \implies \nabla^2 V = \frac{\partial^2 V}{\mathrm{dx}^2} = 0$$

Solution $\implies V = Ax + B$

$$(A \text{ and } B \text{ const. of integration})$$

When x = 0, V = 0 ⟹ 0 = 0 + B ⟹ $\mathrm{B} = 0$

When x = d, $V = V_0$

$\implies$ $V_0 = Ad + 0 \implies A = \frac{V_0}{d}$

So, $V = \frac{V_0}{d} x \quad 0 < x < d$ 4.25

- Electric field $\vec{E} = -\nabla V = \frac{-\partial V}{\partial x}\hat{x} = \frac{-V_0}{d}\,\hat{x}\; 0 < x < d$ 4.26
- Surface charge density

$$[p_s]_{x=d} = [D]_{x=0} \cdot \hat{x} = \frac{-\varepsilon_0 V_0}{d} \hat{x} \cdot \hat{x} = \frac{-\varepsilon_0 V_0}{d} \quad 4.27$$

$$[p_s]_{x=d} = [D]_{x=0} \cdot (-\hat{x}) = \left(\frac{-6V_0}{d} \hat{x}\right) \cdot (-\hat{x}) = \frac{-\varepsilon_0 V_0}{d} \quad 4.28$$

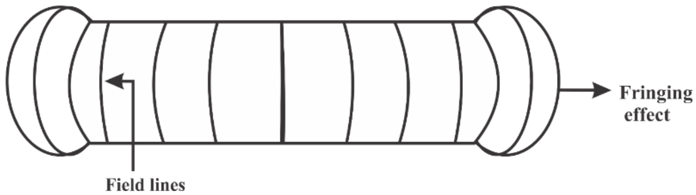


Figure 4.14 Fringing Effect

Example 4.10 Find the potential and electric field intensity for region between two concentric right circular cylinders. Where $V = 0\ a\ r_a = 1mm$ and $V = 100\ V$ at $r_b = 20mm$. Neglect fringing.

Solution: $V = f(r)$ i.e., potential in a function of radius

$So,\qquad \frac{1}{r}\frac{d}{\partial r}\left(r\frac{dV}{dr}\right) = 0$ (From Laplace equations)

Integrating once, $r\frac{dV}{dr} = A$

Integrating again, $V = Alnr + B$

Boundary again, $V = 0$ at $r_b = 20mm\ and$

$V = 100\ V$ at $r_b = 20mm$

Conditions $0 = A \ln 0.001 + \mathrm{B}$

$100 = A \ln 0.020 + B$

$\Rightarrow$ $A = 33.36; \mathrm{B} = 230.49$

$V = 33.36\ ln\, r + 230.49$

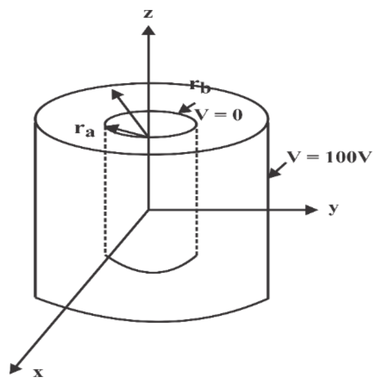


Figure 4.15

$$\vec{E} = -\nabla V = \frac{-dV}{dr}\,\hat{r}$$

$$\Longrightarrow \qquad \vec{E} = -\frac{33.36}{r}\,\hat{r}\;V/m \qquad 4.29$$

Example 4.11 For the configuration of Fig. 4.15, find V and E for $\theta < \emptyset < \alpha$ using Laplace's equation. Hence, find capacitance of system.

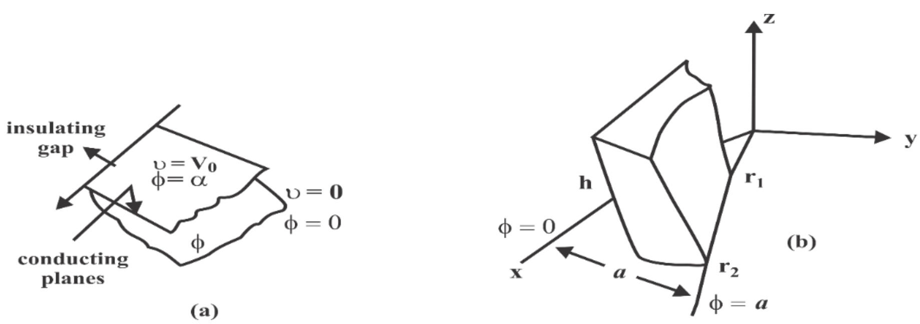


Figure 4.15

Solution: Since the potential is constant w.r.t. 'r' and 'z'. Laplace's equation is

$$\frac{1}{r^2} \times \frac{\partial^2 V}{\partial \emptyset^2} = 0$$

Integrating twice,

$$V = A\emptyset + B$$

B.C $\qquad V = 0\; at\; \emptyset = 0 \quad \Longrightarrow \quad B = 0$

$$V = V_0 \; at \; \emptyset = \alpha \Longrightarrow \; A = \frac{V_0}{\alpha}$$

So, $$V = V_0 \, \emptyset / \alpha \qquad 4.30$$

Taking the gradient,

$$E = -\nabla V = -\frac{1}{r}\frac{\partial}{\partial \emptyset}\left(V_0 \, \frac{\emptyset}{\alpha}\right)\hat{\emptyset} = \frac{aV_0}{\alpha r} \, \hat{\emptyset}$$

$\Longrightarrow$ $$D_s = -\frac{\varepsilon \, V_0}{ar} \, \hat{\alpha} \qquad 4.31$$

Find average density on plates

$$\rho_s = D_n = -D\emptyset = \frac{\varepsilon V_0}{ra}$$

Total charge on plates for $\Longrightarrow 0 \; to \; h \; and \; r = r_1 + r_2$ in (b)

$$Q = \int \rho_s \, ds = \int_0^h \int_{r_1}^{r_2} \frac{\varepsilon V_0}{r\alpha} \, dr \, dz = \frac{\varepsilon \, V_0 \, h}{\alpha} \ln \frac{r_2}{r_1}$$

$$C = \frac{Q}{V_0} = \frac{\varepsilon \mathrm{h}}{\alpha} \ln \frac{r_2}{r_1} \qquad 4.32$$

Example 4.12 Since no new problem are solved by choosing fields which very only with y or with z in rectangular coordinates. We pass on to cylindrical coordinates variations with respect to Z are again nothing new, we next assume variation with respect to ρ only.

Solution: Laplace's equation becomes

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0$$

Nothing the ρ in denominator, we exclude $\rho = 0$ from our solution and then multiply by ρ and integrate

$$\rho \frac{dV}{d\rho} = A$$

Rearrange and integrate again

$$V = A\, ln\, \rho + B$$

The equipotential surfaces are given by $\rho =$ constant and are cylinders and the problem is that of the coaxial capacitor or coaxial transmission line. We choose a potential difference of V_0 by letting $V = V_0$ at $\rho = \alpha, V = 0$ at $\rho = b, b > \alpha$ and obtain.

$$V = V_0\, \frac{ln(b/\rho)}{ln(b/a)}$$

From which $E = \frac{V_0}{\rho} \frac{1}{1n\left(\frac{b}{a}\right)}\, a_p$

$$D_{N\,(p=a)} = \frac{\varepsilon\, V_0}{a\, ln\, \left(\frac{b}{a}\right)}$$

$$Q = \frac{\varepsilon V_0\; 2\pi a L}{a\, ln\, \left(\frac{b}{a}\right)}$$

$$C = \frac{2\pi\varepsilon L}{ln\, \left(\frac{b}{a}\right)}$$

Example 4.13 Solve the above example 4.12 in cylindrical coordinates

Solution: let us assume that V is function only of $\emptyset$ in cylindrical coordinates. See that equipotential surfaces are given by $\emptyset =$ constant. These are radial planes. Boundary conditions might be $V = 0$ at $\emptyset = 0, V = V_0$ at $\emptyset = \alpha$, Laplace's equation is now.

$$\frac{1}{\rho^2} \frac{d^2V}{d\emptyset^2} = 0$$

We exclude $\rho = 0$ and have

$$\frac{d^2V}{d\emptyset^2} = 0$$

$\Longrightarrow$ $V = A\emptyset + B$ is solution

The boundary conditions determine A and B and

$$V = \frac{V_0\ \emptyset}{\alpha}$$

Taking gradient of above produces electric field intensity

$$E = \frac{-V_0\ \hat{a}_{\emptyset}}{\alpha_{\rho}}$$

It is interesting to note that $\vec{E}$ is a function of ρ and not of $\emptyset$

Example 4.14 Solve example 4.12 using spherical coordinates.

Solution: Turn to spherical coordinates, dispose immediately of variations with respect to $\emptyset$ only as having just been solved, and treat first $V = V(r)$.

The potential field is

$$V = \frac{V_0\ \left(\frac{1}{r} - \frac{1}{b}\right)}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

$$\text{So, capacitance} \qquad = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$$

However, when we restrict potential function to $V = V(\theta)$, we get

$$\frac{1}{r^2 \sin\theta\ d\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dV}{d\theta}\right) = 0$$

We exclude $r = 0$ or π and have

$$\sin\theta\ \frac{dV}{d\theta} = A$$

The second integral is then

$$V = \int \frac{A\, d\theta}{\sin\theta} + B$$

Which is not as obvious as previous ones.

$$V = A\ln\left(\tan\frac{\theta}{2}\right) + B$$

4.8 Dirac Delta Representation for an Infinitesimal Dipole

Potential due to infinitesimal dipole found by evaluating the far field of finite dipole as:

$$\rho(r) = 2\,\delta(x)\,\delta(y)\left[\delta\left(z - \tfrac{1}{2}\right) - \delta\left(z + \tfrac{l}{2}\right)\right] \qquad 4.33$$

"Find a representation so that for field approximations become unnecessary?"

Such a distribution may be derived by writing an expression for charge density with finite spacing and then letting the spacing approach zero.

$$\rho(r) = \lim_{l\to 0}\left\{q\delta(x)\delta(y)\left[\delta\left(z - \frac{1}{2}\right) - \delta\left(z + \frac{1}{2}\right)\right]\right\}$$

$$= \lim_{l\to 0}\left\{ql\delta(x)\delta(y)\frac{\left[\delta\left(z - \frac{1}{2}\right) - \delta\left(z + \frac{1}{2}\right)\right]}{l}\right\}$$

$$= \frac{\delta\left(z + \frac{1}{2}\right) - \delta\left(z - \frac{1}{2}\right)}{\mathrm{I}} \qquad \textit{Derivative of Dirac} - \textit{Delta}$$

If dipole moment $'p'$ is defined as $\quad p = \lim_{l\to 0} ql$

Then the charge density of an infinitesimal dipole is

$$\boldsymbol{\rho(r) = -\rho\delta(x)\delta(y)\delta'(z)} \qquad 4.34$$

Frequently the dipole moment of p is expressed as a vector $\vec{p}$ by assigning to it the direction of a line drawn from the negative point to the positive point charge.

$$\Longrightarrow \qquad \vec{p} = \hat{z}p \qquad 4.35$$

FORMULAS TO BE USED

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \qquad \textbf{cartesian coordinates}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \emptyset^2} + \frac{\partial^2 V}{\partial z^2} \qquad \textbf{cylindrical coordinates}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2}\,(sin^2\theta)^2\frac{\partial^2 V}{\partial \emptyset^2} \qquad \textbf{Spherical coordinates}$$

4.9 Exercise

1. Calculate the capacitance between concentric metal spheres of radius r_1 and r_2 with charge Q placed or the outer surface of inner shell.
2. The potential field at any point in a space containing a dielectric material of relative permittivity 2.1 is given by $V = 5x^2y + 3yz^2 + 6xz\,V,$ where $x, y, z,$ are in meters. Calculate the volume charge density at point $P(2, 5, 3)m$.
3. Three-point charges $3, 4\ and\ 5$ coulombs are situated in fill space at 3 corners of an equilateral triangle with sides 5cm. find the energy density due to electric fields in the triangle.
4. (a) consider a parallel plate capacitor occupying planes $x = 0$ and $x = d$ and is kept at potential $v = zero$ and $V = V_0$ respectively. The medium consists of two dielectrics: ε_1 for $0 < x < t$ and ε_2 and for $t < x < d$. Find the potential and electric field intensities in two regions using Laplace's equation.
   (b) Take $d = 4cm$: $\varepsilon_{r1} = 2, 0 < x < 2;\ \varepsilon_{r2} = 4, 2 < x < 4, v_0 = 100V$. Find voltage and field intensities in two regions. Calculate capacitance per unit surface area of system.

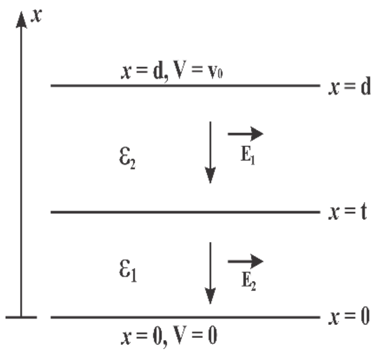


Fig 4.25

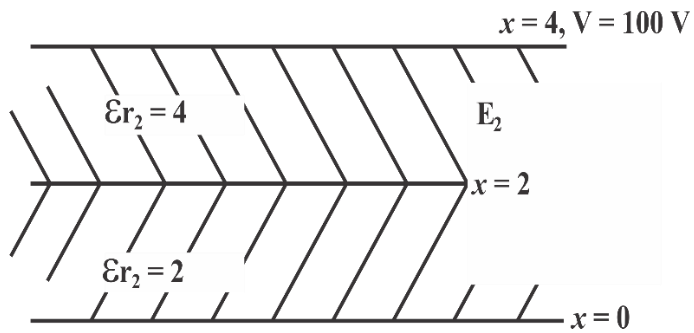


Fig 4.26

5. A boundary exists at $z = 0$ between two dielectrics, $\varepsilon_{r1} = 2.5$ region $=< 0$, and $\varepsilon_{r2} = 4$ region $z > 0$. The field in region $\varepsilon_{r1}\ is\ \vec{E}_1 = -30\,\hat{x}\ 50\hat{y} = 70\hat{z}\ V/m$. Find
    a. Normal component of $\vec{E}_1$
    b. Tangential component of $\vec{E}_1$
    c. The angle $a_1\ \leq 90^0$ between $\vec{E}_1$ and normal to surface
    d. Normal component of O_2
    e. Tangential component of D_2
    f. D_2
    g. Polarization in ε_{r2} material
    h. Angle a_2 between E_2 and normal of surface.
6. The potential distribution at mouth of slot is given by:

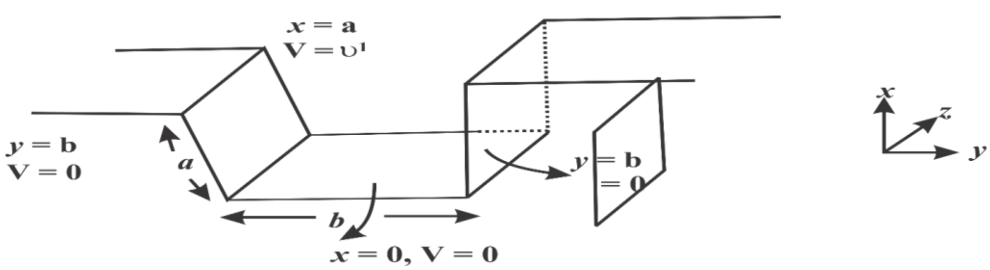


Figure 4.27

$$V = V = V_1 \sin\frac{\pi y}{b} + V_2 \sin\frac{3\pi y}{b}\ for\ x = a, 0 < y < b$$

Where $V_1\ and\ V_2$ are constants. Find the solution for potential distribution in slot.

7. Explain the term capacitance
8. Relate charge and potential by an expression

9. Derive expression for energy stored in capacitor
10. Give alternative expressions for energy stored in capacitor
11. Derive expression foe electrostatic energy in system charges
12. Find force on charged conductor
13. Explain charge and discharge of capacitor
14. Define Poisson's equation
15. Define Laplace's equation
16. Prove Poisson's equation
17. Prove Laplace's equation
18. Relate Laplace equation to Maxwell's equation
19. Conclude from Laplace's equation that "Lines of electric field strength are continuous
20. Derive expression for field inside parallel plate capacitor
21. What do you understand by boundary conditions?
22. Explain "perfect metals are equipotential".
23. Discuss conditions for metals.
24. Derive following boundary conditions:
    a. Dielectric-dielectric
    b. Conductor-dielectric
    c. Conductor-conductor
25. Conclude boundary relations from Gauss's law and Faraday's law
26. What do you mean by Dirac-Delta representation of point charge?
27. Relate Dirac-Delta representation with impulse function
28. Derive Dirac-Delta representation for volume charge density and surface charge density
29. Relate Green's function and Dirac-Delta distribution
30. What do you mean by Dirac-Delta representation for an infinitesimal dipole?
31. Discuss properties of Dirac-Delta distribution

CHAPTER 5

ELECTROMAGNETIC INDUCTION

5.0 Faraday's Law of Electromagnetic Induction

Faraday's law of electromagnetic induction state that "The induction e.m.f. (e volts) is equal to the negative rate of change of flux"

$$e = -\frac{d\emptyset}{dt} \qquad 5.1$$

Note: the negative sign was introduced by Lenz. The sign indicates the direction of e.m.f induced

Faraday's law can be put in the differential equation for as

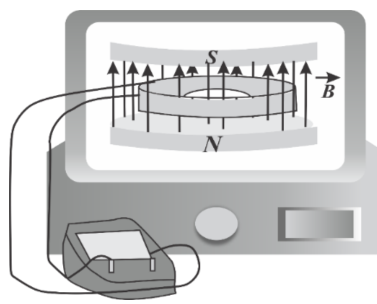


Figure 5.1 Coil in an increasing Magnetic Field

$$e = \oint_S \vec{E} \cdot \vec{dl} = \int_S (\nabla \times \vec{E}) \cdot \hat{n}\, da$$

5.2

(here 'e' is denoting potential)

And

$$\frac{d\emptyset}{dt} = \frac{d}{dt}\int_S \vec{B} \cdot \hat{n}\, da = \int_S \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\, da$$

$$e = -\frac{d\emptyset}{dt}$$

$$\int_S \nabla \times \vec{E} \cdot \hat{n}\, da = \int_S \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\, da$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad 5.3$$

Whenever there is a change in the number of magnetic field lines passing through a loop of wire a voltage (or emf) is generated (or induced) in the loop of wire. This is how an electric generator works. The phenomenon is known as electromagnetic induction and is explained by Faraday's law of induction:

$$V = -d\emptyset/dt \qquad 5.4$$

Where ∅ is the magnetic flux given by the closed integral of the dot product $B. dA$.

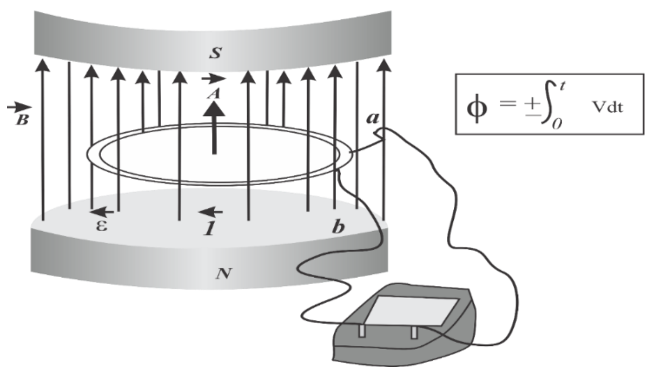


Figure 5.2

$$I_g = \frac{V_{\text{induced}}}{R}$$

⟹ R is total resistance in galvanometer circuit

According to Faraday's law, if there is no change (with time) in the number of lines of B field, or magnetic flux, through a closed loop(s) there will be no induced, or generated, voltage set up in the loop(s).

Note: When flux is increased in the +ve direction", the induced voltage is –ve

$$\Longrightarrow \qquad \theta = \int_0^1 I_g \, dt = \frac{1}{R}\int_0^1 V \ dt \qquad 5.5$$

According to Faraday's law as the strength of a magnetic field (B) passing through a loop of wire increases, there will be an increase in the number of magnetic field line, and therefore an induced voltage, or emf set up in the wire loop. If the strength decreases there will be also an induced emf set up in the loop but with the opposite polarity. Lenz law indicates the polarity of the induced emf.

Example 5.1. Calculate the inductance of co-axial capacitor

$$\vec{B} = \frac{\mu_0 \, i}{2\pi r} \, \hat{a}_\emptyset$$

Solution: Consider the space between r = a and r = b is free space. If l is length of cable, then

$$\emptyset_m = \oint B \cdot ds = \int_0^1 \int_a^b \frac{\mu_0 \, i}{2\pi r} dr \, dl = \frac{\mu_0 \, il}{2\pi r} \ln(r)_a^b = \frac{\mu_0 \, il}{2\pi r} \ln\left(\frac{b}{a}\right)$$

$$\frac{\emptyset_m}{i} = L = \frac{\mu_0 \, l}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$\frac{L}{I} = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right)$$

5.1 Magnetic Flux Density

Another important vector is magnetic flux density B. it is related to H via:

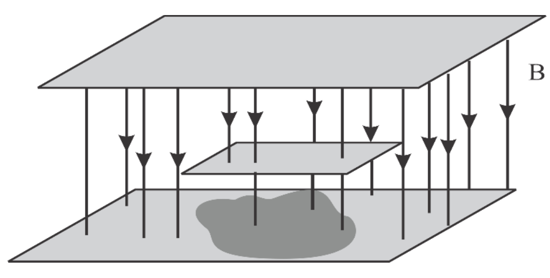


Figure 5.3

$$\vec{B} = \mu_0 H \qquad 5.6$$

The flux associated with a magnetic field is therefore a measure of the number of magnetic field lines penetrating some surface.

- The above pictures show the spherical case of a plane area S and a uniform flux density B. the normal to the field is at an angle with the field. In this case, the flux is given by

$$\Phi_m = BS\cos\theta \qquad 5.7$$

If B is the value of the flux density

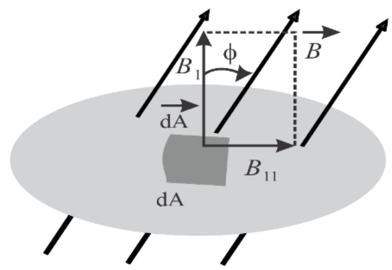


Figure 5.4 $\Phi = \mathrm{f}$ (orientation of loop; its area $\Rightarrow B \Rightarrow$ VECTOR

- Generally, if an element of area Ds on an arbitrarily shaped surface, has a magnetic field running through it, the magnetic flux through this area is BdS, if B is the value of the field at this element. The total magnetic flux is:

$$\Phi_m = \int_S \vec{B}\cdot\vec{n}\,dS \qquad 5.8$$

- Magnetic field lines are continuous and form loops. This is illustrated in the solenoid in the Fig. 5.5.

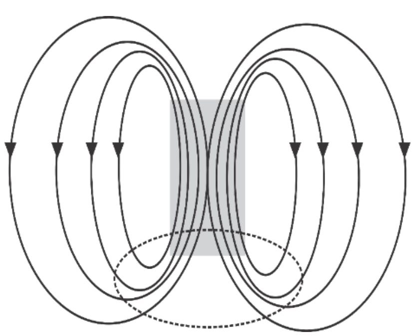


Figure 5.5 Electromagnetic Field Lines

- For any closed surface the number of lines entering that surface is equal to the number leaving it, as shown above in red. That means that the net flux is zero. This is called Gauss's law it is expressed thus:

$$\oint \vec{B} \cdot \vec{n}\, dS = 0$$

- Therefore, at any point:

$$\nabla \cdot \vec{B} = 0 \qquad 5.9$$

Example 5.2: Calculate flux density at center of square loop of 10 turns, 2m on side carrying 10amp. The loop is in air media.

Solutions: $\mu_r = 1, \mu = \mu_0 \mu_r = 4\pi \times 10^{-7}\ \mathrm{H/m}$

$$I = 10\mathrm{amp}; \mathrm{N} = 10; \mathrm{a} = 2\mathrm{m}$$

$$B = \frac{\mu I N\, 2\sqrt{2}}{\pi \alpha} = \frac{4\pi \times 10^{-7} \times 10 \times 10 \times 2\sqrt{2}}{\pi \times 2} = 4\sqrt{2} \times 10^{-5} T$$

$$\boxed{= 5.656 \times 10^{-5}\ T \text{ or } \mathrm{Wb/m^2}}\ Ans$$

Example 5.3. Calculate the magnetic flux density at center of a current carrying loop when radius of loop is 2cm, loop current is 1mA and loop is placed in air.

Solution: $\frac{\boldsymbol{\mu I}}{\boldsymbol{2\pi r}} = \frac{\boldsymbol{4\pi} \times 10^{-7} \times 1 \times 10^{-3}}{\boldsymbol{2\pi} \times 2 \times 10^{-2}} = \times\, 10^{-10} \times 10^{2} = \times\, 10^{8}\ \mathrm{T}$

$$\vec{B}_{center} = \frac{\mu I}{2a} = \frac{4\pi \times 10^{-7} \times 10^{-3}}{2 \times 10^{-2}} = 2\pi \times 10^{-8}$$

$$= 6.28 \times 10^{-8}\ \mathrm{T}$$

5.2 Definition of the Telsa and the Weber

The Telsa is defined as the density of a magnetic field such that a conductor carrying one ampere at right angles to the field has a force of one Newton per meter acting on it.

The Weber can be defined in two ways:

1. The amount of flux, when cut at a uniform rate by a conductor in one second, generates an emf of one volt.
2. The magnetic flux linking one turn induces in it as emf of one volt when the flux is reduced to zero at a uniform rate in one second.

The flux and Flux density are related by the following formula.

$$(\text{phi}) = \text{B} \times A$$

Where

(phi) is the flux in Webers. Wb.

B is the flux density in Teslas, T

A is the cross-sectional area, meters squared the induced voltage in a coil therefore depends on the total flux, the number of turns, and the time for the field to be reversed.

$$\text{conversion of neper into decibel}$$
$$10\ log_{10}\left(\frac{P_0}{P}\right) = \frac{10}{2.3026} ln\ \frac{P_0}{P} = \frac{20}{2.3026}\ (\alpha z)$$
$$= 8.686\ (\alpha z)$$
$$1Np = 8.686\ dB$$

5.3 Magnetic Field Strength

The magnetic field strength, H is the magnetomotive force per unit length in a magnetic circuit. It is given by:

$$\text{H} = \text{mmf}\ /\text{I}$$

Where, H = magnetic field strength, $\boldsymbol{A/m}$

$\mathbf{mmf}$ = magnetomotive force, $\boldsymbol{A}$

$\mathbf{I}$ =length of magnetic current $\boldsymbol{l}$

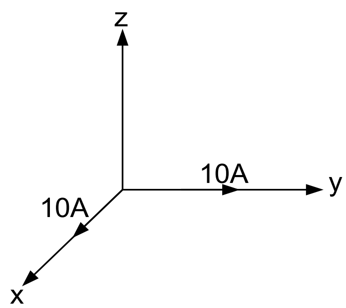


Figure 5.6

Example 5.4: A filamentary current of 10 A is directed in from infinity origin on the positive x-axis and then back out to infinity along the y-axis. Use the Biot-Savart law to find $\vec{\boldsymbol{H}}$ at $\boldsymbol{\rho}(\mathbf{0}, \mathbf{0}, \mathbf{1})$.

Solution: we show that the current 10A is directed from infinity to origin on positive x-axis and then back out to infinity positive Y-axis in Fig 4.1. We consider current which lies on x-axis and we write the expression.

$$d\vec{H} = \frac{I\,\overrightarrow{dl} \times \hat{a}_{12}}{4\pi d^2}$$

$$dH = \frac{IdlR}{4\pi R^3}$$

$$\overrightarrow{dl} = dx\hat{a}_x, \text{point} - 2(0,0,1)and$$

$$\text{Point } 1\ (x, 0, 0)$$

$$R_{12} = -x\hat{a}_x + \hat{a}_z$$

$$d = |\vec{R}_{12}| = \sqrt{x^2 + 1}$$

$$\vec{d}H = \frac{10\ dx\ \hat{a}_x}{4\pi} \times \left[\frac{-x\ \hat{a}_x + \hat{a}_z}{(x^2+1)^{3/2}}\right]$$

$$\vec{d}H = \frac{-10\ dx\ \hat{a}_y}{4\pi\ (x^2+1)^{3/2}}$$

On integrating, we obtain the firm as

$$H = \frac{-10}{4\pi}\int_{\infty}^{0} \frac{dx\hat{a}_y}{(x^2+1)^{\frac{3}{2}}} = \frac{-10}{4\pi}\left[\frac{x}{\sqrt{x^2+1}}\right]_{\infty}^{0} \hat{a}_y$$

$$\vec{H} = \frac{-10}{4\pi}\ [0-1]\hat{a}_y = 0.796\ \hat{a}_y$$

Similarly, for current which lied on Y-axis then we write the expression as

$$\overrightarrow{dH} = \frac{10\ dy\ \hat{a}_y}{(4\pi\sqrt{y^2+1})\ ^2} \times \left(\frac{-y\ \hat{a}_y + \hat{a}_z}{\sqrt{y^2+1}}\right)$$

$$\overrightarrow{dH} = \frac{10\ dy\ \hat{a}_x}{4\pi(y^2+1)^2}$$

Integrating we obtain the form as

$$\vec{H} = \frac{10}{4\pi}\int_{0}^{\infty} \frac{dy\ \hat{a}_y}{(y^2+1)^{\frac{3}{2}}} = \frac{10}{4\pi}\left[\frac{y}{\sqrt{x^2+1}}\right]_{0}^{\infty} \hat{a}_x$$

$$\vec{H} = \frac{10}{4\pi}\ [1-0]\hat{a}_x$$

$$\vec{H} = 0.796\ \hat{a}_x$$

$$\vec{H}_{\text{total}} = 0.796\hat{a}_x + 0.796\hat{a}_y\ \text{A/m}$$

5.4 Magnetomotive Force

A magnetic circuit consisting of a coil wound on either a magnetic or non-magnetic former can be compared with the electric circuit. In the electric circuit:

$$\text{Current} = \text{emf/resistance}$$

Or

$$\text{I} = \text{E/R}$$

In the magnetic circuit:

$$\text{Flux} = \text{mmf/Reluctance}$$

Or

$$(\text{phi}) = \text{F/Rm}$$

Magnetomotive force is measured in amperes, A and is produced by the current in the magnetizing current where

$$\text{mmf} = \text{NI}$$

where

mmf is the magnetomotive force in amperes A

N is the number of turns

I is the magnetizing current in amperes A

5.5 Points to Note

Note

Experiment shows that for homogeneous medium $\vec{B}$ is related to current I,

$$B = \frac{\mu I}{r} \quad r \text{ distance from wire}$$

$\mu \rightarrow$ permeability of medium $\Longrightarrow \mu = \mu_r \mu_0$

$\mu_0 =$ absolute permeability of vacuum $\Longrightarrow 4\pi \times 10^{-7}$ H/m

$\mu_r =$ relative permeability

Proportionality factor $= \frac{1}{2\pi}$

$$B = \frac{1}{2\pi} \times \frac{1}{r} \times \mu$$

Where

$$\mu = \frac{1}{2\pi r}$$

$$\vec{B} = \mu\vec{H}$$

Note 2

Magnetic field strength H is thus defined in terms of current which produces it and the geometry of system.

It's a vector quantity, having same dir^n as $\vec{B}$ (in isotopic media).

$*$ $\quad H \neq f(\mu) \quad B = f(\mu)$

$\Longrightarrow$ $\quad H \equiv E \quad and\ B \equiv D$

Note 3

Magnetomotive force

$$\mathrm{f} = \int_{a}^{b-1} \mathrm{H} \cdot \mathrm{ds} = \oint \frac{1}{2\pi r}\ \mathrm{ds} = 1 \qquad 5.10$$

This result will be obtained for any closed path about the current

$\Longrightarrow$ called Ampere's work Law/Ampere's Circuital law.

Note 4

I. For any closed path C around the core inside the winding, the MMF

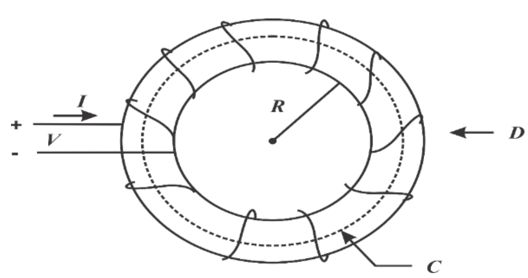


Figure 5.7 Toroidal Coil

$\Longrightarrow \qquad f = nI$

$n \longrightarrow$ no. of turns

$\therefore$ The no. of times the path links with the current I

$$\vec{H} = \frac{f}{2\pi R} = \frac{nI}{2\pi R} = \frac{nI}{I} \quad \text{A turns/m} \quad (D \ll R) \qquad 5.11$$

$l \rightarrow$ length of coil

Note 5

Magnetic field strength is nearly uniform throughout the cross-section of the core and is equal to the ampere turns per unit lengths.

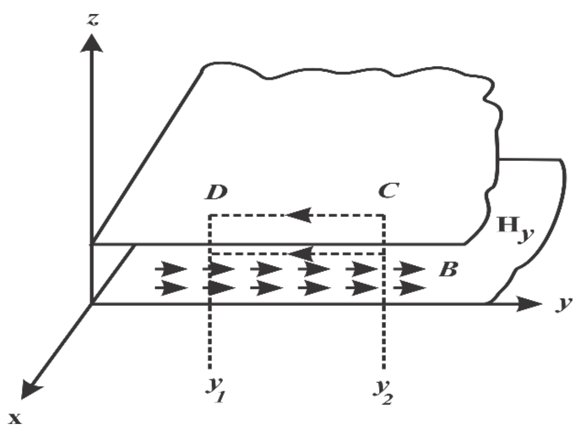


Figure 5.8 Parallel Plane Conductors

II. Two closed spaced parallel planes carrying equal and oppositely directed currents.

Magnetic Field: f (confined to region between places UNIFORM (except area edges)).

$$\neq f(\text{distance apart of planes})$$

- Current is assumed to be flowing in the positive $'x'$ dir (upper plate outwards)

  J_{sx} current per meter width

CHAPTER 6

ELECTROMAGNETIC EQUATIONS

6.0 Maxwell's Equations

The electromagnetic equations are called Maxwell's equations.

Each differential equation has its integral counterpart; one from, may be derived from other with help of Stokes Theorem/Divergence Theorem (dot superscript indicating partial derivatives w.r.t. time).

1. $\nabla \times H = D^0 + J \rightarrow \oint H \cdot ds = \int (D^o + J) \cdot da \rightarrow$ **Ampere's circuital law** 6.1

2. $\nabla \times \varepsilon = -B^0 \quad \oint E \cdot ds = -\int B^0 \cdot \ da \rightarrow$ ***Faraday's*** **law** 6.2

3. $\nabla \times D = \rho \quad \rightarrow \oint D \cdot da = \int_{\rho} dV \rightarrow$ **Gauss's law** 6.3

4. $\nabla \times B = O \quad \rightarrow \oint B \cdot da = O \rightarrow$
   No isolated magnetic charge or monopole 6.4

Contained in the above is the Equ 6.4 of continuity.

$$\nabla \cdot J = -\rho^o \quad \oint J \cdot ja = -\int \rho^o \ dV \qquad 6.5$$

Statement of Laws used in Deriving Maxwell's Equation

(1) Ampere's Circuital Law: This law states that the circulation of magnetic field intensity around any closed path is equal to the sum of free current and displacement current flowing through the surface bounded by path.

The term $\frac{dD}{dt}$ is displacement current density:

Maxwell's Equation-Modified Ampere's law

From Faraday's experiment, $\nabla \times E = -\frac{dB}{dt}$

Ampere's circuital law applied to steady magnetic fields gives

$$\nabla \times H = J$$

Taking divergence on both sides

$$\nabla \cdot (\nabla \times H) = \nabla . J$$

But $\nabla \times H = 0$ for static

$$\boxed{\nabla . J = \frac{-d\rho_v}{dt}} \quad \text{(from equation of continuity)} \qquad 6.6$$

Let us take an arbitrary $'G'$ quantity for solving above unrealistic limitation of $\frac{-d\rho_v}{dt} = 0$

$\therefore \qquad \nabla \times H = J + G$

Taking divergence

$$\nabla \cdot (\nabla \times H) = \nabla \cdot J + \nabla \cdot G$$

$\Longrightarrow \qquad \nabla \cdot G = -\nabla \cdot J = \frac{d\rho_v}{dt}$

We know that $\nabla . D = \rho_v$ (Gauss's law)

$\Longrightarrow \qquad \nabla \cdot G = \frac{d}{dt}(\nabla \cdot D) = \nabla \cdot \frac{dD}{dt}$

We get $\qquad G = \frac{dD}{dt}$

Thus, Ampere's law becomes

$$\boxed{\nabla \times H = J + \frac{dD}{dt}}$$

It also agrees with continuity equation

(2) Faraday's law of electromagnetic induction:

This law states that an emf is nearly a voltage that arises from conductors moving in a magnetic field or from changing magnetic fields. (Fig 6.1 a, b)

$$\boxed{\text{emf} = V_e = -\frac{\text{d}\emptyset}{\text{dt}}}$$

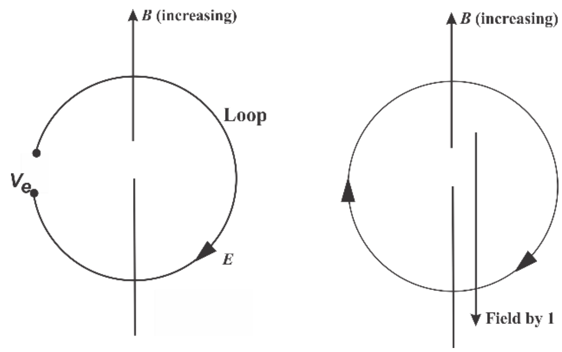


Figure 6.1 (a) Emf induced in an open circuited loop (b) Current induced in Loop

The emf induced in loop (V_e) is equal to emf producing field $\vec{E}$ (associated with induced current) integrated around the loop.

$$V_e = \oint E \cdot dl$$

The total flux through the circuit is equal to integral of normal component of flux density B over the surface bounded by circuit.

$$\emptyset = \int_s B \cdot ds$$

Thus, $V_e = \oint E \cdot dl = -\frac{d}{dt} \int_s B \cdot ds$ (transformer induction equation)

Faraday's law in integral form:

$$\oint E \cdot dl = -\int_s \frac{dB}{dt} \cdot ds$$

By Stokes's theorem we get differential form of Faraday's law:

$$\nabla \times E = -\frac{dB}{dt}$$

(3) Gauss's law states that the total outward electric flux over any closed surface is equal to the total free charge enclosed in the volume surrounded by surface.

(4) It is postulate of magnetostatics which states that there are no magnetic flux sources, and magnet flux lines always close upon themselves. It is the law of conservation of magnetic flux (no monopole)

Word statement of the field equations:

1. The magnetomotive force around a closed path is equal to the conduction current plus the time derivative of electric displacement through any surface bounded by path.
2. The electromotive force around a closed path is equal to time derivatives of the magnetic displacement through any surface bounded by path.
3. The total electric displacement through surface enclosing a volume is equal to total charge within volume.
4. The net magnetic flux emerging through any closed surface is zero.

Analogies:

1. Electric current = both conduction and displacement currents
2. Time derivative of electric displacement = electric current
3. Time derivative of magnetic displacement =magnetic current
4. Electromotive force = electric voltage
5. Magnetomotive force = magnetic voltage

Restatement:

1. The magnetic voltage around a closed path is equal to electric current through the path.
2. The electric voltage around a closed path is equal to magnetic current through the path.

Maxwell's equation for fields varying harmonically with time when the fields are harmonically varying with time, we write

$$E = E_o \cos(\cos + \emptyset)$$

Or $$E = E_o\, e^{j(\cos + \emptyset)} \quad \text{(as phasor)}$$

Differentiating w.r.t time

$$\frac{dE}{dt} = j\omega\, E_o e^{j(\omega t + \emptyset)} = j\omega E$$

$$\frac{d}{dt} = j\omega$$

Rewriting Maxwell's equations using Stokes's and Divergence theorem we get,

Differential form — Integral Form!

(1) $\nabla \times \vec{E} = -j\omega \vec{B}$ — $\oint_C \vec{E}\ dl = -j\omega \int_S B \cdot ds$ (6.7)

(2) $\nabla \times \vec{H} = j + j\omega D$ — $\oint_C \vec{H}\ dl = (\sigma + j\omega E) \int_S E \cdot ds$ (6.8)

(3) $\nabla \cdot D = \rho_v$ — $\oint_C D\, ds = \int_V \rho_v\, dV$ (6.9)

(4) $\nabla \cdot B = 0$ — $\oint_S B \cdot ds = 0$ (6.10)

Example 6.1: A coaxial capacitor has parameters $a = 10mm, b = 15mm$

$l = 20cm, \varepsilon_r = 8,\ \sigma = 10^{-6}\Omega/m$. If $\mathrm{J}_c = \frac{2}{r}\sin 10^6 t\ \hat{a}_r\ \mathrm{A/m^2}$ then find out

i. The maximum instantaneous value of displacement current density.

j. Total displacement current

Solution. (i) The conduction current density is expressed as

$$\vec{\mathrm{J}}_C = \sigma \vec{E};\ \ \vec{E} = \frac{\vec{\mathrm{J}}_C}{\sigma}$$

∴ We can write as

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon_0 \varepsilon_r \frac{\vec{J}_C}{\sigma}$$

$$\vec{J}_p = \frac{\overrightarrow{dD}}{dt} = \left(\frac{\varepsilon_0 \varepsilon_r}{\sigma}\right)\left(\frac{d\,\vec{J}_C}{dt}\right)$$

$$= \frac{8 \times 8.854 \times 10^{-12}}{10^{-6}} \times \frac{2}{r} \times 10^6 \times \cos 10^6 t \;\; \hat{a}_r$$

$$\vec{J}_p = \frac{141.664}{r} \cos(10^6 t)\hat{a}_r \;\; A/m^2$$

At $r = 10mm = 0.1m$ the $\left(J_p\right)_{max}$ is obtained as

$$\left(\vec{E}_0\right)_{max} = \frac{14.1664}{r} \cos(10^6 t)\hat{a}r$$

$$\left(\vec{J}_D\right)_{max} = 14166.4 \cos(10^6 t) \;\; \hat{a}_r$$

(ii) the total displacement current in terms of displacement current density expressed as

$$I_D = \vec{J} \cdot \overrightarrow{ds}$$

$$= \int_{2\pi} \left(\frac{14.1664}{r} \cos(10^6 t)\hat{a}r\right) \cdot (r\, d\emptyset\, dz)\hat{a}_r$$

$$= \int_0^{2\pi} \int_0^{0.2} 141.664 \;\; \cos(10^6 t)\, d\emptyset\, dz$$

$$= 141.664 \cos(10^6 t) \;\; [\emptyset]_0^{2\pi} \; [z]_0^{0.2}$$

$$= 178.02 \cos(10^6 t) \; \frac{\mathrm{A}}{\mathrm{m}}^2 \qquad \textit{Answer}$$

Example 6.2 Calculate the value of K so that each of the following pairs of fields satisfies Maxwell's equations in a region at $\sigma = 0$ and $\rho_v = 0$.

i. $E = (k_x - 100\,t)\hat{a}_y\ V/m,\ \ H = (x + 20t)\hat{a}_z\ A/m$, if $\mu = 0.25\mu/\mathrm{m}$ and $\varepsilon = 0.01F/m$

ii. $D = 5\mathrm{x}\ \hat{a}_x - 2y\ \hat{a}_y + Kz\ \hat{a}_z\ \mu C/m^2$, $B = 2ay\ \mathrm{mT}$, $\mu = \mu_0$ and $\varepsilon = \varepsilon_0$

iii. $E = 60 \sin 10^6\ \ t \sin 0.01\ z\ \hat{a}_x\ \ V/m$, $H = 0.6 \cos 10^6\ \ t \cos 0.01\ z\ \hat{a}_y\ \ \mu = k$ and ε

solution. The differential form of Faraday's law is expressed in form as

$$\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$$

On taking the left-hand side of the Equ (i)

$$\vec{\nabla} \times \vec{E} = -\frac{d}{dz}\ (k_z - 100\ t)\ \hat{a}_x + \frac{d}{dx}(kx - 100)\hat{a}_z = k\hat{a}_z \qquad i$$

From the right side of Equ (i), we can write as

$$\frac{d\vec{B}}{dt} = \mu\ \frac{d\vec{H}}{dt} = 0.25\ (20)\hat{a}_z = 5\hat{a}_z \qquad ii$$

On substituting these values in Equ (i), we obtain as

$$k\hat{a}_z = -5\hat{a}_z$$

$$k = -5\mathrm{V/m^2}$$

(ii), Now we write differential form of Gauss's law for electric field as

$$\nabla \cdot D = \rho_v = 0$$

$$= \frac{dD_x}{dx} + \frac{dD_y}{dy} + \frac{dD_z}{dz} = 0$$

$$= (5 - 2 + K)(10^{-6})$$

$$= 0$$

$$K = -3 \times 10^{-6}\ \mathrm{C/m^3}$$

$$K = -3\mu\frac{\mathrm{C}}{\mathrm{m}}^3$$

(iii) Again, from the differential form of Faraday's law we can write as

$$\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt} \qquad iii$$

First take left hand side of Equ (iii) then we obtain as

$$\vec{\nabla} \times \vec{E} = \frac{d}{dz}\ (60\sin(10^6\ t)\sin(0.01\ z)\hat{a}_y - (60\sin(10^6 t)\ \sin(0.01\ z)\hat{a}_z$$

$$= 0.6\sin(10^6\ t)\cos(0.01z)\ \hat{a}_y$$

Now we take R. H. S. of Equ (iii)

$$\frac{d\vec{B}}{dt} = \mu\frac{d\vec{H}}{dt} = K\ (0.6)(-10^6) \times \sin(10^6\ t)\cos(0.01\ z)\hat{a}_y$$

$$= -K \times 6 \times 10^5 \sin(-10^6\ t)\cos(0.01z)\hat{a}_y$$

On substituting these values in Equ (iii), we obtain

$$= 0.6\sin(10^6\ t)\cos(0.01z)\hat{a}_y$$

$$= K \times 6 \times 10^5 \times \sin(10^6\ t)\cos(0.01z)\hat{a}_y$$

$$0.6 = K \times 6 \times 10^5$$

$$K = 10^{-6}\ \ \mathrm{H/m}$$

Example 6.3 Given the field $\vec{E} = E_m \sin(\mathrm{wt} - \beta\mathrm{z})\hat{a}_y$ in free space. Find out $\vec{D}, \frac{d\vec{B}}{dt}, \vec{H}$

Solution. First write the expression for $\vec{D}$ interms of E as,

$$\vec{D} = \varepsilon_0\ \vec{E}$$

$$\vec{D} = \varepsilon_0\ \ E_m\ (\omega t - \beta z)\hat{a}_y$$

The Maxwell equation is also expressed as,

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & E_m \sin(\omega t - \beta z) & 0 \end{vmatrix} = -\frac{\partial \vec{B}}{\partial t}$$

$$-\frac{\partial \vec{B}}{\partial t} = \beta \mathrm{E}_m \cos(\omega t - \beta z)\hat{a}_x$$

On taking integral of the above expression to obtain the value of $\vec{B}$ as,

$$\vec{B} = -\frac{\beta\, \vec{E}_m}{\omega} \sin(\omega t - \beta z)\hat{a}_x$$

Now, we obtain the value of $\vec{H}$ by the relation as,

$$\vec{B} = \mu_0 \vec{H}$$

$$\vec{H} = \frac{\vec{B}}{\mu_0}$$

Therefore, we obtain the value of $\vec{H}$ in the form as,

$$\vec{H} = -\frac{\vec{B}\vec{E}_m}{\omega \mu_0} \sin(\omega t - \beta z)\hat{a}_x$$

We note that $\vec{E}$ and $\vec{H}$ are mutually perpendicular

6.1 Derivation Conditions at Boundary Surface

Maxwell's equation which exits within a continuous medium are represented by differential equations.

Maxwell's equation used to determine what happen at the boundary surface between different media are represented by Integral equations.

At surface of Discontinuity (Fig. 6.2)

(a) Tangential component of $'\vec{E}'$ is continuous at surface

(b) Tangential components of $'\vec{H}'$ is continuous across a surface except at the surface of a "perfect conductor". At surface of perfect conductor, the tangential component of $'\vec{H}'$ is discontinuous by amount equal to surface current per unit width.

(c) Normal components of $'\vec{B}'$ is continuous at surface discontinuity

(d) Normal component of $'\vec{D}'$ is continuous if there is no surface charge density.

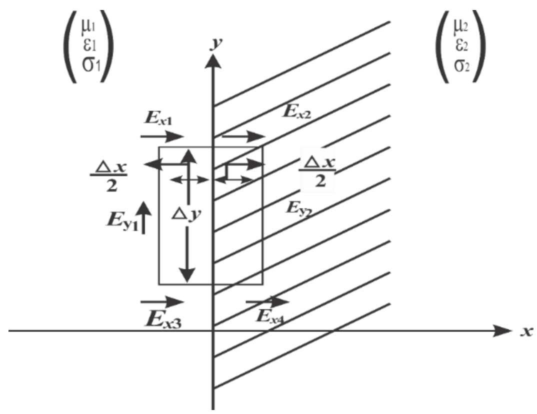


Figure 6.2 Surface of Discontinuity

Otherwise $\vec{D}$ is discontinuous by an amount equal to surface charge density.

Suppose: Surface of discontinuity is plane $x = 0$

Consider: Rectangle width $\Delta x, lenght\ \Delta y$

Two media (1) and (2)

$$\oint \vec{E} \cdot ds = -\int_S B^o \cdot da \tag{6.11}$$

Applying above for elemental rectangle:

$$E_{y_2}\,\Delta y - E_{y_2}\,\frac{\Delta x}{2} - E_{y_1}\frac{\Delta x}{2} - E_{y_1}\,\Delta y + E_{y_3}\frac{\Delta x}{2} + E_{y_4}\frac{\Delta x}{2} = -B^o\,\Delta x\,\Delta y$$

Where "B_z" is average magnetic flux density through rectangle $\Delta x \times \Delta y$.

Conditions: 1. Area of rectangle is made of approach zero by reducing width $'\Delta x'$ of rectangle, always keeping the surface of discontinuity between the sides of rectangle.

2. it is assumed that $\vec{B}$ is always finite, then RHS = 0

3. $\left(\frac{\Delta x}{2}\right)$ terms of LHS = 0

$\Longrightarrow$ $$E_{y_1}\,\Delta y - E_{y_1}\,\Delta y = 0$$

$\therefore$ $$E_{y_2} = E_{y_1}$$

Tangential component of $\vec{E}$ is continuous

Similarly, $$\oint \vec{H} \cdot ds = \int_S (D^o + J) \cdot d\vec{a} \qquad 6.12$$

$$H_{y_2}\,\Delta y - H_{y_2}\frac{\Delta x}{2} - \frac{H_{x_1}\Delta x}{2} H_{y_1}\,\Delta y + 4_{x_3}\frac{\Delta x}{2} + H_{x_4}\frac{\Delta x}{2}$$

$$= (D^o{}_z + J_z)\Delta x\,\Delta y$$

(A) Consider. (1) Rate of charge of electric displacement D^o and current density J are both considered to be finite.

$\Longrightarrow$ $$H_{y_2}\,\Delta y = H_{y_1}\,Dy = 0$$

Or $$H_{y_1} = H_{y_2}$$

Tangential component of $\vec{H}$ is continuous. 6.13

Note: "Current sheet" – as conductivity (of conductor) increases depth of penetration of electric field $(\vec{E})$ reduce a high frequency current will flow in this sheet near the surface $\Longrightarrow$ finite current per unit width $'J_s'$ ampere per meter.

$$\boxed{\underset{\Delta x \to 0}{1\text{imit}}\, \text{J}\Delta\text{x} = \text{J}_s} \;\; \text{A/m}$$

(B) Consider: (1) if current density $'J_z'$ becomes '*Infinite*' as $\Delta x = 0$, $\text{RHS} = 0$

So, $H_{y_2}\,Dy - H_{y_1}\,\Delta y = J_{sz}\,\Delta y$

$\Rightarrow$ $H_{y_1} = H_{y_2} - J_{sz}$ $(\therefore\ \ \Delta x = 0 \Rightarrow J_z = J_{sz})$ 6.14

"if electric field is zero within perfect conductor, the magnetic field must also be zero".

$\Rightarrow$ $H_{y_1} = -J_{sz}$

Current per unit width along surface of a perfect conductor is equal to magnetic field strength (H) just outside the surface.

Magnetic field and surface current will be parallel to surface, but perpendicular to each other.

$\Rightarrow$ $J_{sz} = \hat{n} \times H$ 6.15

"$\hat{n}$" is unit vector along outward normal to surface.

6.2 Conditions on Normal Components of $\vec{B}$ and $\vec{D}$

$$\oint_S D \cdot\, da = \int_V \rho\, dV \quad \text{(From Gauss's law)}$$

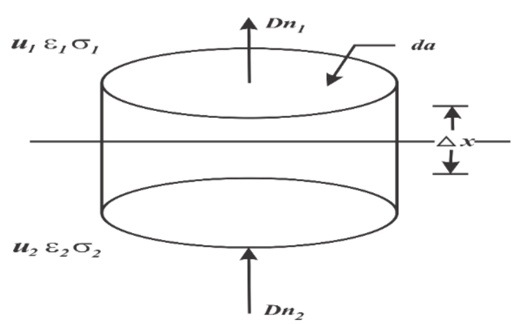


Figure 6.3 Pillbox

With reference to three surfaces of pill-box in Fig 6.3 we can rewrite above as

$$\Rightarrow \quad D_{n_1}\, da - D_{n_2} da + \psi_{\text{edge}} = \rho\, \Delta x\, da$$

Where $\psi_{\text{edge}}{}'$ is outward electric flux through the curved edge surface of pill box.

As $\Delta x \to 0,\ \psi_{\text{edge}} \to 0$

$\Longrightarrow$ $D_{n_1}\, da - D_{n_2}\, da = 0$

$\therefore$ $D_{n_1} = D_{n_2}$ 6.16

If there is no surface charge the normal component of $\vec{D}$ is continuous across the surface.

(A) Consider: $\rho\ \Delta\mathrm{x} = \rho\mathrm{s}$

In use of metallic surface, charge resides “on the surface”. If this layer of surface charge has a surface charge density ρ_s $\mathrm{c/m^2}$ the charge density $'\rho'$ of surface layer is given by

$$\rho = \frac{\rho_s}{\Delta x} \qquad 6.17$$

$\Longrightarrow$ $\lim\limits_{\Delta x \to 0} \rho \Delta x = \rho_s \Longrightarrow \Delta x = 0$, charge density approaches infinity.

$\therefore$ $D_{n_1} = \rho_s$ 6.18

Normal component of displacement density in dielectric’ is equal to surface charge density on conductor.

Similarly, in case of magnetic flux density $\vec{B}$, since there are no isolated “magnetic charge”.

$$B_{n_1} = B_{n_2} \qquad 6.19$$

Normal component of magnetic flux density is always continuous across boundary surface.

6.3 Exercise

1. An a.c voltage source $v = V_0 \sin \omega t$ is connected across a parallel plate capacitor C. verify that the displacement current in capacitor is same as conduction current wires.

2. A straight conductor of 0.4m lies on x-axis with one and at origin. The conductor is subjected to magnetic flux density $B = 0.08\hat{y}\ T$ and velocity $v =$

$2.5 \sin 10^3\ t\ \hat{z}\ m/s$. Calculate motional electric field intensity and emf induced in conductor.

3. The magnetic flux density in given is cylindrical coordinate by

$$\vec{B} = \begin{bmatrix} 4\ B_0 \cos \omega t\ \hat{z} & r < a \\ 0 & r < a \end{bmatrix}$$

Where B_o and ω are constants. Calculate the induced electric field at all values of r.

4. Fig. 6.4 is a rectangular loop moves toward origin at velocity $v = -200\bar{y}\ \text{m/s}$ in a magnetic field $\vec{B} = 0.75\ e^{-0.5y}\hat{z}\ T$. Find current at the instant coil sides are at $y = 0.5m$ and $0.6\ m$, if $R = 3\Omega$

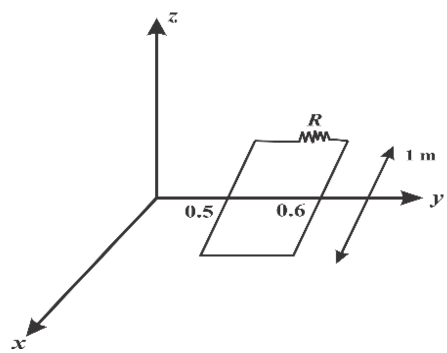


Figure 6.4

5. Two regions are separated by a surface $3x - 2y + 5z = 0$. Region 1 has permeability $\mu_1 = 2\mu_0$ and region 2 has $\mu_2 = 5\mu_0$. The point $P(2, 2, 2)$ lies in region 2. For a field $H_1 = 4\hat{x} + 6\hat{y} - 3\hat{z}\ \text{A/m}$. find H_2.

6. There exists a boundary between two magnetic materials at $Z = 0$, having permittivities $\mu_1 = 4\mu_0\ \text{H/m}$ for region 1 where $z > 0$ and $\mu_2 = 7\mu_0\ \text{H/m}$ for region 2 where $z < 0$. There exists a surface current of density $K = 60\ \hat{x}\ \text{A/m}$ at boundary $Z = 0$. For a field $B_1 = 2\hat{x} - 3\hat{y} + 2\hat{z}\ m\text{T}$ in region 1. Find value of flux density B_2 in region 2.

7. What are Maxwell's equations?
8. Write Maxwell's equation in differential and integral form

9. Write corresponding electromagnetic laws used for derivation of Maxwell's equation.
10. State all four Maxwell's equation
11. Rewrite Maxwell's equation for harmonically varying field.
12. Which theorem help to convert differential to integral form of Maxwell's equation.
13. What is physical significance of differential (Maxwell's) equation?
14. What is physical significance of integral (Maxwell's) equation?
15. At the surface of discontinuity, derive relation for
    a. Tangential component of $\vec{\varepsilon}$
    b. Tangential component of $\vec{H}$
    c. Normal component of$\vec{B}$
    d. Normal component of $\vec{D}$
16. Which Maxwell's equation is used for derivation of above continuity expression in Q.15 (a), (b), (c), (d)?

CHAPTER 7

ELECTROMAGNETIC WAVES ANALYSIS

7.0 Constitutive Relations of E.M Waves

Relations that concern the characteristics of medium in which field exists are called

Constitutive Relations, given below are true

$$D = \varepsilon E \qquad \text{(Permittivity)} \qquad (7.1)$$

$$B = \mu H \qquad \text{(Permeability)} \qquad (7.2)$$

$$\mathrm{J} = \sigma \mathrm{E} \qquad \text{(Conductivity)} \qquad (7.3)$$

Provided the medium is assumed to be homogenous, isotopic and source free.

i. In homogeneous medium, ε, μ and σ are constant throughout medium.
ii. In Isotopic medium, if ε is a scalar constant, $\vec{\mathrm{D}}$ and $\vec{\mathrm{E}}$ everywhere have the same direction.
iii. In source-free regions, there are no impressed voltages or currents. (No generators).

7.1 Solutions for Free Space Conditions

From free space condition, $\sigma = 0$ (source free) and $\rho = 0$ (no free charges

$$\therefore \quad \vec{\mathrm{J}} = \sigma \vec{\mathrm{E}} \qquad 7.4$$

$$\Longrightarrow \quad \mathrm{J} = 0$$

From Maxwell's equation:

$\nabla \times \vec{\mathrm{H}} = \dot{D} \rightarrow$ Taking time derivative on both sides we get

$$\frac{\partial}{\partial t}\left(\nabla \times \vec{\mathrm{H}}\right) = \nabla \times \vec{\mathrm{H}} = \vec{\mathrm{D}} = \varepsilon \vec{\mathrm{E}}$$

From Maxwell's equation

$\nabla \times \vec{\mathrm{E}} = -\dot{\vec{\mathrm{B}}}$ → Taking curl on both sides we get $\nabla \times \nabla \times \vec{\mathrm{E}} = -\mu \nabla \times \dot{\vec{\mathrm{H}}} = -\mu(\varepsilon \ddot{\vec{\mathrm{E}}})$ 7.5

From assumed conditions we infer,

$$\nabla \cdot \vec{\mathrm{D}} = 0 \qquad 7.6$$

$$\nabla \cdot \vec{\mathrm{B}} = 0 \qquad 7.7$$

From Maxwell's equation Using vector identity

$$\nabla \times \nabla \times \vec{\mathrm{E}} = \nabla(\nabla \cdot \vec{\mathrm{E}}) - \nabla^2 \vec{\mathrm{E}} = -\mu\varepsilon \ddot{\vec{\mathrm{E}}} \qquad \left[\nabla \cdot \vec{\mathrm{E}} = \frac{1}{\varepsilon} \nabla \cdot \vec{\mathrm{D}} = 0\right]$$

∴ $\nabla^2 \vec{\mathrm{E}} = \mu\varepsilon \ddot{E}$ Law that $\vec{\mathrm{E}}$ must obey 7.8

Parallelly $\nabla^2 \vec{\mathrm{H}} = \mu\varepsilon \ddot{H}$ "Wave equations" 7.9

$\vec{\mathrm{E}}$ and $\vec{\mathrm{H}}$ must satisfy "wave equations"

Example 7.1 Let us express $E_y\ (\mathrm{z,t}) = 200\cos(10^8\, t - 0.5z + 30^o)\ \mathrm{V/m}$ as phasor.

Solution: We first go to exponential notation.

$$\overrightarrow{E_y}\,(z,t) = R_e\left[200\, e^{j(10^8 t - 0.5z + 30^o)}\right]$$

And then drop R_e and suppress $e^{j(10^8\, t)}$, obtaining the phasor

$$E_{ys}\,(z) = 200 e^{-j0.5z + j30^o}$$

7.2 Uniform Plane-Wave Propagation

A uniform plane wave is a particular solution of Maxwell's equations with $\vec{\mathrm{E}}$ assuming the direction, same magnitude and same phase in infinite planes perpendicular to direction of propagation (same applies to $\vec{\mathrm{H}}$). A uniform plane wave does not exist in practice because a source infinite in extent would be required to create it, and practical waves sources are always finite in extent. But if we one for enough away from source, the wave front (surface of constant phase) becomes almost spherical and a very small portion of the surface of a giant sphere is very nearly a plane. The characteristics of uniform plane waves are particularly simple, and their study is of fundamental and theoretical, as well as practical, importance from equation $\vec{E} = A\sin(\omega t - \beta z)$ we can infer following characteristics:

1. It is time harmonics because we assumed time dependence of form $e^{j\omega t}$ to arrive at above equation.
2. The amplitude of wave is A has same units as E.
3. The phase (in radius) of wave depends on time 't' and space variables z, it is the form $(\omega t - \beta z)$.
4. The angular frequency ω is given in radians per second; β is phase constant or wave number is given in radians per meter.

7.2.1 Wave Equation

$\vec{E}$ and $\vec{H}$ are considered to be independent of two dimensions, y and z.

So, we can write $\nabla^2 E = \frac{\partial^2 E}{dx^2} \Longrightarrow \frac{\partial^2 E}{\partial x^2} = \mu\epsilon\,\frac{\partial^2 E}{\partial t^2}$ (Wave equation)

For uniform-wave propagation in the x direction, $\vec{E}$ may have components E_y and E_z

$$\frac{\partial^2 E_y}{dx^2} = \mu\varepsilon\frac{\partial^2 E_y}{dt^2} \qquad 7.10$$

For above differentials is equation,

General solution is $\vec{E} = f_1\,(x - v_0 t) + f_2\,(x + v_0 t)$ 7.11

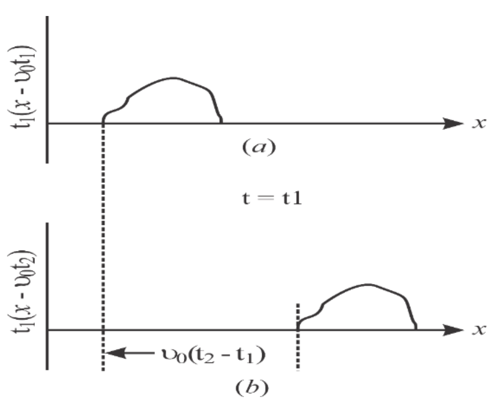


Figure 7.1 Uniform Plane Wave

If a physical phenomenon that occurs at one place at a given time is reproduced at other places at later times, time delay being proportional to the space separation from first location, then group of phenomena constitute a wave as in Fig 7.1 (a).

For fixed time t_2 . $v_0 t_1 and\ v_0 t_2$ are constant

f_1 and f_2 are functions of $'x'$ only

Displacement of curve to right $= v_0\ (t_2 - t_1)$

Phenomenon has travelled in the positive $'x'$ direction with velocityv_0. $\left(f_1\ (x - v_0\ t_1)\right)$ Fig 7.1(b)).

$f_2\ (x - v_0 t)$ wave traveling in –ve x direction

General solution of wave equation consists of two waves, one travelling to right (away from source) and other travelling to left (back from source).

If there is no reflecting surface present to reflect the wave back to source solution is given by

$$\vec{E} = f_1(x - v_0 t)$$

Example 7.2 Given the complex amplitude of the electric field of a uniform plane wave, $\vec{E}_o = 100\hat{a}_x + 20 < 30^o\ \hat{a}_y$ V/m construct the phasor and real instantaneous fields if the wave is known to propagate in forward z direction in free space and has frequency of 10 MHz.

Solution: We begin by constructing the general phasor expression:

$$\vec{E}_s\ (z) = \left[100\hat{a}_x + 20e^{j30^o}\hat{a}_y\right]e^{-jk_0\, z}$$

Where

$$\beta = \frac{\omega}{V} = 0.21$$

The real instantaneous form is them found through rule

$$\vec{E}_{(z,t)} = Re\left[100e^{-j0.21z}\ e^{j\ 2\pi\times 10^7}\hat{a}_x + 20\ e^{j\ 30^o}e^{-j0.21z}e^{j\ 2\pi\times 10^7}\hat{a}_y\right]$$

$$= 100\cos(2\pi\times 10^7\ t - 0.21z)\ \hat{a}_x + 20\cos\ (2\pi\times 10^7 t - 0.21z + 30^o)\hat{a}_y$$

Proceeding further for uniform plane wave, we get

$$\vec{E} = f_1(x - v_0 t)$$ (From Equ 7.12)

Equ 7.12 says that $\vec{E} \neq f(y, z)$

$$\vec{E} = f(x, t)$$

Such a wave is called uniform plane wave.

$$\text{Rewriting wave equation,}\frac{\partial^2 E}{\partial x^2} = \frac{\mu\varepsilon\partial^2 E}{\partial t^2}$$

In terms of components of $\vec{E}$, $(i.e., E_x, E_y \text{ and } E_z$ along X, Y and Z axis respectively)

$$\frac{\partial^2 E_x}{\partial x^2} = \frac{\partial^2 E}{\partial t^2}\ \mu\varepsilon \qquad 7.13$$

$$\frac{\partial^2 E_y}{\partial x^2} = \mu\varepsilon\frac{\partial^2 E}{\partial t^2} \qquad 7.14$$

$$\frac{\partial^2 E_z}{\partial x^2} = \mu\varepsilon\frac{\partial^2 E}{\partial t^2} \qquad 7.15$$

We know that $\nabla . \vec{E} = \frac{1}{\varepsilon}\left(\nabla . \vec{D}\right) = 0$ (From Gauss's law)

$$\Longrightarrow \qquad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \qquad 7.16$$

As $\vec{E} = f(x, t)$

$\therefore \qquad \frac{\partial E_x}{\partial x} = 0 \Longrightarrow$ there is variations of E_x in x direction

$$\frac{\partial^2 E_x}{\partial x^2} = 0 \Longrightarrow E_x = 0/\text{Constant in time/increasing with time}$$

Important. Uniform plane wave progressing in x direction has no x component of $\vec{\mathrm{E}}$ and $\vec{\mathrm{H}}$

∴ Uniform plane electromagnetic waves are transverse and have component of $\vec{E}$ and $\vec{H}$ only in directions perpendicular to the direction of propagation.

7.3 Relation Between $\vec{E}$ and $\vec{H}$ in a Uniform Plane Wave

We know that by retaining components variations along x direction and keeping $E_x = 0$, we get

$$\nabla \times E = -\frac{\partial E_z}{\partial x}\hat{y} + \frac{\partial E_y}{\partial x}\hat{z}$$
$$\nabla \times H = -\frac{\partial H_z}{\partial x}\hat{y} + \frac{\partial E_y}{\partial x}\hat{z}$$

Expanding Equ (7.17) and Equ (7.18), we get

(I) $$-\frac{\partial H_z}{\partial x}\hat{y} + \frac{\partial H_y}{\partial x}\hat{z} = \varepsilon\left(\frac{\partial E_y}{\partial t}\hat{y} + \frac{\partial E_z}{\partial t}\hat{z}\right)\left(as\ \vec{\nabla} \times \vec{H} = \frac{\overrightarrow{dD}}{\overrightarrow{dt}}\right) \qquad 7.19$$

(II) $$-\frac{\partial E_z}{\partial x}\hat{y} + \frac{\partial E}{\partial x}\hat{z} = -\mu\left(\frac{\partial H_y}{\partial t}\hat{y} + \frac{\partial H_z}{\partial t}\hat{z}\right)\left(as\ \vec{\nabla} \times \vec{E} = \frac{\overrightarrow{dB}}{\overrightarrow{dt}}\right) \qquad 7.20$$

Therefore, by comparing coefficients in Equs (7.19), (7.20) we get

$$\frac{-\partial H_z}{\partial x} = \varepsilon\frac{\partial E_y}{\partial t} \qquad 7.21$$

$$\frac{\partial E_z}{\partial x} = \frac{\mu H_y}{\partial t} \qquad 7.22$$

$$\frac{\partial H_y}{\partial x} = \frac{\varepsilon \partial H_z}{\partial t} \qquad 7.23$$

$$\frac{\partial E_y}{\partial x} = -\mu\frac{\partial H_z}{\partial t} \qquad 7.24$$

$$E_y = f_1(x - v_0 t);\ v_0 = \frac{1}{\sqrt{\mu\varepsilon}} \qquad \text{(where } v_0 \text{ is speed of light)} \qquad 7.25$$

Differentiating w.r.t time we get

$$\frac{\partial E_y}{\partial t} = \frac{\partial f_1}{\partial(x - v_0 t)} \frac{\partial(x - v_0 t)}{\partial t} = -v_0 \frac{\partial f_1}{\partial(x - v_0 t)}$$

$$\frac{\partial E_y}{\partial t} = f_1'(x - v_0 t) \frac{\partial(x - v_0 t)}{\partial t} = -v_0 f_1'(x - v_0 t)$$

Where

$$f_1(x - v_0 t) = \frac{\partial f_1\ (x - v_0 t)}{\partial(x - v_0 t)}\ of\ f_1'$$

⟹ Putting Equ (7.25) in Equ (7.21) we get

⟹ $$-\frac{\partial H_z}{\partial x} = \varepsilon \frac{\partial E_y}{\partial t} = E\left(-v_0 f_1'(x - v_0 t)\right)$$

⟹ $$\boxed{\frac{\partial H_z}{\partial x} = v_0 \varepsilon \mathrm{f}_1'} \qquad 2.26$$

Then after integration w.r.t. x,

$$H_z = \sqrt{\frac{\varepsilon}{\mu}} \int f_1' dx + C \qquad 7.27$$

Where C is constant of integration

Now, $$\frac{\partial f_1}{\partial x} = f_1' \frac{\partial(x - v_0 t)}{\partial x} = f_1'$$

Hence, $$H_z = \sqrt{\frac{\varepsilon}{\mu}} \int \frac{df_1'}{\partial x} dx + C = \sqrt{\frac{\varepsilon}{\mu}} \int f_1' + C$$

$$= \sqrt{\frac{\varepsilon}{\mu}} \int f_1' dx + C$$

[C indicates that field independent of x would be present]

$\Longrightarrow$ $$\boxed{H_z = \sqrt{\frac{\varepsilon}{\mu}}\, E_y} \qquad 7.28$$

Parallelly $$\boxed{\frac{E_z}{H_y} = -\sqrt{\frac{\mu}{\varepsilon}}} \qquad 7.29$$

And $$\boxed{\frac{E_y}{H_z} = -\sqrt{\frac{\mu}{\varepsilon}}} \qquad 7.30$$

Also, $$\vec{E} = \sqrt{E_y^2 + E_z^2} \ \ and \ \ \vec{H} = \sqrt{H_z^2 + H_y^2}$$

$\vec{E}$ and $\vec{H}$ are total electric and magnetic field strength

$$\frac{\mathrm{E}}{H} = \sqrt{\frac{\mu}{\varepsilon}} \qquad 7.31$$

Equ 7.31 is called Characteristics impedance/intrinsic impedance.

In a travelling plane, EM wave there is a definite ratio between amplitude of $\vec{E}$ and $\vec{H}$ and that this ratio is equal to square root of ratio of permeability to dielectric constant of medium.

$$\frac{E}{H} = \sqrt{\frac{4\pi \times 10^{-7}}{{}^{1}/_{36}\,\pi \times 10^{9}}} = 377\Omega \ \ or \ 120\pi \qquad 7.32$$

Example 7.3. Show that $\vec{E}_y = E_0 \sin(\omega t - \beta z) and \ \vec{H}_x = -\frac{\beta E_0}{\mu_0 \omega} \sin(\omega t - \beta z)$ travels with velocity of light in free space. Also find $\frac{E}{H}$ ratio.

Solution: $\frac{E}{H} = \frac{\mu_0 \omega}{\beta}$ where ω/β is velocity of light

$$\frac{E}{H} = \mu_0 \times \frac{1}{\sqrt{\mu\varepsilon}}$$

$$\frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \eta = 377\Omega$$

From Maxwell's equations

$$\vec{\nabla} \times \vec{E} = -\mu \frac{d\vec{H}}{dt} = -\mu \times \frac{\beta \varepsilon_0}{\omega \mu_0} \cos(\omega t - \beta z) \times \omega$$

$$= \frac{\mu_0 \beta E_0}{\omega \mu_0} \cos(\omega t - \beta z) \times \omega$$

$$\vec{\nabla} \times \vec{E} = \beta \varepsilon_0 \cos(\omega t - \beta z) \times \omega$$

Expanding L. H. S.

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ 0 & \varepsilon_y & 0 \end{vmatrix} = \hat{\imath}\left(\frac{-d\varepsilon_y}{dx}\right) - \hat{\jmath}(0) + \hat{k}\left(\frac{d\varepsilon_y}{dx}(0) - 0\right)$$

$$= \hat{\imath}\, E_0 \cos(\omega t - \beta z) \times \beta$$

$$= \beta E_0 \cos(\omega t - \beta z)$$

Example 7.4. Find magnetic field intensity for a TEM wave with electric field intensity of $4\ \mu V/m$ in air, lossless dielectric with $\varepsilon_r = 5$.

Solution: $\frac{E}{H} = \eta = 377\Omega$ in free space or

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

$$H = \frac{E}{\eta} = \frac{E}{377} = 10.6 \times 10^{-9}\ A/m$$

Lossless dielectric with $\varepsilon_r = 5$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{377}{\sqrt{5}}\ 168.6$$

$$H = \frac{E}{\eta} = \frac{1}{168.6} \times 4 \times 10^{-6} = 23.72 \times 10^{-9}\ \mathrm{A/m}$$

7.4 Wave Equation for a Conducting Medium

In a conducting medium, $\vec{J} = \sigma\vec{E}$ i.e., conduction current is present. Therefore, from Maxwell's equation:

$$\nabla \times H = \varepsilon\dot{E} + J \quad (I) \quad (\vec{J} = \sigma\vec{E}) \qquad 7.33$$

$$\nabla \times \vec{E} = -\mu\dot{H} \qquad 7.34$$

(I) $\nabla \times \vec{H} = -\varepsilon\dot{E} + \sigma E$ (as $J = \sigma E$)

(II) $\nabla \times \nabla \times \vec{E} = \mu\nabla \times \vec{H} = -\mu(\varepsilon\vec{E} + \sigma\vec{E}) = -\mu(\varepsilon\vec{E} + \sigma\vec{E})$

$$= -\mu\varepsilon\vec{E} + \mu\sigma\vec{E}$$

(By taking curl of Equ (7.34) and substituting Equ (7.33)

$\Longrightarrow \quad \nabla(\nabla.\vec{E}) - \nabla^2\vec{E} = -\mu\varepsilon\vec{E} - \mu\sigma\vec{E}$

(Using identity on L.H.S) $\quad (\nabla \times \nabla \times E = \nabla(\nabla \cdot E) - \nabla^2 E)$

$\Longrightarrow \quad \nabla\frac{1}{\varepsilon}(\nabla \cdot \vec{D}) - \nabla^2\vec{E} = -\mu\varepsilon\vec{E} - \mu\alpha\vec{E} \qquad (D = \varepsilon E)$

$\Longrightarrow \quad \nabla - \nabla^2\vec{E} = -\mu\varepsilon\vec{E} - \mu\alpha\vec{E}$ (E in conductor is zero)

$\Longrightarrow \quad \boxed{\nabla^2 E = -\mu\varepsilon\ddot{H} + \mu\alpha\dot{E}}$ wave equation for $\vec{E}$ 7.35

Parallelly $\nabla \times \nabla \times \vec{H} = \varepsilon\nabla \times \dot{E} + \sigma\nabla \times E = \varepsilon\nabla \times \dot{E} - \mu\alpha\dot{H}$

(Taking curl of 7.33 and substituting 7.34)

$\Longrightarrow \quad \nabla(\nabla \cdot \vec{H}) - \nabla^2\vec{H} = -\mu\varepsilon\vec{H} - \mu\alpha\vec{H}$ (Using identity on L.H.S)

$\boxed{\nabla^2 H = \mu\varepsilon\ddot{H} + \mu\alpha\dot{H}}$ wave equation for $\vec{H}$ 7.36

7.5 Sinusoidal Time Variations

Any time/periodic variations can always be analyzed in terms of sinusoidal variations with fundamental and harmonic frequencies

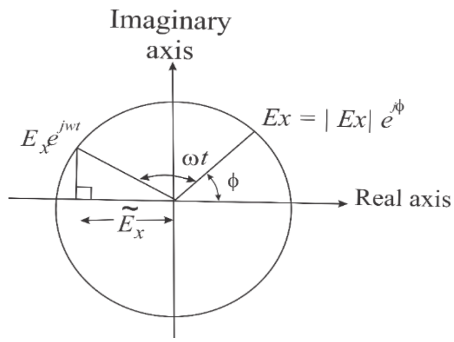


Figure 7.2 Sinusoidal Time Variation

∴ $E = E_0 \cos \omega t$

$E = E_0 \sin \omega t$

Or $E(r,t) = Re\{E(r)e^{j\omega t}\}$ 7.37

$(\sim)$ time varying quantity to distinguish it from phasor quantity.

$\tilde{E}_x\,(r,t) = Re\,\{E_x(r)e^{j\omega t}\}$ 7.38

$E_x(r)$ is a complex number (represented as point r in complex plane)

Multiplication by $e^{j\omega t}$ results in rotation through angle $'\omega t'$ measured from $'\phi'$ shown in Fig 7.2

As time progress, the point $E_x e^{j\omega t}$ traces out a circle with center at origin.

$\tilde{E}_x = Re\,\{|E_x|\; e^{j\emptyset}\; e^{j\omega t}\} = |E_x| \cos(\omega t + \emptyset)$ 7.39

E_x is Peak value

$[\sqrt{2}\;\; |E_x| \rightarrow \text{rms value}]$

Maxwell's equations using phasor notation.

In time varying form,

$$\nabla \times \widetilde{H} = \frac{\partial \breve{D}}{dt} + \tilde{J}$$

$$\Longrightarrow \qquad \nabla \times Re\left(\vec{H}\, e^{j\omega t}\right) = \frac{\partial}{dt}\, Re\left\{\vec{D} e^{j\omega t}\right\} + Re\left\{\mathrm{J}e^{\mathrm{j\omega t}}\right\}$$

{for sinusoidal steady state}

$$\Longrightarrow \qquad Re\left\{\left(\nabla \times \vec{H} - j\omega D - J\right)e^{j\omega t}\right\} = 0$$

$$\Longrightarrow \qquad \boxed{\nabla \times \vec{H} = j\omega\vec{D} + \vec{J}}$$ required differential equation in phasor form

It means that $\boxed{\frac{d}{dt} = jw,}$ Rewritting Maxwell's equations we get,

$$\therefore \quad \nabla \times \vec{H} = j\omega\vec{D} + \vec{J} \qquad \oint \vec{H} \cdot \overrightarrow{ds} = \int (j\omega D + J) \cdot \overrightarrow{da} \qquad 7.40$$

$$\nabla \times \vec{E} = -\omega j\vec{B} \qquad \oint \vec{E} \cdot \overrightarrow{ds} = -\int j\omega B \cdot da \qquad 7.41$$

$$\nabla \times \vec{D} = \rho \qquad \oint \vec{D} \cdot \overrightarrow{da} = \int \rho\, dV \qquad 7.42$$

$$\Longrightarrow \quad \nabla \times \vec{B} = 0 \qquad \oint \vec{B} \cdot \overrightarrow{da} = 0 \qquad 7.43$$

And $\nabla . \vec{\mathrm{J}} = -j\omega\rho$ or $\oint \underset{J}{\Rightarrow} . \underset{da}{\rightarrow} = -\int j\omega\rho\, dV$ 7.44

[Equation of Continuity]

If $\frac{d}{dt} = j\omega$, then $\frac{d^2}{dt^2} = -w^2$, putting in

"Helmholtz equation": $\nabla^2\vec{E} = -\omega^2\varepsilon E$

In a conducting medium: $\nabla^2 E + (\omega^2\mu\varepsilon - j\omega\mu\sigma)E = 0$ 7.45

Example. 7.5. In free space, $\vec{E}\,(z,t) = 10^3 \sin(\omega t - \beta z)\hat{a}_y$ (V/m). Obtain $\vec{H}\,(z,t)$ and determine the propagation constant γ.Given that the frequency is 90MHz.

Solution: $\omega t - \beta z$ phase (shows that the direction of propagation is + z)

Since $\vec{E} \times \vec{H}$ must also be in the + z direction,

$\vec{H}$ most have the direction $-\hat{a}_x$.

Consequently

$$\frac{E_y}{-H_x} = \eta = 120\pi\Omega$$

Or

$$H_x = \frac{-10^3}{120\pi}\sin(\omega t - \beta z)\ (\mathrm{A/m})$$

$$\gamma = \sqrt{j\omega\mu\ (\sigma + j\omega\varepsilon)}$$

In free space, $\sigma = 0$, so that

$$\gamma = j\omega\sqrt{\mu_0\varepsilon_0}$$

$$\gamma = j\frac{2\pi f}{C}$$

$$\omega = 2\pi f$$

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

$$\gamma = j\ \frac{2\pi\ (90 \times 10^6)}{3 \times 10^8} = j(1.884)\mathrm{m}^{-1}$$

$$\alpha + j\beta = j(1.884)$$

Hence, attenuation factor $\alpha = 0$

Phase shift constant $\beta = 1.884$

7.5.1 Time Harmonic Fields

A time-harmonic field is one that varies periodically or sinusoidally with time.

Let first discuss phasor representation of vector fields. A phasor is a complex number that contains the amplitude and phase of a sinusoidal oscillation. A phasor z as a complex number, is given by

$$z = x + j\,y = r < \emptyset \qquad \text{(rectangular form)}$$

Or $$z = re^{j\emptyset} = r\cos\emptyset + x_y \sin\emptyset \qquad \text{(Polar form)}$$

Where $j = \sqrt{-1}$ and x is neat part and $y$is imaginary part ω,

$$\mathrm{r} = |\mathrm{z}| = \sqrt{x^2 + y^2}$$

and $\emptyset$ is phase of $z, \emptyset = \tan^{-1} y/x$

given complex numbers,

$$z = x + jy = r < \emptyset, z_1 = x_1 + jy_1 = r_1 < \emptyset_1$$

And $$z_2 = x_2 + jy_2 = r_2 < \emptyset_2$$

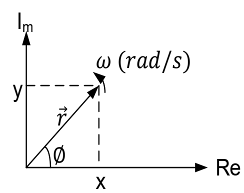


Figure 7.3 Representation of phasor $z = x + jy = r < \emptyset$

The following properties can be inferred as

1. Addition: $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$
2. Subtraction: $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$
3. $z_1 - z_2 = r_1 r_2\ \emptyset_1 - \emptyset_2$
4. $\frac{z_1}{z_2} = \frac{r_1}{r_2}\ \emptyset_1 - \emptyset_2$
5. Square root: $\sqrt{z} = \sqrt{r} < \emptyset/2$
6. Complex conjugate: $z^{\emptyset} = x - jy = r < -\emptyset = re^{j\emptyset}$

Let us now introduce time element,

$$\emptyset = \omega t + \theta$$

Where θ may be function of time or space coordinates or a constant.

$$re^{j\emptyset} = re^{j\emptyset}.re^{j\omega t}$$

$$\therefore \qquad Re\left(re^{j\emptyset}\right) = r\sin(\omega t + \theta$$

$$I_m\left(re^{j\emptyset}\right) = r\sin(\omega t + \theta)$$

Let us consider a complex sinusoidal current $I(t) as\ I_o e^{j\theta}.e^{j\omega t}$, where

$I_o = I_o e^{j\theta} = I_o < \theta$ $\rightarrow$ Phasor current

And $I(t) = I_o\cos(\omega t + \theta)$ has instantaneous form

$$I(t) = Re\left(I_s e^{j\omega t}\right)$$

In general, phasor can be scalar or vector. If a vector $\vec{A}(x, y, z, t)$ is time harmonic field, then phasor form is

$$\vec{A} = Re\left(Ase^{j\omega t}\right)$$

$$\Longrightarrow \qquad \frac{d\vec{A}}{dt} = \frac{d}{dt}\,Re\left(A_s e^{j\omega t}\right) = R_e\left(j\omega A_s e^{j\omega t}\right)$$

Showing that taking the time derivative of instantaneous quantity is equivalent to multiplying states that the algebraic sum of all magnetic fluxes flowing of out of a junction in a magnetic circuit is zero.

7.6 Polarization

Polarization refers to time varying behaviour of electric field strength vector at some fixed point in space as shown in Fig 7.3.

1. If $\tilde{E}_y = 0$ and only $\tilde{E}_x$ is present
   POLARIZED IN 'X' DIRECTION
2. If $\tilde{E}_x = 0$ and only $\tilde{E}_y$ is present
   POLARIZED IN 'Y' DIRECTION
3. If both $\tilde{E}_x\ and\ \tilde{E}_y$ are present and in phase
   LINEARLY POLARIZED

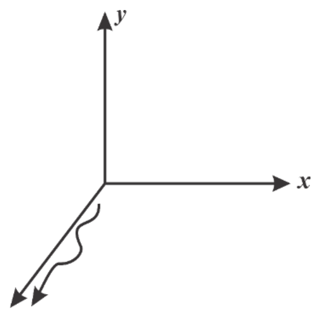


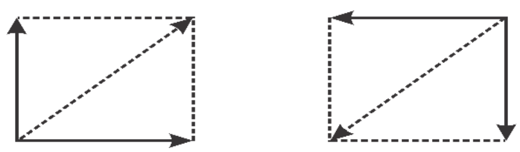
Figure 7.4 Polarized in z dir^n

Direction is dependent on relative magnitude

Angle is what the direction makes with x-axis $\tan^{-1}\left(\frac{E_y}{E_x}\right)$

(constant with time)

Uniform plane wave



4. If $\tilde{E}_x \, and \, \tilde{E}_y$ are not in phase, they reach their maximum values at different instant time.

In Fig 7.4, locus of END POINT $\Longrightarrow$ CIRCLE

$\Longrightarrow$ ELLIPTICALLY POLARIZED

$\Downarrow$ if $\tilde{E}_y = \tilde{E}_y \Longrightarrow < 90^o$

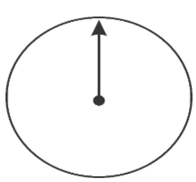


Fig 7.5 (a) ellipse

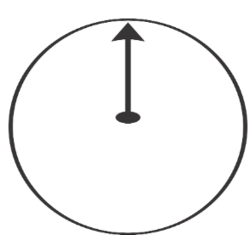


Fig 7.5 (b) Circle

In Fig 7.5, LOCUS of END POINT $\Longrightarrow$ CIRCLE

$\Longrightarrow$ CIRCULARLY POLARIZED

The electric field of a uniform plane wave travelling in 'Z' direction

$$E(z) = E_0 e^{-j\beta z} \tag{7.46}$$

In time varying form as $\tilde{E}(z,t) = Re\left\{E_0 e^{-j\beta z}\, e^{j\omega t}\right\}$

$$\tilde{E}(0,t) = Re\left\{(E_r + jE_i)\, e^{j\omega t}\right\}$$

$$\tilde{E}(0,t) = E_r \cos\omega t - E_i \sin\omega t \tag{7.47}$$

$\tilde{E}$ not only charges its magnitude but also changes its direction as time varies.

7.6.1 Circular Polarization

If y component leads x by 90^o

$$E_0 = (\hat{x} + j\hat{y})E_a \;\Longrightarrow\; \hat{E}(0,t) = (\hat{x}\cos\omega t - \hat{y}\sin\omega t)\, E_a$$

$$\left.\begin{aligned}\hat{E}_x &= E_a \cos\omega t\\ \hat{E}_y &= -E_{as}\sin\omega t\end{aligned}\right\} \quad \boxed{\hat{E}_x^2 + \hat{E}_y^2 = E_a^2}$$

7.48

$\therefore$ Endpoint of $\hat{E}(0,t)$ traces out a circle of radius $'E_a'$

$$E_0 = (\hat{x} - j\hat{y})E_a$$

7.6.2 Elliptical polarization

$$E_0 = \hat{x}A + j\hat{y}B \;\Longrightarrow\; \hat{E}(0,t) = \hat{x}A\cos\omega t - \hat{y}B\sin\omega t$$

$$\left.\begin{aligned}\hat{E}_x &= A\cos\omega t\\ \hat{E}_y &= -B\sin\omega t\end{aligned}\right\} \quad \frac{\tilde{E}_x{}^2}{A^2} + \frac{\tilde{E}_y{}^2}{B^2} = 1$$

7.49

∴ Endpoint of $\tilde{E}\ (0,t)$ traces out of ellipse

7.6.3 Resolution in Different Polarization

The instantaneous field of a plane-wave travelling in negative z direction may be given as

$$\vec{E}\ (z,t) = E_1\ (z,t)\hat{x} + E_2\ (z,t)\hat{y} \qquad 7.50$$

In complex notation, Equ (7.50) is written as

$$E_1\ (z,t) = R_e\left[E_1 e^{-j(\omega t+\beta z)}\right]$$

$$\Longrightarrow \qquad \boxed{E_1(z,t) = E_1\cos(\omega t + \beta z + \delta_x)} \qquad 7.51$$

||ly

$$\boxed{E_2(z,t) = E_2\cos\left(\omega t + \beta z + \delta_y\right)} \qquad 7.52$$

Where $E_1 \rightarrow$ Maximum amplitude of x component

$E_2 \rightarrow$ Maximum amplitude of y component

$\delta \rightarrow$ time phase

(I) Linear Polarization

For wave to have linear polarization, the time phase difference between two components must be

$$\delta = \delta_y - \delta_x = n\,\pi$$

Where $n = 0, 1, 2, 3, 4\ ....$

A linearly polarized wave can be resolved into a light hand circularly polarized wave and left-hand circularly polarized wave of equal amplitude.

To prove the above statement let's consider a linearly polarized plane wave propagating in z direction, we can assume, with no loss of generality, that $\vec{E}$ is polarized in x direction. In phasor notation.

$$\vec{E}(z) = E_0 e^{-j\beta z}\,\hat{x}$$

But this can be written as

$$\vec{E}\,(z) = \vec{E}_{rc}(z) + \vec{E}_{lc}\,(z)$$

Where $\vec{E}_{rc}(z) = \frac{E_0}{2}\,(\hat{x} - j\hat{y})e^{-j\beta z}$

And $\vec{E}_{lc}(z) = \frac{E_0}{2}\,(\hat{x} - j\hat{y})e^{-j\beta z}$

Where $\vec{E}_{rc}$ is RHCP wave and

$\vec{E}_{lc}$ is LHCP wave

Having amplitude $\frac{E_0}{2}$

The converse statement that sun of two oppositely rotating circularly polarized waves of equal amplitude is a linearly polarized wave is true.

(II) Circular Polarization

This can be achieved when magnitudes of two components are same and time phase difference between them is odd multiples of $\pi/2$ i.e.,

$$|\vec{E}_1| = |\vec{E}_2| \qquad \text{or } E_1 = E_2$$

$$\delta = \delta_y - \delta_x = \begin{cases} +\left(\frac{1}{2} + 2n\right)\pi \ldots \ldots \text{in CW} \\ -\left(\frac{1}{2} + 2n\right)\pi \ldots \ldots \text{in CCW} \end{cases}$$

Where $n = 0, 1, 2 \ldots ..$

In case direction of propagation is reversed (i.e., $+\,z$ direction) the phase in above two for CW and CCW can be interchanged.

(III) Elliptical Polarization

This is achieved only when the time difference between the two components is odd multiple of $\pi/2$ and their magnitude are not the same or when the time difference between two components is not equal to multiples of $\pi/2$ (irrespective of magnitudes) i.e.

$$|\vec{E}_1| \neq |\vec{E}_2| \text{ or } E_1 \neq E_2$$

$$\delta = \delta_y - \delta_x = \begin{cases} +\left(\frac{1}{2} + 2n\right)\pi \ldots\ldots \text{in CW} \\ -\left(\frac{1}{2} + 2n\right)\pi \ldots\ldots \text{in CCW} \end{cases}$$

Where $n = 0, 1, 2 \ldots\ldots$

Or

$$\delta = \delta_y - \delta_x \neq \pm\frac{\dot{n}\pi}{2}\begin{cases} > 0 & \text{for CW} \\ < 0 & \text{for CCW} \end{cases}$$

Here, the curved traced at a given position as a function of time is a tilt, ellipse for elliptical polarization ratio of major to minor axis is called axial ratio (AR) i.e

$$AR = \frac{\text{major axis}}{\text{minor axis}} = \frac{OP}{OR})$$

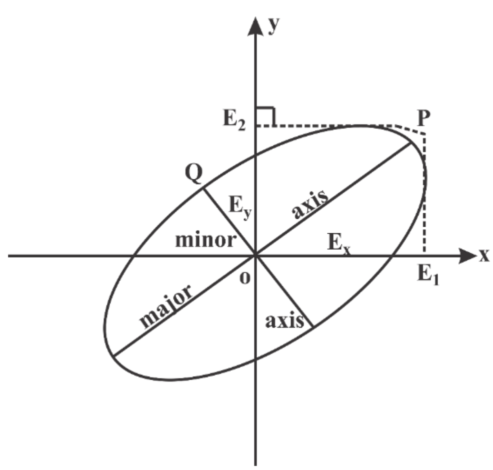


Fig 7.6. Polarization ellipse at an angle.

AR lies between 1 to ∞

$$OP = \sqrt{\frac{1}{2}\left\{E_1^2 + E_2^2 + \sqrt{E_1^4 + E_2^4 + 2E_1^2E_2^2\ \cos 2\delta}\right\}}$$

And

$$OQ = \sqrt{\frac{1}{2}\left\{E_1^2 + E_2^2 + \sqrt{E_1^4 + E_2^4 + 2E_1^2E_2^2}\cos 2\delta\right\}}$$

And tilt of ellipse is (w.r.t. y-axis)

$$\tau = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left\{\frac{2E_1E_2\cos\delta}{E_1^2 - E_2^2}\right\}$$

When ellipse is aligned with principal axis then $\tau = \frac{n\pi}{2}$ $(n = 0, 1, 2, \dots.)$

7.7 Mathematical Analysis

Let $E_x =$ instantaneous electric field of horizontal polarized wave (component)

$E_y =$ instantaneous electric field component of vertically polarized wave (component)

Then, electric field (Fig 7.6) components as a function of time and distance in x and y directions are:

$$E_x = E_1 \sin(\omega t - \beta z) \qquad 7.54$$

$$E_y = E_2 \sin(\omega t - \beta z + \delta) \qquad 7.55$$

δ is time phase angle by which E_y loads E_x

Now $\vec{E} = E_x\,\hat{x} + E_y\,\hat{y} = E_1\sin(\omega t - \beta z)\hat{x} + E_2\sin(\omega t - \beta z + \delta)\hat{y}$

At $z = 0$ (eqn 7.54)

$$E_x = E_1 \sin\omega t\ \ and\ E_y = E_2\sin(\omega t + \delta)$$

Or

$$E_y = E_2(\sin \omega t \cos \delta + \cos \ \omega t \sin \delta$$

So, $\frac{E_x}{E_1} = \sin \omega t$

$$\cos \omega t = \sqrt{1 - \sin^{-1} \omega t} = \sqrt{1 - \left(\frac{E_x}{E_1}\right)^2}$$

$\Longrightarrow$ $\boxed{\frac{E_y}{E_2} = \sin \omega t \cos \delta + \cos \omega t \sin \delta}$

Putting values of $\cos \omega t \sin \omega t)$ we get

$$\frac{E_y}{E_2} = \frac{E_x}{E_1} \cos \delta + \sqrt{1 - \left(\frac{E_x}{E_1}\right)^2} \times \sin \delta$$

Rearranging and squaring

$$\left(\frac{E_y}{E_1}\right)^2 + \left(\frac{E_x}{E_1}\right)^2 \cos^2 \delta - \frac{2E_x E_y \cos \delta}{E_1 E_2} = \sin^2 \delta - \left(\frac{E_x}{E_1}\right)^2 \sin^2 \delta$$

As $\sin^2 \delta + \cos^2 \delta = 1$

$$\left(\frac{E_y}{E_2}\right)^2 - \frac{2E_x E_y \cos \delta}{E_1 E_2} + \left(\frac{E_x}{E_1}\right)^2 = \sin^2 \delta$$

Dividing throughout by $\sin^2 \delta$, we get

$$\frac{E_y^2}{E_1^2 \sin^2 \delta} - \frac{2E_x E_y \cos \delta}{E_1 E_2 \sin^2 \delta} + \frac{E_x^2}{E_1^2 \ \sin^2 \delta} = 1$$

Or $\boxed{\alpha\, E_x^2 - bE_x E_y + cE_y^2 = 1}$ 7.56

Where $\alpha = \frac{1}{E_1^2 \ \sin^2 \delta}$

$$b = \frac{2 \cos \delta}{E_1 E_2 \sin^2 \delta}$$

$$c = \frac{1}{E_2^2\ \sin^2\delta}$$

OP →semi-major axis, OQ →semi-minor axis

(I) Let E_y be in phase or 180^o out of phase with E_x then $\delta = KA, K = 0, 1, 2$.

$$\left(\frac{E_x}{E_y}\right)^2 + \frac{2E_xE_y}{E_1E_2} + \left(\frac{E_x}{E_1}\right)^2 = 0$$

Or $\frac{E_x}{E_2} = m\frac{E_x}{E_1}$ 7.57

Or $E_y = mE_x$ where $\mathrm{m} = \frac{E_2}{E_1}$ or slope of line

Hence when two linearly polarized waves are in phase or out of phase the resultant wave is a plane polarized wave with $\vec{E}$ i.e. If $E_1 = 0$, the wave is polarized in y direction or if $E_2 = 0$, wave is polarized in x direction or when $E_1 = E_2$ and $\delta = 0$ where is still linearly polarized but at 45°.

(II) Let $E_1 = E_2$ and $\delta = \pm 90^o$

⟹ $\frac{E_x^2}{E_1^2} - 2\ \frac{E_xE_y \cos 90^o}{E_1E_2} + \frac{E_y^2}{E_1^2} = \sin 90^o$

⟹ $\frac{E_x^2}{E_1^2} + \frac{E_y^2}{E_1^2} = 1$

⟹ $\boxed{E_x^2 + E_y^2 = E_1^2}$ or $\boxed{E_x^2 + E_y^2 = E_2^2}$ 7.58

Hence when too linearly polarized components are in time phase quadrature of 90° and also are equal in magnitude, then resultant wave is circularly polarized.

Hence, when $E_1 = E_2$ and $\delta = \pm\frac{\pi}{2}$ the wave is circularly polarized

Also, $\delta = +90^o$ → Light circularly polarized

$\delta = -90^o$ → Right circularly polarized

7.8 Exercise

1. A uniform plane wave in free space is given by $\vec{E} = (200 < 30^o)e^{-j250x}\hat{y}$ V/m find:
    a. Phase constant (b) angular frequency (c) frequency (d) wavelength (e) intrinsic impedance (f) magnetic field intensity (g) $\vec{E}$ at $x = 8\text{mm}, t = 6 \times 10^{-9}s$
2. Given $E(x,t) = 10^3 \sin(3 \times 10^8\ t - \beta x)\hat{y}\ \text{V/m}$ in free space, sketch the wave at $t = 0$ and t_1, when it is travelled $\lambda/4$ along the x-axis. Find t_1, β and λ.
3. A lossless dielectric medium has $\sigma = 0, \mu_r = 1$ and $\varepsilon_r = 4$. An εM wave has magnetic field components expressed as
$$\vec{H} = -0.1\cos(\omega t - z)\ \hat{x} + 0.5\sin(\omega t - z)\ \hat{y}\ \text{A/m}$$
Find the components of electric field intensity of wave.
4. A normally incident electric field has amplitude $E = 1\,\text{V/m}$ in free space just outside sea water in which $\varepsilon_r = 80, \mu_r = 1, \sigma = 2.5\,\text{S/m}$. for a frequency of 120MHz at what depth the amplitude of E be $10^{-3}\ \text{V/m}$
5. A perpendicularly polarized wave propagates from a region having $\varepsilon_r = 4, \mu_r = 1, \sigma = 0$ to free space with angle of incidence of 30°. the incident field is $1\ \mu\ \text{V/m}$, find reflected and transmitted electric field; incident, reflected and transmitted magnetic field.
6. A parallel polarized wave propagates from air to dielectric at Brewster angle of 8.5°. calculate relative dielectric constant of medium.
7. What are constitutive relations of electromagnetic waves?
8. What are constitutive relations of εM waves in homogeneous medium?
9. What are constitutive relations of εM waves in isotropic medium?
10. What are constitutive relations of εM waves in isotropic medium?
11. Assuming free space conditions, derive wave equations?
12. Explain uniform plane wave propagation
13. What do you understand by uniform plane waves?
14. Derive relations between $\vec{E}$ and $\vec{H}$ in a uniform plane wave.
15. Derive expression for intrinsic impedance.
16. Derive wave equation for electric field $(\vec{E})$ in conducting medium.
17. Derive wave equation for magnetic field $(\vec{H})$ in conducting medium.

CHAPTER 8

WAVE PROPAGATION IN A LOSSLESS MEDIUM

Given a uniform plane wave case with no variation in the direction y or z, the wave equation in phaser form will be:

$$\frac{\partial^2 E}{\partial X^2} = -\omega^2 \mu \varepsilon E \quad or \quad \frac{\partial^2 E}{\partial X^2} = -\beta^2 E \tag{8.1}$$

$\Longrightarrow$ $\boxed{\beta = \omega\sqrt{\mu\varepsilon}}$

Solution for different equations gives

E_y component $\Longrightarrow$ $\boxed{E_y = C_1 e^{j\beta x} + C_2 e^{j\beta z}}$ (C_1, C_2 are arbitrary constants) 8.2

$$\Longrightarrow \quad \tilde{E}_y(x,t) = Re\left\{E_y(x)e^{j\omega t}\right\}$$

$$= Re\left\{C_1 e^{j\,(\omega t - \beta x)} + C_2 e^{j\,(\omega t + \beta x)}\right\}$$

$$= C_1 \cos(\omega t - \beta x) + C_2 \cos(\omega t + \beta x) \tag{8.3}$$

Equ 8.3 can be interpreted as:

Sum of two waves traveling in opposite directions.

If $\boxed{C_1 = C_2 \Longrightarrow \text{ STANDING WAVE}}$

Wave velocity $\quad \mathrm{v} = \frac{\omega}{\beta}$ $\quad \left[\omega t - \beta x = \alpha \Longrightarrow \frac{dx}{dt} = v = \frac{\omega}{\beta}\right]$

Phase constant β measure of phase shift in radians per unit length.

Wavelength λ distance over which sinusoidal waveform passes through a full cycle of 2π radians.

8.0 Wave Propagation in a Conducting Medium

Helmholtz equation: $\quad \nabla^2 E - \gamma^2 E = 0$ 8.4

$$[\therefore\ \gamma^2 = j\omega\mu\,(\sigma + j\omega\varepsilon)]$$

Propagation constant $\gamma = \alpha + j\beta$

(as it has a real part → attenuation and imaginary part → phase shift

Uniform plane wave traveling in x direction: $\frac{\partial^2 E}{\partial x^2} = \gamma^2 E$

$$E(x) = E_0 e^{-\gamma z} \qquad 8.5$$

Time varying form: $\tilde{E}_{(x,t)} = Re\left\{E_0 e^{+j\omega t - \gamma x}\right\}$

$$= e^{-\alpha x}\, Re\left\{E_0 e^{j\,(\omega t - \beta z)}\right\} \qquad (\therefore \gamma = \alpha + j\beta) \qquad 8.6$$

Equ 8.6 is equation of wave travelling in x direction and attenuated by $e^{-\alpha x}$

$$\alpha + j\beta = \sqrt{j\omega\mu\,(\sigma + j\omega\varepsilon)}$$

$$\alpha^2 - \beta^2 + j^2\alpha\beta = j\omega\mu\,(\sigma + j\omega\varepsilon)$$

$$= j\omega\mu \times j\omega\varepsilon\,\left(1 + \frac{\sigma}{j\omega\varepsilon}\right) = -\omega^2\mu\varepsilon\,\left[1 + \frac{\sigma}{j\omega\varepsilon}\right]$$

Equating real and imaginary parts

$\Longrightarrow \qquad \alpha_2 - \beta^2 = -\omega^2\,v\varepsilon$

And $\qquad 2\alpha\beta = \frac{j^2\,\omega^2\,\mu\varepsilon\sigma}{j\omega\varepsilon} = j\omega\mu\sigma$

$\Longrightarrow \qquad \beta = \frac{j\omega\mu\sigma}{2\alpha}$

Now, putting value of β we get

$$\alpha^2 - \left(\frac{j\omega\mu\sigma}{2\alpha}\right)^2 = -\omega^2\mu\varepsilon$$

$\Longrightarrow \qquad \alpha^2 - \frac{j^2\omega^2\mu^2\sigma^2}{4\alpha^2} = -\omega^2\mu\varepsilon$

$\Longrightarrow \qquad 4\alpha^2 + \omega^2\mu^2\sigma^2 = -\omega^2\mu\varepsilon\,(4\alpha^2)$

$\Longrightarrow \qquad 4\alpha^2 + \omega^2\mu\varepsilon 4\sigma^2 + \omega^2\mu^2\alpha^2 = 0$

$$\Longrightarrow \qquad \alpha^2 = -\frac{4\omega^2\,\mu\varepsilon \pm \sqrt{(4\omega^2\,\mu\varepsilon)^2 + 4\times4\times\omega^2\mu^2\sigma^2}}{2\times4}$$

$$= -\frac{4\omega^2\,\mu\varepsilon \pm \sqrt{16\omega^4\,\mu^2\varepsilon^2 + 16\omega^2\mu^2\sigma^2}}{8}$$

$$= -\frac{4\omega^2\,\mu\varepsilon \pm \sqrt{(\omega^2\varepsilon^2 + \sigma^2)\,16\omega^2\mu^2}}{8}$$

$$= -\frac{4\omega^2\,\mu\varepsilon \pm 4\omega^2\mu\varepsilon\sqrt{\left(1 + \frac{\sigma^2}{\omega^2\,\mu^2\varepsilon^2}\right)}}{2\times4}$$

$$\Longrightarrow \qquad \alpha^2 = -\frac{4\omega^2\,\mu\varepsilon \pm \omega^2\mu\varepsilon\sqrt{\left(1 + \frac{\sigma^2}{\omega^2\,\varepsilon^2}\right)}}{2}$$

Therefore, we get,

From above two equations we get;

$$\boxed{\alpha = \sqrt[\omega]{\frac{\mu\varepsilon}{2}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right)};} \qquad 8.7$$

$$\boxed{\beta = \sqrt[\omega]{\frac{\mu\varepsilon}{2}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right)};} \qquad 8.8$$

8.1 Conductors and Dielectric

$\left(\frac{\sigma}{\omega\varepsilon}\right)$ is ration of conduction current density to displacement current density

GOOD CONDUCTORS: $\frac{\sigma}{\omega\varepsilon} \gg 1$ (constant over frequency)

GOOD DIELECTRICS: $\frac{\sigma}{\omega\varepsilon} \ll 1$ ($\sigma, \varepsilon = f$ (frequency))

Dissipation factor D is power factor of dielectric

$\therefore$ $\quad \text{P.F.} = \sin \emptyset$

Where $\quad \emptyset = \tan^{-1} D$

Where $D = \frac{\sigma}{\omega\varepsilon}$ i.e ratio of conduction to displacement current

8.2 Wave Propagation in Good Dielectrics

For dielectrics, denominator will be greater in $\frac{\sigma}{\omega\varepsilon}$. Refer to Equs (8.7) and (8.8), we get

$\frac{\sigma}{\omega\varepsilon} \ll 1 \quad \therefore \quad \sqrt{1 + \frac{\sigma^2}{\omega^2\,\varepsilon^2}} \cong \left(1 + \frac{1}{2}\,\frac{\sigma^2}{\omega^2\,\varepsilon^2}\right)$ (from Binomial expansion)

$$\alpha \cong \omega\sqrt{\frac{\mu\varepsilon}{2}\left(1 + \frac{\sigma^2}{\omega^2\,\varepsilon^2}\right)} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

8.9

$$\beta \cong \omega\sqrt{\frac{\mu\varepsilon}{2}\left(1 + \frac{\sigma^2}{2\omega^2\,\varepsilon^2} + 1\right)} = \omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma^2}{8\omega^2\,\varepsilon^2}\right)$$

8.10

$$\mathrm{v} = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}\left(1+\frac{\sigma^2}{8\omega^2\,\varepsilon^2}\right)} \simeq \mathrm{c}\left(1 - \frac{\sigma^2}{8\omega^2\,\varepsilon^2}\right)$$

8.11

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j^{\omega t}}}$$

8.12

$$= \sqrt{\frac{\mu}{\varepsilon} \times \frac{1}{\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)}}$$

$\boxed{\eta \approx \sqrt{\frac{\mu}{\varepsilon}}\left(1 + \frac{\sigma}{2j\omega\varepsilon}\right)}$ intrinsic independence of good dielectric $\sigma = 0.$

Example 8.1 For a non-magnetic material drawing $\varepsilon_r = 2.25, \sigma = 10^{-4}\ \mathrm{s/m}$ find (i) loss tangent (ii)attenuation constant (iii) phase constant (iv) intrinsic impedance for a wave having a frequency of 2.5MHz. assume the material to be good dielectric.

Solution: (i) Loss tangent $= \frac{\sigma}{\omega\varepsilon}$

$$= \frac{10^{-4}}{2\pi \times 2.5 \times 10^{6} \times 2.25 \times 8.854 \times 10^{-12}} = 0.38$$

$$\frac{\sigma}{\omega\varepsilon} < 1$$

(ii) Attenuation constant

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = \frac{10^{-4}}{2}\sqrt{\frac{4\pi \times 10^{7}}{2.25 \times 8.854 \times 10^{-12}}}$$

$$= 0.01256\ \ \mathrm{Np/m}$$

(iii) phase constant

$$\beta = \omega\sqrt{\mu\varepsilon}\left[1 + \frac{\sigma^2}{8\omega^2\,\varepsilon^2}\right]$$

$$= 2\pi \times 2.5 \times 10^{6}$$

$$\times \sqrt{4 \times 10^{-7} \times 2.25 \times 8.854 \times 10^{-12}} \times \left[1 + \left(\frac{0.320}{8}\right)^2\right]$$

$$= 0.0796\ \ \mathrm{rad/m}$$

(iv) Intrinsic impedance

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\left(1 + j\frac{\sigma}{2\omega\varepsilon}\right)$$

$$= \sqrt{\frac{4\pi \times 10^{7}}{2.25 \times 8.854 \times 10^{-12}}}\left(1 + j\frac{0.320}{2}\right)$$

$$= \boxed{254.35 < 9.09^o \ \Omega}$$

8.3 Wave Propagation in Good Conductor

For good conductors, numerator will be greater in $\left(\frac{\sigma}{\omega\varepsilon}\right)$

As $\frac{\sigma}{\omega\varepsilon} \gg 1 \quad \therefore \gamma = \sqrt{(j\omega\mu\sigma) \ \left(1 + j\frac{\omega\varepsilon}{\sigma}\right)} \ \cong \ \sqrt{(j\omega\mu\sigma)} = \sqrt{\omega\mu\sigma} < 45^o$

$\therefore \qquad \alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$

8.13

$$v = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{2}}$$

8.14

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}} < 45^o$$

8.15

We conclude that for good conductor $\sigma \gg$, so, $\alpha \gg$ and $\beta \gg$ $v \ll$ and $Z \ll$

Example 8.2 A uniform plane wave in medium having $\sigma = 10^{-3}$ s/m $\varepsilon = 80\ \varepsilon_0$ and $\mu = \mu_0$ is having a frequency of 10kHz. Calculate the different parameters of the wave.

Solution: $\frac{\sigma}{\omega\varepsilon} = \frac{10^{-3}}{2\pi\times10^4\times80\times8.854\times10^{-12}} = \frac{10^{-3}}{4.448\times10^{-5}} = 22.48 \gg 1$

The medium is a good conductor, so attenuation constant

$$\alpha = \sqrt{\pi\ f\ \mu\sigma} = \sqrt{\pi \times 10^4 \times 4\pi \times 10^{-7} \times 10^{-3}}$$

$$= 2\pi \times 10^{-3} \ \text{Np/m}$$

$$\alpha = \beta = 2\pi \times 10^{-3} \ \text{N}\emptyset\text{/m}$$

Intrinsic impedance

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}} < 45^o = \sqrt{\frac{2\pi \times 10^4 \times 4\pi \times 10^{-7}}{10^{-3}}} < 45^o = 2\pi\,(1+j)$$

Wavelength $\lambda = \frac{2\pi}{\beta} = 100m$

Velocity of wave $= \frac{\omega}{\beta} = \frac{2\pi\times10^4}{2\pi\times10^{-3}} = 10^7\ \text{m/sec}$

8.4 Depth of Propagation

In a good conductor at ratio frequencies, the rate of attenuation is very great.

Wave may penetrate over a small distance before being reduced to a negligibly small percentage of original strength.

8.5 Depth of Penetration (δ)

Depth of penetration (δ) is the depth in which wave is attenuated to $\left(\frac{1}{e}\right)$ or approximately 37% of its original value (as shown in Fig 8.1).

$$\delta = \alpha^{-1} = \frac{1}{\sqrt{\pi f \mu\sigma}} \qquad 8.16$$

Electromagnetic waves, j, E, B, only penetrate a distance δ into a metal. Check the magnitude of δ in lab and web exercises.

The wave equation for match simplifier to

$$\frac{\partial^2 E_y(z)}{\partial_z{}^2} = j\omega\sigma\,\mu_0\,E_y\,(z)$$

The solution $E_y(z) = \exp\left(-\frac{1+j}{\delta} z\right)$

Where 'd' the skin depth is given be

$$\delta = \sqrt{\frac{2}{\omega\sigma\mu_0}}$$

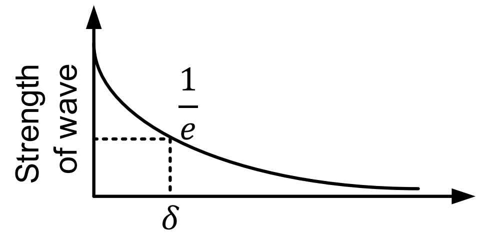


Figure 8.1 (a)

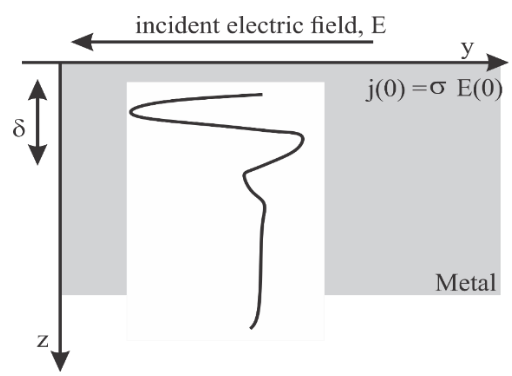


Figure 8.1 (b) Skin Depth

8.5.1 Impedance per square

By integrating the formula for the electric field inside a metal,

$$E_y(z) = \exp\left(-\frac{1+j}{\delta} z\right)$$

To find the current per unit width I_s we defined the impedance per square as

$$Z_s = E_y(0) I_s = \frac{1+j}{\sigma\delta} = \sqrt{\frac{\pi\mu_0 f}{\sigma}}\ (1+j)$$

For a wire of radius, a length L and circumference $2\pi\alpha$, we obtain

$$Z = \frac{L}{2\pi\alpha}\ Z_s$$

Example 8.3 A 160 MHz plane wave penetrates through A1 of $s = 10^5 \text{mho}, \varepsilon_2 = \mu_r = 1$. Calculate skin depth and also depth at which the wave amplitude decreases to 13.5% of initial value.

Solution: Skin depth = loops tangent = $\frac{\sigma}{\omega\varepsilon}$

$$\frac{10^5}{2\pi \times 160 \times 10^6 \times 8.854 \times 10^{-12}} = \frac{10^5}{0.008896} \gg 1$$

$$8 = \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{1}{\sqrt{\pi f \mu\sigma}} = \frac{1}{\sqrt{160 \times 10^6 \times 4\pi^2 \times 10^{-7} \times 10^5}}$$

$$\delta = 0.000125886\, m$$

Given, $$e^{\alpha x} = \frac{1}{0.135} = 7.407$$

$$\alpha_x = \ln(7.407) = 2.0025$$

$$x = \frac{2.0025}{\alpha} \text{ where } \alpha = \sqrt{\frac{\omega\mu\sigma}{2}}$$

Example 8.4 The electric field intensity of a linearly polarized uniform plane wave propagating in the positive z direction in sea water is $\vec{E} = 100\cos(10^7\, \pi t)\, \hat{a}_x$ V/m at $z = 0$. The parameter of sea water are $\varepsilon_r = 72, \mu_r = 1$, and $\sigma = 4$ s/m.

i. Determine the attenuation constant (α), phase constant (β), intrinsic impedance (η), phase velocity $(V\rho)$, wavelength (λ), and skin depth (δ).
ii. Find the distance at which the amplitude of E is one percent of its value of $z = 0$.
iii. Write the expression for $E(z,t)$ and $H(z,t)$ at $z = 0.8m$ as a function of t.

Solution: Electric field at $z = 0$ is

$$\vec{E} = 100\cos(10^7\, \pi t)\hat{a}_z \text{ V/m}$$

$$\omega = 10^7\, \pi\, rad/sec$$

$$\omega = 2\pi f$$

$$f = \frac{\omega}{2\pi} = \frac{10^7\pi}{2\pi} = 5 \times 10^6 \text{ Hz}$$

In this case

$$\frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{\omega\varepsilon_0\varepsilon_r} = \frac{4}{10^7\pi\left[\frac{1}{36\pi} \times 10^{-9}\right] \times 72} = 200 \gg 1$$

Hence, we have to use the formulae for good conductor:

(i). Attenuation constant

$$\alpha = \sqrt{\pi\mu\sigma f} = \sqrt{\pi \times 5 \times 10^6 \times 4\pi \times 10^{-7} \times 4}$$

$$= 8.88 \text{ N}\rho/\text{m}$$

Phase constant: $\beta = \alpha = 8.88 \text{ N}\rho/\text{m}$

Intrinsic impedance: $\eta = \sqrt{\frac{\omega\mu}{\sigma}} < 45^o$ or $(1+j)\sqrt{\frac{\pi f \mu}{\sigma}}$

$$\eta = (1+j)\sqrt{\frac{\pi \times 5 \times 10^6 \times 4\pi \times 10^{-7} \times 1}{4}}$$

$$= (1+j)(2.22)$$

$$2.22 + j(2.22)$$

$$= 3.14 < 45^o\Omega$$

Phase velocity

$$V = \frac{\omega}{\beta} = \frac{10^7\pi}{8.88} = 0.707 \text{ m/sec}$$

Skin depth: $\delta = \frac{1}{\alpha} = \frac{1}{8.88} = 0.1126$

(ii). The distance Z_1 at which the amplitude of wave decreases to one percent of its value at $z = 0$.

$$e^{-az_1} = 0.01$$

$$e^{-az_1} = \frac{1}{0.001} = 100$$

$$Z_1 = \frac{1}{\alpha} \ln 100 = \frac{4.605}{8.88} = 0.5186m$$

(ii). The value of electric field in phasor notation is given as;

$$\vec{E}(z) = 100e^{\alpha z}\, e^{-j\beta z}\, \hat{a}_x$$

Then the instantaneous expression for $\vec{E}$ is expressed as

$$\vec{E}(z,t) = R_e\left[\vec{E}(z)e^{j\omega t}\right] = R_e\left[100e^{\alpha z}\, e^{j(\omega t-\beta z)}\hat{a}_x\right]$$

$$= 100e^{-\alpha z}\cos(\omega t - \beta z)$$

At $z = 0.8m$, the above expression can be written in the form as,

$$\vec{E}\,(0.8,t) = 100e^{-0.80\alpha}\cos(10^7\,\pi t - 0.8\beta)\hat{a}_x$$

$$= 0.082\cos(10^7\pi t - 7.11)\hat{a}_x \quad \text{V/m}$$

A uniform plane wave is a **TEM** wave with $\vec{E}$ perpendicular to $\vec{H}$ and that both are normal to the direction of wave propagation $\hat{a}_z$. Thus $\vec{H} = H_y a_y$.

Hence $\boxed{H_y\,(z,t) = \frac{E_x\,(z,t)}{\eta}}$

$$H_y(z) = R_e\left[\frac{E_x(z)}{\eta}e^{j\omega t}\right]$$

In this case

$$H_y\,(0.8) = \frac{100e^{-80\alpha}\, e^{-j0.8\beta}}{\pi e^{-j\pi/4}} = \frac{0.082e^{-j7.11}}{\pi e^{j\pi/4}}$$

$$= 0.026e^{-j1.61}$$

As both angles must be in radians before combining the instantaneous expression for $\vec{H}$ at $z = 0.8m$ is then expressed as

$$H(0{,}8, t) = 0.026\cos(10^7\pi t - 1.61)\hat{a}_y \text{ A/m}$$

8.6 Properties of Uniform Plane Wave

It is necessary to write the expression for plane wave i.e., travelling in some arbitrary direction w.r.t fixed set of axes.

This is done in terms of direction cosines of the normal to plane of wave

By definition of uniform plane wave, the “equiphasic surfaces” are “planes”

$$\boxed{E(x) = E_0\, e^{J\beta x}} \qquad 8.17$$

(for wave travelling in x direction)

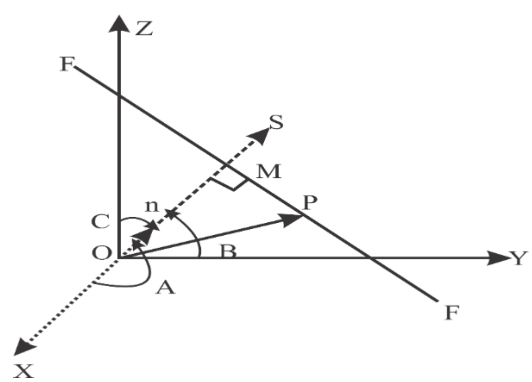


Figure 8.2 Direction Cosines

The planes of constant phase are given by, $x = a$ constant

The equation of plane:

$$\boxed{\hat{n}\,.\,r = a} \qquad 8.18$$

Constant see Fig 8.2

Where r is radius vector from origin to any point P on plane $\hat{n}$ is unit vector normal to plane (wave normal)

$\hat{n}\,.\,r$ is projection of radius vector $'r'$ along normal to plane

Constant value OM for all points in plane

$$\Longrightarrow \quad \boxed{\hat{n}\,.\,r = x\cos A + y\cos B + z\cos C} \qquad 8.19$$

x, y, z are components of vector r, and cos A, cos B, cos C are the components of unit vector $\hat{n}$ along x, y and z axes.

A, B, and C are the angles that unit vector $\hat{n}$ makes with positive x, y and z axes, respectively. Their cosine are termed the 'direction cosine or direction components of vector.

$$\Longrightarrow \quad E(r) = E_0 e^{j\beta\hat{n}.r} = E_0 e^{-j\beta(x\cos A + y\cos B + z\cos C)}$$

In time varying form: $E_0 = E_r + jE_i$

$$\tilde{E}_{(r,t)} = Re\left\{E_0 e^{-j(\beta\hat{n}.r - \omega t)}\right\}$$

$$E_r\cos(\beta\hat{n}.r - \omega t) + E_i\sin(\beta\hat{n}.r - \omega t) \qquad 8.20$$

Wavelength Uniform plane wave expression:

$$\mathbf{e^{-jhu}} \text{ where} \qquad 8.21$$

$'h'$ is some real constant

$'u'$ distance measured along a straight line $\left.\begin{matrix}\leftrightarrow & h\\ \leftrightarrow & \omega\end{matrix}\right\}$ for distance $\hat{u}$

$$\boxed{\lambda_u = \frac{2\pi}{h}}$$

Phase velocity $\quad \boxed{\mathrm{v}_u = \frac{\omega}{h}} \qquad 8.22$

Parallelly $\quad \lambda_x = \frac{2\pi}{\beta\cos A} = \frac{\lambda}{\cos A} \qquad 8.23$

And $\quad v_x = \frac{\omega}{\beta\cos A} = \frac{\lambda}{\cos A} \qquad 8.24$

As $\lambda = \frac{2\pi}{\beta},\ v = \frac{\omega}{\beta}$

As long as angle A is not zero, both wavelength and phase velocity measured along x axis are greater than when measured along wave normal.

For small angles $'\theta'$ in Fig 8.3 velocity v_y with which a crest moves along y axis, becomes very great,

$v_y = \infty \ as\, \theta = 0$

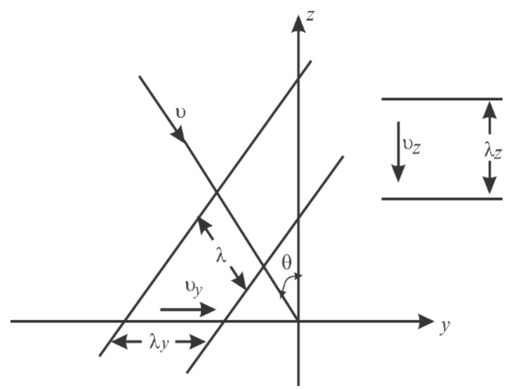


Figure 8.3 Uniform Plane Wave

8.7 Reflection of Uniform Plane Waves by Perfect Dielectric-Normal Incidence

When a plane EM wave is incident normally on the surface of a perfect dielectric, part of energy is transmitted and part of it is reflected. Fig 8.4

A perfect dielectric is one with zero conductivity so that there is no loss of power in propagation through the dielectric. Consider. Plane wave travelling is x direction

Incident on a boundary i.e., parallel to $x = 0$ plane

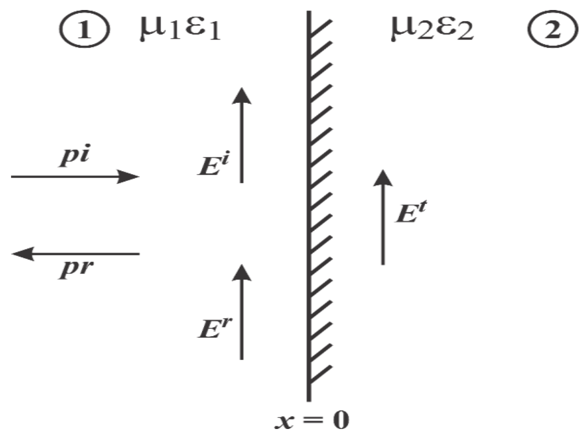


Figure 8.4 Normal incidence

Medium: $\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$ 8.25

(1) Medium: $\eta_1 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$ 8.26

Relationships for electric and magnetic fields

$$E^i = \eta_i H^i \quad 8.27$$

$$E^r = -\eta_i H^r \quad 8.28$$

$$E^t = \eta_2 H^t \quad 8.29$$

Continuity of tangential components of $\vec{E}$ and $\vec{H}$ require that

$$H^i + H^r = H^t, \; E^i + E^r = E^t \quad 8.30$$

$\Longrightarrow$ $$H^t + H^r = \frac{1}{\eta_1}\left(E^i - E^r\right) = \frac{1}{\eta_2}\left(E^i + E^r\right)$$

$\Longrightarrow$ $$\eta_2\left(E^i - E^r\right) = \eta_1\left(E^i + E^r\right)$$

(A) Reflection coefficient

$$\Gamma = \frac{E^r}{E^i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad 8.31$$

(B) Transmission coefficient

$$\tau = \frac{E^t}{E^i} = \frac{E^i + E^r}{E^i} = 1 + \frac{E^r}{E^i} = \frac{2\eta_2}{\eta_1 + \eta_2} \qquad 8.32$$

$\Longrightarrow$ $$\frac{H^r}{H^i} = -\frac{E^r}{E^i} = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} \qquad 8.33$$

And $$\frac{H^t}{H^i} = \frac{\eta_1 E^t}{\eta_2 E^i} = \boxed{\frac{2\eta_1}{\eta_1 + \eta_2}} \qquad 8.34$$

And $$\eta_0 \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

(C) For perfect dielectrics, $\mu_1 = \mu_2 = \mu_0$

$$\frac{E^r}{E^i} = \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} \quad \frac{E^t}{E^i} = \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} \quad \frac{H^r}{H^i} = \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} \quad \frac{H^t}{H^i} = \frac{2\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}$$

Note: Important points

- $1 + \Gamma = \tau$
- $\Gamma \ and\ \tau \rightarrow$dimensionless and may be complex
- $0 \le |\Gamma| \le 1$

If medium 2 is perfect conductor, $\eta_2 = 0 \Longrightarrow \Gamma = -1$ and $\tau = 0$

$\therefore$ $E^r = -E^i$ and $E^t = 0$

Incident wave will be totally reflected and standing wave will be produced in medium (1)

(D) Cases: (a) $\Gamma > 0\ (\eta_2 > \eta_1) \rightarrow SW$ in (1)
(b) $\Gamma > 0\ (\eta_2 > \eta_1) \rightarrow SW$ in (2)

Example 8.5 Determine the amplitude of reflected and transmitted $\vec{E}$ and $\vec{H}$ at the interface between two regions. The characteristics of region 1 are $\varepsilon_r = 0, \mu r_1 = 1$ and $\sigma_1 = 0$; region 2 is free spaces.

The incident $E_{\omega i}$ in region 1 is of 4.5 V/m.

Assume normal incidence also, find average power in two regions. (Fig 8.5).

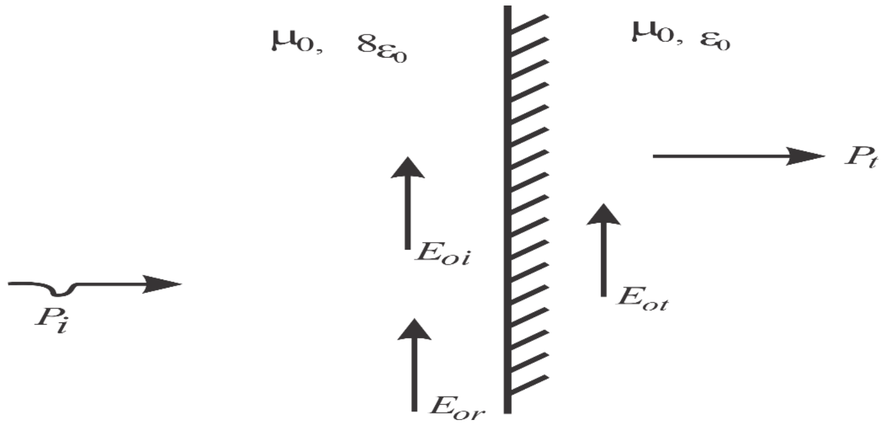


Figure 8.5

solution: $$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{4\pi\times10^{-7}}{8\times8.854\times10^{-12}}} = 133.3\Omega$$

$$\eta_2 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\Omega$$

$$E_{or} = \left(\frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}\right) E_{0i} = \frac{377 - 133.3}{377 + 133.3} \times 4.5 = 2.148 \text{ V/m}$$

$$E_{ot} = \left(\frac{2\eta_2}{\eta_1 + \eta_2}\right) E_{0i} = \frac{2 \times 377}{377 + 133.3} \times 4.5 = 6.648 \text{ V/m}$$

$$H_{oi} = \frac{E_{oi}}{\eta} = 3.39 \times 10^{-2} \text{ A/m}$$

$$H_{or} = \left(\frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}\right) H_{0i} = \frac{133.3 - 377}{133.3 + 377} \times 3.39 \times 10^{-2}$$

$$= -16.2 \times 10^{-3} \text{ A/m}$$

$$H_{ot} = \left(\frac{2\eta_1}{\eta_1 + \eta_2}\right) H_{0i} = \frac{2 \times 133.3}{133.3 + 377} \times 3.49 \times 10^{-2} = 17.7 \times 10^{-3} \ A/\text{m}$$

The incident average power densities in two regions

$$P_{oi} = \frac{1}{2} E_{oi} H_{oi} = 7.63 \times 10^{-3} \text{ W/m}^2$$

$$P_{ir} = \frac{1}{2} E_{or} H_{or} = 1.74 \times 10^{-2} \ \omega/\text{m}^2$$

$$P_{it} = P_{2i} = \frac{1}{2} E_{ot}\, H_{ot} = 5.88 \times 10^{-2} \ \ \omega/\mathrm{m}^2$$

8.8 Reflection by a Perfect Dielectric-Oblique Incidence

When a plane wave is incident upon a boundary surface i.e, not parallel to plain containing $\vec{E}$ and $\vec{H}$, $B.C.$ are more complex, see Fig 8.6.

Part of wave will be transmitted and part of it is reflected, but in this case the transmitted wave will be refracted: i.e, direction of propagation will be altered.

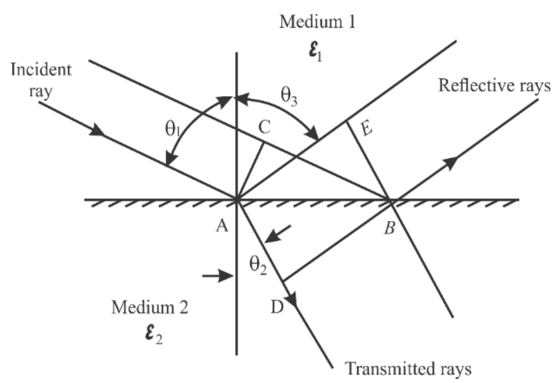


Fig 8.6 Oblique Incidence

Incident ray travels distance CB

f_{xd} ray travels distance AD

Reflected ray travels from A to E

v_1 and v_2 are velocities of medium 1 and 2

$$\frac{CB}{AD} = \frac{v_1}{v_2}$$

$CB = AB = \sin\theta_1$ and $AD = AB\sin\theta_2$

$\Longrightarrow$ $\quad \frac{CB}{AB} = \sin\theta_1;$ $\quad \frac{AD}{AB} = \sin\theta_2$

So, $\quad \frac{\sin\theta_1}{\sin\theta_2} = \frac{CB}{AD} = \frac{v_1}{v_2}$ $\quad$ 8.35

$$v_1 = \frac{1}{\sqrt{\mu_1 \varepsilon_1}} = \frac{1}{\sqrt{\mu_0 \varepsilon_1}}; \qquad v_2 = \frac{1}{\sqrt{\mu_2 \varepsilon_2}} = \frac{1}{\sqrt{\mu_0 \varepsilon_2}}$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

We, have $AE = CB$

So, $\sin\theta_1 = \sin\theta_3 \implies \theta_1 = \theta_3$ 8.36

Angles of incidence = angle of reflection

8.9 Snell's Law

Snell's law relates angle of incidence with angle of refraction

By conservation of energy:

$$\boxed{\frac{1}{\eta_1} E^{i2} \cos\theta_1 = \frac{1}{\eta_1} E^{r^2} \cos\theta_1 + \frac{1}{\eta_2} E^{t^2} \cos\theta_2}$$

$$\frac{E^{r^2}}{E^{i^2}} = 1 \frac{\eta_1 E^{t^2} \cos\theta_2}{\eta_2 E^{i^2} \cos\theta_1}$$

$$\boxed{\frac{E^{r^2}}{E^{i^2}} = 1 - \frac{\sqrt{\varepsilon_2}\, E^{t^2} \cos\theta_2}{\eta_2\, E^{i^2} \cos\theta_1}}$$ 8.37

Example 8.6 the amplitude of E_i in free space (region 1) at the interface with region 2 is 1 V/m. if $H_{or} = -1.41 \times 10^{-3}$ A/m, $\varepsilon_{re} = 18.5$ and $\sigma_2 = 0$, find μr_2.

Solution:

$$\frac{E_{or}}{H_{or}} = -120\pi\ \Omega = -377\Omega$$

And $\frac{E_{oi}}{H_{oi}} = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2} = \frac{\eta_2 - 377}{\eta_2 + 377}$

$$\frac{E_{oi}}{H_{oi}} = \frac{1.0}{-1.41 \times 10^{-3}} = \frac{-377(377 + \eta_2}{\eta_2 - 377}$$

Or $\eta_2 = 1234\Omega$

Then $\frac{\eta\sqrt{\mu r_2}}{\sqrt{18.5}} = 1234$

$\Longrightarrow$ $\mu r_2 = 198.4$

(i) Horizontal polarization

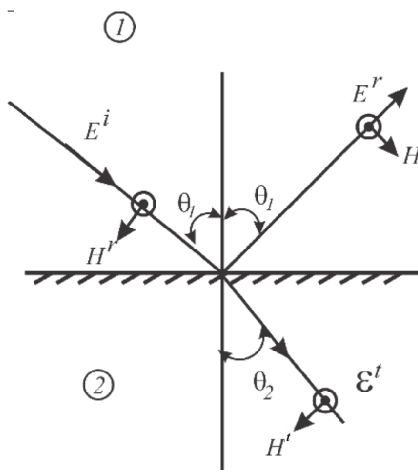


Figure 8.7 Horizontal Polarization

Electric field vector 'I' is parallel to boundary surface or ⊥ to plane of incidence in horizontal polarization as shown in in Fig 8.7.

Horizontal Polarization:

Example 8.7 A perpendicularly polarized wave propagates from region 1 ($\varepsilon_{r1} = 8.5, \mu_{r1} = 1, \sigma_1 = 0$) to region 2, free space, with an angle of incidence 15^0.

Given $E_{oi} = 1\mu\text{V/m}$. find $E_{or}, H_{or}, \theta_t, H_{oi}, E_{ot}$ and H_{ot}.

Solution: The intrinsic impedances are:

$$\eta_1 = \frac{\eta_0}{\sqrt{\varepsilon_{r1}}} = \frac{120\pi}{\sqrt{8.5}} = 129.33\Omega$$

$$\eta_2 = \eta_0 = 377\Omega$$

And the angle of transmission is given by

$$\frac{\sin 15^o}{\sin \theta_t} = \sqrt{\frac{\varepsilon}{8.5\varepsilon_0}} \quad or \;\; \theta_t = 48.99^o$$

$$\frac{E_{or}}{E_{oi}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t} = \frac{120\,\pi \cos 15^o - 129.33 \cos 48.99^o}{120\pi \cos 15^o + 129.33 \cos 48.99^o}$$

$$= 0.623$$

Or $\quad E_0{}^r = 0.623 \times 1.0\,\mu\,\mathrm{V/m}$

$$= 1.623\,\mu\,\mathrm{V/m}$$

$$\frac{E_{ot}}{E_{oi}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t} = \frac{2 \times 120\pi \cos 15^o}{120\pi \cos 15^o + 129.33 \cos 48.99^o}$$

$$= 1.623$$

$$= 1.623 \times 1\mu\mathrm{V/m}$$

$$= 1.623\mu\,\mathrm{V/m}$$

Finally, $\quad H_{oi} = \frac{E_{oi}}{\eta_1} = \frac{1\,\mu\mathrm{V/m}}{129.33} = 7.732n\mathrm{A/m}$

$$H_{oi} = \frac{E_{or}}{\eta_1} = \frac{0.623\,\mu\mathrm{V/m}}{129.33} = 4.817n\mathrm{A/m}$$

Similarly, $\quad \boxed{H_{ot} = 4.31\,n\mathrm{A/m}}$

(ii) Vertical Polarization

(II) Magnetic vector is parallel to boundary surface and electric vector is parallel to plane of incidence in vertical polarization as shown in Fig 8.8

MATHEMATICAL EXPLANATION

(I) Horizontal polarization

$E^i + E^r = E^t$ (B.C

∴ Tangential component of $\vec{E}$ is continuous)

$$\frac{E^t}{E^i} = 1 + \frac{E^r}{E^i}$$

Or $\frac{E^{r2}}{E^{i2}} = 1 - \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}} \frac{\cos\theta_2}{\cos\theta_1} \cdot \left(1 + \frac{E^r}{E^i}\right)^2$

$\Longrightarrow$ $1 - \left(\frac{E^t}{E^i}\right)^2 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \left(1 + \frac{E^r}{E^i}\right)^2 \frac{\cos\theta_2}{\cos\theta_1}$

$\Longrightarrow$ $1 - \frac{E^r}{E^i} = \sqrt{\frac{\varepsilon_2}{\varepsilon_2}} \left(1 + \frac{E^r}{E^i}\right) \frac{\cos\theta_2}{\cos\theta_1}$

$$\frac{E^r}{E^i} = \frac{\sqrt{\varepsilon_2}\cos\theta_1 - \sqrt{\varepsilon_2}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_1 + \sqrt{\varepsilon_2}\cos\theta_2} = \sigma \qquad 8.38$$

(II) Vertical Polarization

$\left(E^i - E^r\right)\cos\theta_1 = E^t \cos\theta_2$

(B.C. ∴ tangential component of $\vec{E}$ is continuous)

$$\frac{E^t}{E^i} = \left(1 - \frac{E^r}{E^i}\right) \frac{\cos\theta_1}{\cos\theta_2}$$

$\Longrightarrow$ $\left(\frac{E^r}{E^i}\right)^2 = 1 - \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \times \left(1 + \frac{E^r}{E^i}\right)^2 \frac{\cos\theta_1}{\cos\theta_2}$

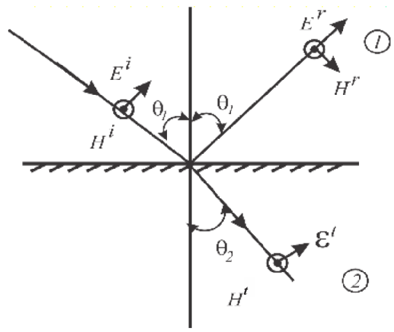


Figure 8.8 Vertical Polarization

$\Longrightarrow \quad 1-\left(\frac{E^r}{E^i}\right)^2 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\left(1-\frac{E^r}{E^i}\right)^2 \frac{\cos\theta_1}{\cos\theta_2}$

$\Longrightarrow \quad 1+\frac{E^t}{E^i} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\left(1-\frac{E^r}{E^i}\right)\frac{\cos\theta_1}{\cos\theta_2}$

$\Longrightarrow \quad \frac{E^r}{E^i}\left(1+\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\frac{\cos\theta_1}{\cos\theta_2}\right) = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\frac{\cos\theta_1}{\cos\theta_2} - 1$

$$\boxed{\frac{E^r}{E^i} = \frac{\sqrt{\varepsilon_2}\cos\theta_1 - \sqrt{\varepsilon_2}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_1 + \sqrt{\varepsilon_2}\cos\theta_2} = \Gamma} \qquad 8.39$$

$$\frac{E^r}{E^i} = \frac{\sqrt{\varepsilon_2}\cos\theta_1 - \sqrt{\varepsilon_1\,(1-\sin^2\theta_2)}}{\sqrt{\varepsilon_1}\cos\theta_1 + \sqrt{\varepsilon_2(1-\sin^2\theta_2)}} \quad (\text{Using} \cos^2\theta + \sin^2\theta = 1)$$

Also, $\quad \sin^2\theta_2 = \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_1$

$$\therefore \quad \frac{E^r}{E^i} = \frac{\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_1 - \sqrt{\left(\frac{\varepsilon_2}{\varepsilon_1}\right) - \sin^2\theta_1}}{\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_1 + \sqrt{\left(\frac{\varepsilon_2}{\varepsilon_1}\right) - \sin^2\theta_1}} \qquad 8.40$$

No Reflection $\sqrt{\varepsilon_2}\cos\theta_1 - \sqrt{\varepsilon_1}\cos\theta_2 = 0$

$\Longrightarrow \quad \frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$

Or $$\boxed{\tan\theta_1 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}} \qquad 8.41$$

Equ (8.25) refers to Brewster's angle

$$\tau = \frac{E^t}{E^i} = \frac{2\sqrt{\varepsilon_1}\cos\theta_1}{\sqrt{\varepsilon_2}\cos\theta_1 + \sqrt{\varepsilon_1}\cos\theta_2} \quad \text{(Transmission coefficient)} \qquad 8.42$$

Example 8.8. A normally incident E field has amplitude $E_{0i} = 1.0\ \mathrm{V/m}$ in free space outside of sea water which $\varepsilon_r = 80, \mu_r = 7$ and $\sigma = 2.5\ \mathrm{s/m}$. for a frequency of 30MHz, at what depth will the amplitude of $\vec{E}$ be 1.0 m/V?

Solution:	Free Space	Sea Water
	Region 1	Region 2
	$\eta_1 = 377\Omega$	$\eta_2 = 9.73 < 43.5^o\Omega$

Then the amplitude of E_{0t} just inside the sea water is E_{0t}

$$\frac{E_{0t}}{E_{0i}} = \frac{2\eta_2}{\eta_1 + \eta_2}$$

Or $\quad E_{0t} = 5.07 \times 10^{-2}\ \mathrm{V/m}$

From $\quad \gamma = \sqrt{j^{\omega}\mu\,(\sigma + j\omega\varepsilon)}$

$$\alpha = 24.36\cos 46.53^o$$

$$= 16.76\ \mathrm{Np/m}$$

Then $\quad 10^{-3} = (5.07 \times 10^{-2})e^{-16.76z}$

$$E_z = E_{0t}\, e^{-16.76z}$$

$$\boxed{z = 0.234\ \mathrm{m}}$$

8.10 Reflection by a Perfect Conductor-Normal Incidence

Wave is entirely reflected se Fig 8.9

$\vec{E}|\vec{H}$ can't exist so none of the energy of incident wave can be transmitted no energy is absorbed by perfect conductor (lossless).

$\vec{E}$ of incident wave $E^i e^{j\,(\omega t - j\beta x)}$

Expression of reflected wave $= E^r\ e^{j\,(\omega t + j\beta x)}$ (charge in direction of power flow)

Tangential component of $\vec{E}$ must be continuous across boundary and $\vec{E}$ is zero within conductor

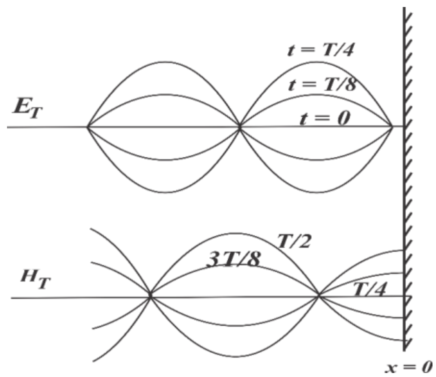


Figure 8.9 Free Space

$\Longrightarrow \qquad E^i + E^r = 0$

[Tangential component of $\vec{E}$ just outside conductor must be zero]

$\Longrightarrow \qquad E^r = -E^i$

(at odd multiples of $\lambda/4 \quad E_{max} = 2E^i$ at multiples of $\lambda/2$ zeroes)

$$\Longrightarrow \qquad E_T = E^i\ e^{j\,(\omega t - \beta x)} + E^r e^{j\,(\omega t + \beta z)} \qquad 8.43$$

$$= E^i \left[e^{j(\omega t - \beta x)} - e^{j\,(\omega t + \beta x)}\right]$$

$$= -2j\ E^i \sin\beta x\ e^{j\,\omega t}$$

With E^i to be real $\Longrightarrow E_T = 2E^i$

$\sin\beta x \sin\omega t$ standing wave

Parallelly, $\qquad H_T = H^i\ e^{J\,(\omega t - \beta x)} + H^r e^{J\,(\omega t + \beta x)}]$

$$= H^i \left[e^{J\,(\omega t - \beta x)} + e^{J\,(\omega t + \beta x)}\right]$$

$$H_T = 2H^i \cos\beta x \;\; e^{J\omega t}$$

With E^i in phase with $H^e \Longrightarrow$ $\boxed{H_T = 2H^i \cos\beta x \cos\omega t}$ 8.44

(max at surface and multiples of $\lambda/2$)

E_T and H_T are 90^o out of time phase ∴ of factor $'f'$.

8.11 Surface Impedance of Conductor

Surface impedance of conductor is defined as ratio of tangential component of electric field strength on the surface of conductor to the linear surface current density k_s

$$\boxed{Z_s = \frac{E_t}{K_s}} \quad 8.45$$

Current density at face of conductor ($J = \sigma E$) decreases exponentially with distance.

$$J = J_0\, e^{-rz}$$

J_0 conduction current density

Average surface current density

$$K_s = \int_0^\infty J\, dz = \int_0^\infty J_0\;\, e^{rz}\;\, dz$$

$\Longrightarrow$ $K_s = J_0\; \int_0^\infty e^{-rz}\;\, dz = \frac{J_0}{r}$

And $E_t = \frac{J_0}{\sigma}$

∴ $\boxed{Z_s = \frac{J_0}{\sigma\,(J_0/r)} = \frac{r}{\sigma}}$ 8.46

For perfect conductors $r = \sqrt{J\omega\mu\sigma}$

$$Z_s = \sqrt{\frac{J\omega\mu}{\sigma}} = \eta$$

Or $$Z_s = \sqrt{\frac{J\omega\mu}{\sigma}} < 45^o \qquad 8.60$$

$$R_s = \sqrt{\frac{\omega\mu}{2\sigma}} \qquad 8.61$$

And $$X_s = \sqrt{\frac{\omega\mu}{2\sigma}} \qquad 8.62$$

Skin effect resistance per unit length

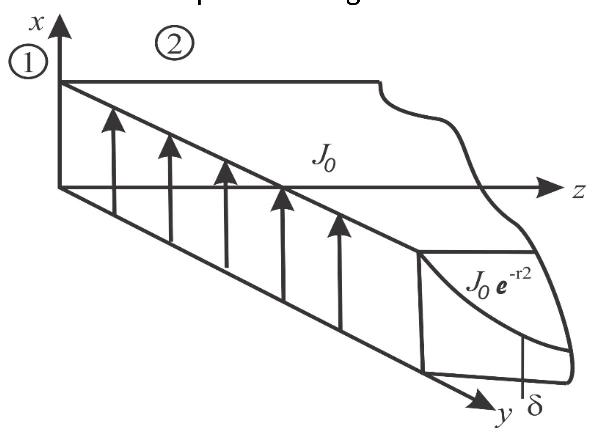


Figure 8.10 Surface Impedance

Skin depth $$\boxed{\delta = \sqrt{\frac{2}{\omega\mu\sigma}}} \quad \therefore \boxed{R_s = \frac{1}{\sigma\delta}} \qquad 8.47$$

8.12 Poynting's Theorem

The Derivation of Poynting Theorem i.e., "Rate of energy transfer" refers to Poynting's Theorem as can be seen in Fig 8.11 shows power from a source to three receivers at different location

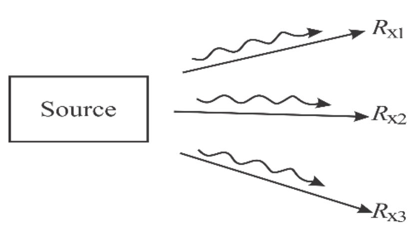


Fig. 8.11 Shows power flow from a source to three Receiver at Different Location

M.M.F. (magnetomotive force) can be written as:

$$J = \nabla \times H - \varepsilon \dot{E}$$ (Using Maxwell's modified Ampere's law 8.48

Dimension of current density

Taking dot product with electric field of Equ (8.25)

$\therefore$ $$E \cdot J = E \cdot \nabla \times H - \varepsilon \dot{E} \cdot \dot{E} \quad 8.49$$

Dimensions of power per unit volume

Identity: $$\nabla \cdot E \times H = H \cdot \nabla \times E - E \cdot \nabla \times H \quad 8.50$$

Using above identity in Equ (8.2) we get

$\cdots$ $$E \cdot J = -\nabla \cdot E \times H + H \cdot \nabla \times E - \varepsilon E \cdot \dot{E}$$

Also, $$\nabla \times E = -\mu \dot{H} \quad 8.67$$

(Using faraday's law in Maxwell's equation)

We get

$\therefore$ $$E \cdot J = -\mu H \cdot \dot{H} - \varepsilon E \cdot \dot{E} - \nabla \cdot E \times H \quad 8.68$$

Now we can write

$$H \cdot \dot{H} = \frac{1}{2}\frac{\partial}{\partial t} H^2 \quad \text{and} \quad E \cdot \dot{E} - \frac{1}{2}\frac{\partial}{\partial t} E^2 \quad 8.69$$

$\therefore$ $$E \cdot J = -\frac{\mu}{2}\frac{\partial}{\partial t} H^2 - \frac{\varepsilon}{2}\frac{\partial}{\partial t} E^2 - \nabla \cdot E \times H \quad 8.70$$

Integrating over volume,

$$\int_V E \cdot J \, dV = -\frac{\partial}{\partial t}\int_V \left(\frac{\mu}{2} H^2 + \frac{\varepsilon}{2} E^2\right) dV - \int_V (\nabla \cdot \mathrm{E} \times \mathrm{H}) dV \quad or$$

(Using divergence theorem) in R.H.S we get

$\Longrightarrow$ $$\int E \cdot I \, dV = \frac{-\partial}{\partial t}\int_V \left(\frac{\mu}{2} H^2 + \frac{\varepsilon}{2} E^2\right) dV - \oint_S E \times H \cdot da \quad 8.71$$

The Poynting theorem explanation of terms I, II, III in Poynting equation (8.69) are:

(I) (Instantaneous) Power dissipated in Volume 'V' Joule's law

A conductor of cross-sectional area A, carrying current I,

Powerless of EI watts per unit length

Power dissipated per unit volume $= \frac{EI}{A} = EJ$ watts per unit volume or $E \,.\, J$

Total power dissipated in a volume $V \rightarrow \int_V E \,.\, j \; dV$

Or dissipated at ohmic (I^2R)loss

$\text{Note: if}'E''\text{is due to to source of power, e.g., batter, then} \int E \cdot J \; dV$ would be used up in driving the current against the battery voltage CHARGING BATTERY NEGATIVE

(II) $\frac{1}{2}\varepsilon E^2$ stored energy (electric) per unit volume ELECTROSTATIC; $\frac{1}{2}\,\mu H^2$ stored energy (magnetic) per unit volume MAGNETOSTATIC ($-ve$ time derivative of this represents rate at which stored energy in volume is decreasing).

(III) Law of conservation of energy rate of energy dissipated in volume V must equal rate at which stored energy in V decreasing plus rate at which energy is entering the volume V from outside.

∴ $-\oint_S \vec{E} \times \vec{H} \cdot da$ rate of flow of energy outward through the surface of volume.

∴ $-\oint_S \vec{E} \times \vec{H} \cdot da$ rate of flow of energy inward through the surface enclosing volume.

$$\boxed{P = \vec{E} \times \vec{H}}$$

It is Poynting's theorem that this vector product at any point is a measure of the rate of energy flow per unit area at that point.

Example 8.9 In free space $E\,(x,t) = 150\cos(\omega_t - \beta_z)\,\hat{a}_x$ V/m. find the average power crossing a circular area of circuit 5m in the plane Z =constant.

Solution: In complex form $\vec{E} = 150\, e^{j\,(\omega t - \beta z)}\, \hat{a}_x \;\; \text{V/m}$

$$\eta = 120\pi\ \Omega$$

With propagation in Z direction

$$\vec{H} = \frac{150}{120\pi}\, e^{j\,(\omega t - \beta z)}\, \hat{a}_x \;\; A/\text{m}$$

Average pointing vector

$$P_{av} = \frac{1}{2}\, R_e\big(\vec{E} \times \vec{H}\big)\text{W/m}^2 = \frac{1}{2}\,(150)\left(\frac{150}{120\pi}\right)\hat{a}_x$$

With flow normal to area

Average power,

$$P_{av} = \frac{1}{2}\,(150)\left(\frac{150}{120\pi}\right)\pi\,(5)^2 = \boxed{2343.6\ \text{W}}\ \ Ans$$

Example 8.10 A medium has the following characteristics: Propagation Constant $r = 520 + j2443.5$ per m and intrinsic impedance $\eta = 50 < 12^o\Omega$ at a wave frequency of $f = 300\text{MHz}$. the electric field is given by $\vec{E}_x = 200\, e^{\alpha x} \cos(6\pi\, 10^8\, t - \beta z)\hat{a}_x$. Calculate the expression for magnetic intensity and the average power/m^2 at $Z = 1mm$, given by electromagnetic wave.

Solution: Given that:

$$\boxed{\begin{array}{c} \vec{E}_x = 200\, e^{\alpha x} \cos(6\pi\, 10^8\, t - \beta z)\hat{a}_x \\ \gamma = 520 + j2443.5 = \alpha + j\beta \\ \alpha = 520\ \text{Np/m}, \beta = 2443.5\ \text{rad/m} \\ \eta = 50 < 12^o\Omega, \beta z = 2.443.5\ rad \\ \alpha_z = 520\text{Np} \end{array}}$$

$$\vec{E}_x = 200\, e^{-0.520} \cos(6\pi\, 10^8\, t - 140^o)\hat{a}_x$$

$$= \boxed{118.9 \cos(6\pi\, 10^8 t - 140^o)\hat{a}_x \;\; \text{V/m}}$$

$$\vec{H}_y = \frac{200}{50}\ e^{-0.520}\cos(6\pi\ 10^8\ t - 140^o - 120^o)\hat{a}_y$$

∴ $\eta = |\eta| < \theta_\eta$ and $|\eta| = 50$

∴ $\boxed{\vec{H}_y = 2.378\cos(6\pi\ 10^8\ t - 152^o)\hat{a}_x\ \ \text{A/m}}$

The pointing vector in Z direction

$$P_z = E_x H_y$$

$$= 282.74\ \cos(6\pi\ \times 10^8 t - 140^o)\ \cos(6\pi\ \times 10^8 t - 152^o)$$

Using the identity $A\cos B = \frac{\cos(A+B)+\cos(A-B)}{2}$

∴ $P_z = E_x H_y = \frac{282.74}{2}\ \cos(12\pi\ \times 10^8 t - 292^o) + \frac{282.74}{2}\cos 12^o$

$$= \boxed{138.28 + 141.37\cos\ (12\pi\ \times 10^8 t - 292^o)\ \ \omega/m^2}$$

Average power

Note that on interpretation of $\vec{E}\times\vec{H}$: Radiation problems

CLASSIC ILLUSTRATION: Bar magnet on which electric charge is placed.

Static electric field crossed with static magnetic field and Poynting's theorem seems to require a continuous circulation of energy around magnet.

Note: surface integral is over closed surface surrounding volume. If any closed surface is taken about bar magnet, $\vec{E}\times\vec{H}$ is always zero.

Net power flow away from magnet is zero

$$\oint(E\times H + F)\cdot da = \oint(E\times H)\cdot da + \int\nabla\cdot F\ dV = \oint(E\times H)\cdot da^{-23.9} \quad 8.72$$

Although $\vec{E}\times\vec{H}$ correctly gives power flow "at each point?"

It is seen that it may be possible to write an expression that gives correctly the net flow at power through closed surface, it is still not possible to state first where the energy is

Note: (vp) Phase Velocity: is velocity of particles that constitute a wave (vg)group Velocity: is velocity of wave or the entire envelop

$\mathrm{v}_p \times \mathrm{v}_g = \mathrm{c}^2$ where c is speed of light and ($c = 3 \times 10^8\ m/s$)

$$P_{av} = \frac{1}{2}\ \frac{E_0^2}{|\eta|}\ e^{a\alpha z} \cos\theta_\eta$$

$$= \frac{1}{2}\ \frac{(200)^2}{50}\ e^{-1.04} \cos 12$$

$$= \boxed{138.28\ \ \mathrm{W/m^2}}$$

8.13 Power and Energy Relations

Consider a region of space represented by an array of field cell transmission lines of total width W and total height H as in Fig. 8.12 with a plane wave traveling from left to right. The electric field E_y is vertical and the magnetic field Hz is horizontal. The voltage $V = E_y H$ and the current $I = H_z W$. By analogy to circuits, the power conveyed is

$$P = VI = E_y H_z HW = E_y H_z A \quad (W) \qquad 8.73$$

Where $A = HW =$ area of field-cell array. The power (surface) density is then

$$S = \frac{P}{A} = E_y H_z \quad (W\ m^{-2}) \qquad 8.74$$

Equ 8.74 relates the scalar magnitudes. The power flow is perpendicular to E and H and it can be shown that in vector notation the power density is given by.

$$\boxed{S = E \times H \quad (W\ m^{-2})\ \ Poynting\ vector} \qquad 8.75$$

Turning E into H and proceeding as with a right-handed screw gives the direction of S perpendicular to both E and H. S is a power surface density called the Poynting vector. Its value in 8.75 is the instantaneous Poynting vector. The average Poynting vector is obtained by integrating the instantaneous Poynting vector over one period and dividing by one period. It is also readily obtained in complex notation from.

$$S_{av} = \frac{1}{2} Re\, E \times H = \frac{1}{2}\, \hat{x}\, |E_y||H_z| \cos\xi \quad (W\, m^{-2}) \qquad 8.76$$

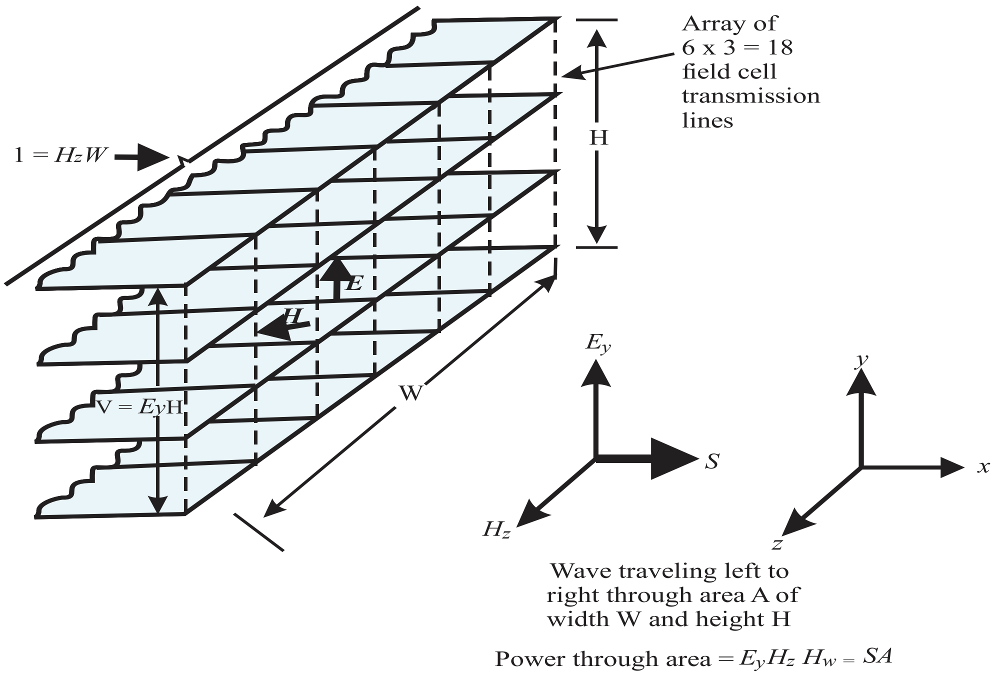


Figure 8.12

Power flow of wave traveling left to right through area of width W and height H is equal to E_y H_z HW.

Where $S_{av} = \hat{x}S =$ average Poynting vector W m^{-2}

$E = \hat{y}E_y = \hat{y}|E_y|e^{j\omega t},\ V\, m^{-1}$

$H = \hat{z}Hz = \hat{z}|Hz|e^{-j(\omega t - \xi)}, A\ \ m^{-1}$

$\xi =$ time phase angle between E_y and H_z rad or deg

H is called the complex conjugate of H, where

$$H = \hat{z}Hz = \hat{z}|Hz|e^{j(\omega t - \xi)}, A\ \ m^{-1}$$

The quantities H and its complex conjugate H have the same direction but they differ in sign in their phase factors. Note that if $E_y\ and\ H_z$ in 8.76 are rms values instead of (peak) amplitudes, the factor ½ in 8.76 is omitted.

The magnitude of the average Poynting vector.

$$S_{av} = \frac{1}{2} Re E_y H_z = \frac{1}{2}|E_y||H_z|\cos\xi \qquad (W\ m^{-2}) \qquad 8.77$$

The relation corresponding to 8.77 for the average power for a travelling wave on a transmission line is.

$$P_{av} = \frac{1}{2} Re\ VI = \frac{1}{2}|V||I|\cos\theta \qquad (W\ ) \qquad 8.78$$

Where V =voltage between conductors of transmission line, V

I = current through one conductor, A

I^* = complex conjugate if I

θ = time phase angle between V and I, rad or deg

Since the intrinsic impedance of the medium

$$Z_0 = \frac{E}{H} = \frac{|E|}{|H|} < \xi = |z_0| < \xi$$

The magnitude of the average Poynting vector can also be written

$$\boxed{S_{av} = \frac{1}{2} Re H_z H_z^* Z_0 = \frac{1}{2}|H_z|^2 Re Z_0\ \ (W\ m^{-2})} \qquad 8.79$$

Or

$$S_{av} = \frac{1}{2} Re\ \frac{E_y E_y^*}{Z_0} = \frac{1}{2}|E_y|^2 Re \frac{1}{Z_0} \quad (W\ m^{-2}) \qquad 8.80$$

Equ (8.79) is very useful, since if the intrinsic impedance Z_0 of a conducting medium and also the magnetic field H_z at the surface are known, it gives the average Poynting vector (or average power per unit area) into the conducting medium.

Example 8.11 Power into copper sheet. A plane 1-GHz traveling wave in air with peak electric field intensity of V m^{-1} is incident normally on a large copper sheet. Find the average power absorbed by the sheet per square meter of area.

Solution. From (8.76-8.88-8.85), the intrinsic impedance of copper at 1GHz is

$$Z_0 = \sqrt{\frac{\omega\mu}{\sigma}} < 45^0$$

For copper $\mu_r = \varepsilon_r = 1 \ \ and\ \sigma = 58\ M℧\ m^{-1}$. Hence the real part of Z_0 is

$$R_e Z_0 = \cos 45^0 \sqrt{\frac{2\pi \times 10^9 \times 4\pi \times 10^{-7}}{5.8\ \times\ 10^7}} = 8.2\ m℧$$

Next, we find the value of H at the sheet (tangent to the surface). This is very nearly double H for the incident wave. Thus,

$$H = 2\frac{E}{Z} = \frac{2 \times 1}{377}\ \ A\, m^{-1}$$

From 8.79, we find that the average power per square meter into the sheet is tiny:

$$S_{av} = \frac{1}{2}\left(\frac{2}{377}\right)^2 8.2 \times 10^{-3} = 115\text{n}(W\, m^{-2})$$

Class work: Poynting vector into aluminum sheet. A 3-GHz wave is incident on a large sheet of aluminum ($\sigma = 3.5 \times 10^{-7}\ ℧/m$). If the field E = 15 V/m, find the average power absorbed by the sheet (W/m^2). Ans. 366n W/m^2.

The relation corresponding to 8.79 and 8.80 for the average power of a traveling wave on a travelling wave on a transmission line are.

$$P_{av} = \tfrac{1}{2} Re\, II^* Z_0 = \tfrac{1}{2}|I|^2 Re\, \mathrm{Z}_0 \qquad (W\ ) \qquad (8.81a)$$

$$P_{av} = \frac{1}{2} Re \, \frac{VV^*}{Z_0} = \frac{1}{2} |V|^2 \, \mathrm{Re} \frac{1}{Z_0} \qquad (W\ ) \qquad (8.81b)$$

When Z_0 is real $(\xi = 0)$ and E and H are rms values, we have for the traveling space wave.

$$S_{av} = EH = H^2 Z_0 = \frac{E^2}{Z_0} \qquad (W\, m^{-2}) \qquad 8.82$$

And for the traveling wave on a transmission line $(\theta = 0 \text{ and } V \text{ and } I \text{ rms})$

$$P_{av} = VI = I^2 Z_0 = \frac{V^2}{Z_0} \qquad (W) \qquad 8.83$$

From Equs (8.74-8.81) the energy density w_e at a point in an electric field is

$$w_e = \frac{1}{2}\, \varepsilon E^2 \qquad (Jm^{-3}) \qquad 8.84$$

Where ε =permittivity of medium, FM^{-1} and E = electric field intensity, Vm^{-1}

From Equs (8.74-8.84-8.89) the energy density w_m at a point in a magnetic field is

$$w_e = \frac{1}{2} \mu H^2 \qquad (Jm^{-3}) \qquad 8.85$$

Where μ =permeability of medium, H m^{-1}, and H = magnetic field, A m^{-1}

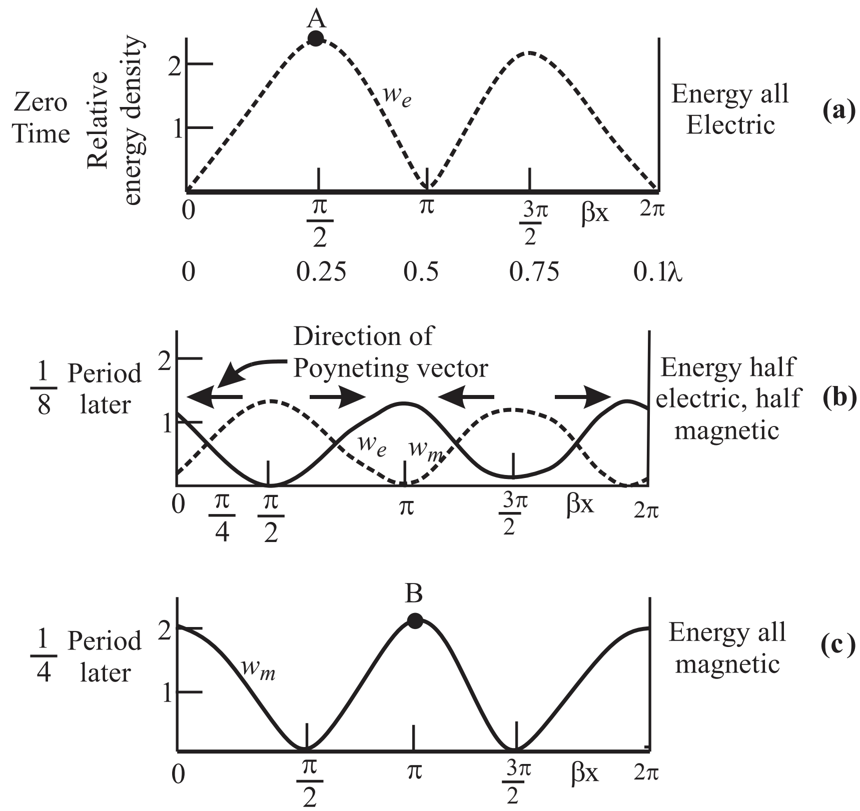


Figure 8.13

Total electric and magnetic energy densities at three instants of time for a pure standing wave. Conditions are shown over a distance of $1\lambda\,(\beta x = 2\pi)$. There is no net transmission of energy in a pure standing wave, but locations of energy oscillate back and forth. The situation here (pure standing wave) is identical with that in a short-circuited transmission line or in a resonator.

In a traveling wave in an unbounded lossless medium

$$\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} \qquad 8.86$$

Substituting from H from 8.86 in 8.85, we have

$$W_m = \frac{1}{2}\mu H^2 = \frac{1}{2}\varepsilon E^2 = W_e \qquad 8.87$$

Thus, the electric and magnetic energy densities in a plane traveling wave are equal and the total energy density w is the sum of the electric and magnetic energies. Thus,

$$\boxed{w = w_e + w_m = \frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2 \quad (J\ m^{-3} \qquad \textbf{Energy density}} \qquad 8.88$$

Or

$$w = \varepsilon E^2 = \mu H^2 \quad (J\ m^{-3}) \qquad 8.89$$

Two waves of equal magnitude traveling in opposite direction produce a standing wave. There is no net transfer of energy in a pure standing wave but energy does oscillate back and forth like water slops in a pail. At one instant the energy is all electric as in Fig 8.13a with maximum at point A. one quarter of a period later, the energy is all magnetic, as in Fig 8.13c, with maximum at B, which is at a distance $\lambda/8$ from A. one quarter period later the energy is all back at A. at intermediate times the energy is moving and is half electric and half magnetic as in Fig 8.13b.

8.14 Linear, Elliptical, and Circular Polarization

Consider a plane wave traveling out of the page (positive z direction) as in Fig 8.14a, with the electric field at all times in the y direction. This wave is said to be linearly polarized (in the y direction). As a function of time and position, the electric field is given by

$$E_y = E_2 \sin(\omega t - \beta z)$$

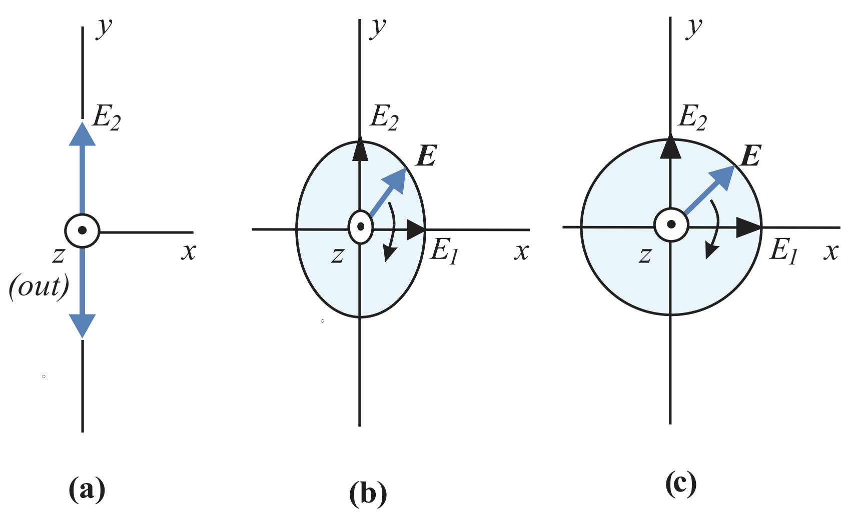


Figure 8.14

(a) Linear, (b) elliptical, and (c) circular polarization for left circularly polarized wave approaching

Electric field is given by

$$E_y = E_2 \sin(\omega t - \beta z) \tag{8.90}$$

In general, the electric field of a wave traveling in the z direction may have both a y component and an x component as suggested in Fig. 8.14b. in this more general situation, with a phase difference δ between the components, the wave is said to be elliptically polarized. At a fixed value of z the electric vector E rotates as a function of time, the tip of the vector describing an ellipse called the polarization ellipse. The ratio of the major to minor axes of the polarization ellipse is called the axial ratio (AR). Thus, for the wave Fig 8.14b, $AR = E_2/E_1$. Two extreme cases of elliptical polarization correspond to circular polarization E_1=E_2 and AR = 1, while for linear polarization E_1 = 0 and AR =∞.

In the most general case of elliptical polarization, the polarization ellipse may have any orientation, as suggested in Fig. 8.15. The elliptically polarized wave may be expressed in terms of two linearly polarized components, one in the x direction and one in the y direction. Thus, if the wave is traveling in the positive z direction (out of the page), the electric field components in the x and y direction are

$$E_x = E_1 \sin(\omega t - \beta z) \tag{8.91}$$

$$E_y = E_2 \sin(\omega t - \beta z + \delta) \tag{8.92}$$

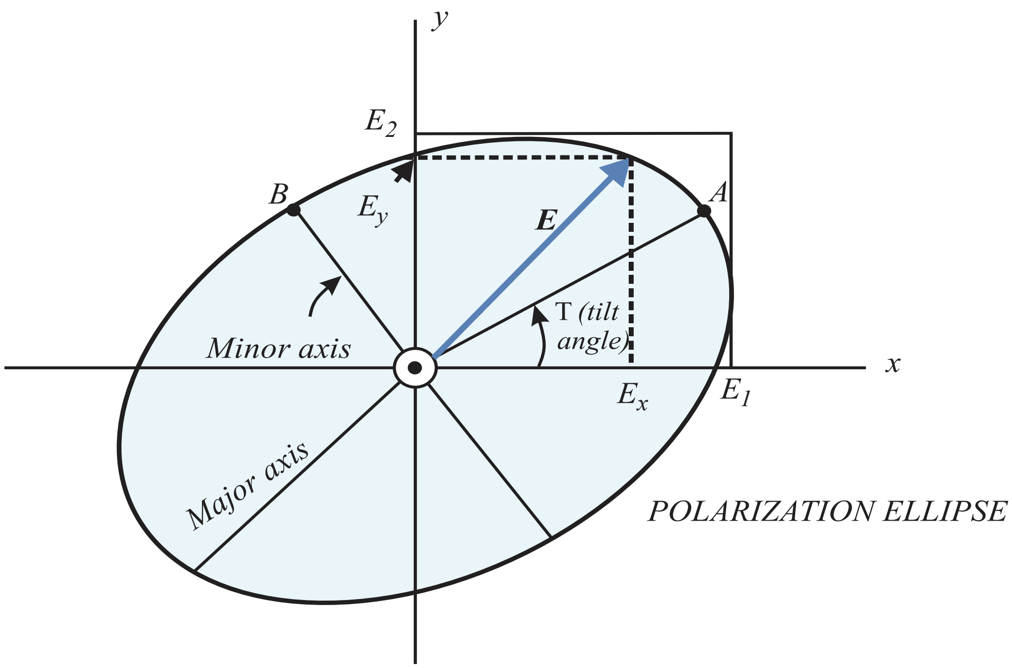


Figure 8.15 Polarization ellipse at tilt angle τ showing instantaneous components E_x and E_y amplitudes (or peak values) E_1 and E_2.

Where $E_1 =$ amplitude of wave linearly polarized in x y direction

$E_2 =$ amplitude of wave linearly polarized in y direction

$\delta =$ time-phase angle by which E_y leads E_x

Combining Equs (8.91 and 8.92) gives the instantaneous total vector field **E.**

$$\boldsymbol{E} = \hat{\boldsymbol{x}}E_1 \sin(\omega t - \beta z) + \hat{y}E_2 \sin(\omega t - \beta z + \delta) \qquad 8.93$$

$At\ z = 0, E_x = E_1 \sin \omega t\ \ and\ E_y = E_2 \sin(\omega t + \delta).$ Expanding E_y yields

$$E_y = E_2(\sin \omega t \cos \delta + \cos t \sin \delta) \qquad 8.94$$

From the relations for E_x we have $\sin \omega t = E_x/E_1$ and $\cos \omega t = \sqrt{1 - (E_x/E_1)^2}$.

Introducing these in Equ 8.94 eliminates ωt, and on rearranging we obtain

$$\frac{E_x^2}{E_1^2} - \frac{2E_xE_y\cos\delta}{E_1E_2} + \frac{E_y^2}{E_2^2}\sin^2\delta \qquad 8.95$$

$$aE_x^2 - bE_xE_y + cE_y^2 = 1 \qquad 8.96$$

Or

$$\text{Where} \quad a = \frac{1}{E_1^2 sin^2\delta} \qquad b = \frac{2\cos\delta}{E_1E_2 sin^2\delta} \qquad c = \frac{1}{E_2^2 sin^2\delta}$$

Equ 8.96 describes a (polarization) ellipse, as in Fig 8.15. The line segment OA is the semimajor axis, and the line segment OB is the semi minor axis. The tilt angle of the ellipse is τ. The axial ratio is.

$$\boxed{AR = \frac{OA}{OB} \ \ (1 \le AR \le \infty) \ \ Axial\ ratio} \qquad 8.97$$

If $E_1 = 0$, the wave is linearly polarized in the y direction. If $E_2 = 0$, the wave is linearly polarized in the x direction. If $\delta = 0$ and $E_1 = E_2$, the wave is also linearly polarized but in a plane at an angle of 45^0 with respect to the x axis ($\tau = 45^0$).

If $E_1 = E_2\ and\ \delta = \pm 90^0$, the wave is circularly polarized. When $\delta = +90^0$, the wave is left circularly polarized, and when $\delta = -90^0$, the wave is right circularly polarized. For the case $\delta = -90^0$ and for $z = 0,\ and\ t = 0$, we have from Equs (8.91 and 8.92) that $E = \hat{y}E_2$, as in Fig 8.16a. one quarter cycle later ($\omega t = 90^0$), $E = \hat{x}E_1$, as in Fig 8.16b. thus at a fixed position ($z = 0$) the electric field vector rotates clockwise (viewing the wave approaching). According to the IEEE definition, this corresponds to left circular polarization. The opposite direction ($\delta = +90^0$) corresponds to right circular polarization.

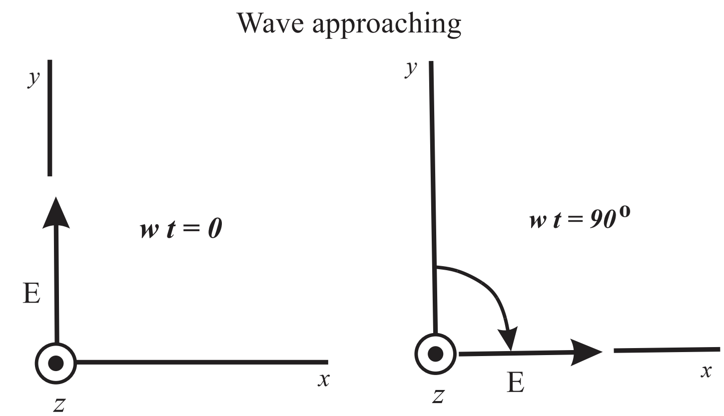


Figure 8.16 Instantaneous orientation of electric field vector E at two instants of time for a left circularly polarized wave which is approaching (out of page).

If the wave is viewed receding (from negative z axis in Fig. 8.16), the electric vector appears to rotates in the opposite direction. Hence clockwise rotation of E with the wave approaching is the same as counterclockwise rotation with the wave receding. Thus, unless the wave direction is specified, there is a possibility of ambiguity as to whether the wave is left or right-handed. This can be avoided by defining the polarization with the aid of an axial mode helical antenna. Thus, a right-handed helical antenna radiates (or receives) right circular (IEEE) polarization. A right-handed helix, like a right-handed screw, is right-handed regardless of the position from which it is viewed. There is possibility here of ambiguity.

The institute of Electrical and electronics engineers (IEEE) definition is opposite to the classical optics definition which had been in use for centuries. The intent of the IEEE standards committee was to make the IEEE definition agree with the classical optic definition, but it got turned around so now we have two definitions. In this book we use the IEEE definition, which has the advantage of agreement with helical antennas as noted above.

8.15 Poynting Vector for Elliptically and Circularly Polarized Waves

In complex notation the Poynting vector is

$$S = \frac{1}{2}\, E \times H^* \qquad 8.98$$

The average Poynting vector is the real part of (1), or

$S_{av} = Re\ S = \frac{1}{2} Re\ E \times H^*$
8.99

We can also write

$$S_{av} = \frac{1}{2}\hat{z}\frac{E_1^2 + E_2^2}{Z_0} = \frac{1}{2}\hat{z}\frac{E^2}{Z_0} \quad Average\ Poynting\ vector \qquad 8.100$$

Where $E = \sqrt{E_1^2 + E_2^2}$ is the amplitude of the total E field.

Example 8.12 Elliptically polarized wave power. An elliptically polarized wave traveling in the positive z direction in air has x and y components.

$$E_x = 3\sin(\omega t - \beta x) \qquad (V\ m^{-1})$$

$$E_y = 6\sin(\omega t - \beta x + 75^0) \qquad (V\ m^{-1})$$

Find the average power per unit area conveyed by the wave.

Solution. The average power per unit area is equal to the average Poynting vector, which from Equ (8.100) has a magnitude

$$S_{av} = \frac{1}{2}\frac{E^2}{Z} = \frac{1}{2}\frac{E_1^2 + E_2^2}{Z}$$

From the stated conditions, the amplitude $E_1 = 3\ Vm^{-1}$ and the amplitude $E_2 = 6\ Vm^{-1}$. Also, for air $z = 377\Omega$. Hence

$$S_{av} = \frac{1}{2}\frac{3^2 + 6^2}{377} = \frac{1}{2}\frac{45}{377} \approx 60m\ \ Wm^{-2} \quad Ans$$

Class work: EP wave power. An elliptically polarized (EP) wave in a medium with constant $\sigma = 0,\ \mu_r = 2,\ \varepsilon_r = 5$ has H-field components (normal to the direction of propagation and normal to each other of amplitudes 3 and 4 A/m. find the average power conveyed through an area of 5 m^2 normal to the direction of propagation. Ans 14.9kW.

8.16 The Polarization Ellipse and the Poincare Sphere

In the Poincare sphere representation of wave polarization, the polarization state is described by a point on a sphere where the longitude and latitude of the point are related to parameters of the polarization ellipse (see Fig. 8.17) as follows.

$$\text{Longitude} = 2\tau \qquad 8.101$$

$$\text{Latitude} = 2\varepsilon$$

Where $\tau = \text{tilt angle}, 0^o \leq \tau \leq 180^o\ (see\ footnote) and\ \varepsilon = tan^{-1}\ (1/\mp AR), -45^o \leq \varepsilon \leq +45^o$. The axial ration (AR) and angle ε are negative for right handed and positive for left handed (IEEE) polarization.

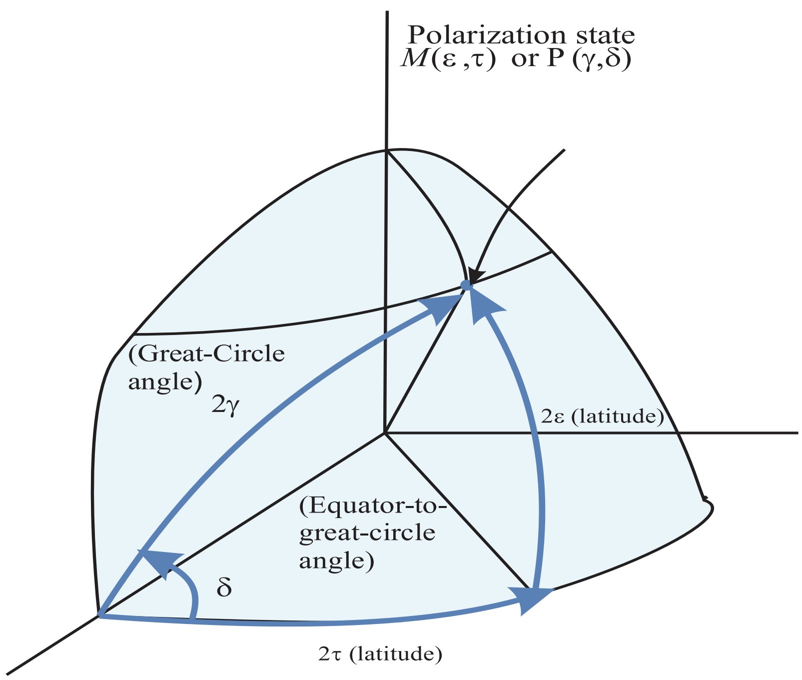


Figure 8.17 Poincare sphere showing relation of angles $\varepsilon, \tau, \delta, \gamma$

The polarization state described by a point on a sphere here can also be expressed in terms of the angle subtended by the great circle drawn from a reference point on the equator and the angle between the great circle and the equator as follows.

$$\text{Great circle angle} = 2\gamma$$

$$\text{Equator} - \text{to} - \text{great} - \text{circle angle} = \delta \qquad 8.102$$

Where$\gamma = tan^{-1}\left(\frac{E_2}{E_1}\right), 0^o \leq \gamma \leq 90^o, and\ \delta =$ phase difference between $E_y\ and\ E_x\ -180^0 \leq \delta \leq 180^0$.

The geometric relations of $\tau, \varepsilon\ and\ \gamma$ to the polarization ellipse is illustrated in Figure 8.18. The trigonometry interrelations of $\tau, \varepsilon, \gamma\ and\ \delta$ are as follows:

$$\boxed{\begin{matrix} \cos 2\gamma = \cos 2\varepsilon \cos 2\tau \\ \tan \delta = \frac{\tan 2\varepsilon}{\sin 2\tau} \\ \tan 2\tau = \tan 2\gamma \cos \delta \\ \sin 2\varepsilon = \sin 2\gamma \sin \delta \end{matrix} \qquad Polarization\ parameters} \qquad 8.103$$

Knowing $\tau\ and\ \varepsilon$ can determine $\gamma\ \ and\ \delta$or vice versa. It is convenient to describe the polarization state by either of the two sets of angles $(\varepsilon, \tau)\ or\ (\gamma, \delta)$ which describe a point on the Poincare sphere (Fig. 8.18). Let the polarization state as a function of $\varepsilon\ and\ \tau$ be designated by $M(\varepsilon, \tau)$, or simply M and the polarization state as a function of $\gamma\ \ and\ \delta$ be designated by $P(\gamma, \delta)$, of simply P, as in Fig. 8.19.

As an application of the Poincare sphere representation (see Fig 8.20) it may be shown that the voltage response V of an antenna to a wave of arbitrary polarization is given by

$$\boxed{V = k \cos \frac{M\, M_a}{2} \qquad Antenna\ voltage\ response}$$

8.104

Where MM_a = angle subtended by great circle line from polarization state M to M_a

M = Polarization state of wave

M_a = Polarization state of antenna

K = Constant

The polarization state of the antenna is defined as the polarization state of the wave radiated by the antenna when it is transmitting. The factor k in (4) involves the field strength of the wave and the size of the antenna. An important result to note is that, if $MM_a = 0^0$, the antenna is matched to the wave (polarization state of wave same as for antenna) and the response is maximized. However, if $MM_a = 180^0$, the response is zero. This can occur, for example, if the wave is linearly polarized in the y direction while the antenna is linearly polarized in the x direction; or if the wave is left circularly polarized while the antenna is right circularly polarized. More generally we may say that an antenna blind to a wave of opposite (or antipode) polarization state.

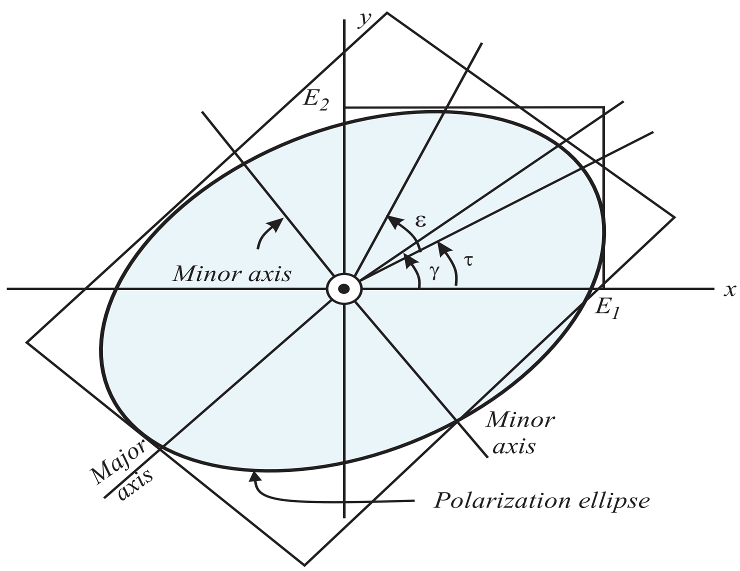


Figure 8.19 Polarization ellipse showing relation of angles $\varepsilon,\ \gamma, and\ \tau$,

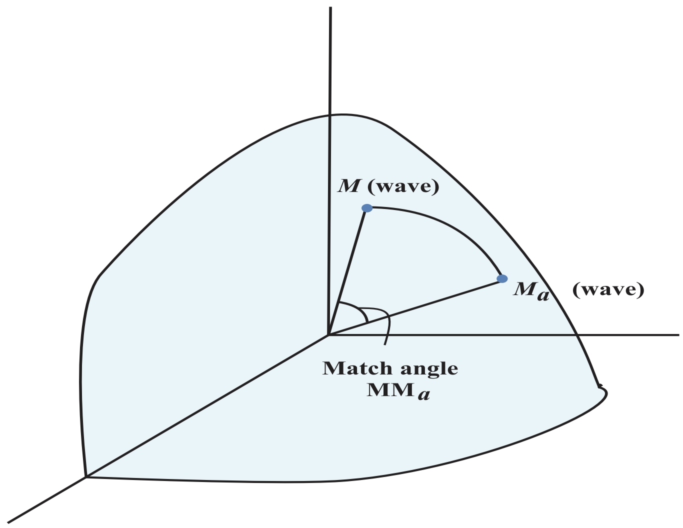


Figure 8.20 The match angle MM_a between the polarization state of wave (M) and antenna (M_a). for $MM_a = 0^0$, the match is perfect. For $MM_a = 180^0$, the match is zero.

Referring to 8.104 a polarization matching factor F (for power) is given by

$$F = cos^2 \frac{MM_a}{2} \qquad 8.105$$

Thus, for a perfect match the match angle $MM_a = 0^0$ and F = 1 (states of wave and antenna the same). For a complete mismatch the match angle $MM_a = 180^0$ and F = 0 (Fig. 8.20).

For a linearly polarization, $\frac{MM_a}{2} = \Delta\tau$ and 8.105 reduces to

$$F = cos^2 \Delta\tau \qquad 8.106$$

Where $\Delta\tau$ difference between the tilt angles of wave and antenna.

In the above discussion we have assumed a completely polarized wave, that is one where E_x, E_y, and δ are constant. In an unpolarized wave they are not. Such a wave result when the vertical component is produced by one noise generator and the horizontal component by a different noise generator. Most cosmic radio sources are unpolarized and can be received equally well with an antenna of any polarization. If the wave is completely unpolarized, $F = \frac{1}{2}$ regardless of the state of polarization of the antenna.

Example 8.13 Polarization matching. Find the polarization matching factor F for a left elliptically polarized wave (w) with (AR) = 4 and $\tau = 15^o$ incident on a right elliptically polarized antenna (a) with AR = -2 and $\tau = 45^o$.

From (1), $2\varepsilon\ (w) = 28.1^o$ and $2\varepsilon\ (a) = 53.1^o$. Thus, the wave polarization state M is at latitude $+28.1^o$ and longitude 30^0 while the antenna polarization state M_a is at latitude $--53.1^o$ and longitude 90^0. Locate these positions on a globe and measure M M_a but also illustrate the geometry. Then compare this result with an analytical one as follows: From proportional triangles obtain $2\tau(w) = 20.7^o$ along the equator and $2\tau(a) = 39.3^0$ further along the equator. Next from (3), obtain $2\gamma(w) = 34.3^o$ and $2\gamma(a) = 62.4^o$. Thus, the total great-circle angle $M\ M_a = 2\gamma(w) + 2\gamma(a) = 96.7^o$ and the polarization matching factor.

$$F = cos^2\left(\frac{96.7}{2}\right) = 0.44$$

Or the received power is 44 percent of the maximum possible value. Ans

Class work: Antenna response. Find the relative voltage response for an antenna oriented to receive

a wave traveling in the +x direction if the wave is given by $E = \hat{z}\sin(\omega t - \beta x)\ mV(rms)/m$ and the parameters for the antenna are: (a) AR = -1; (b) $AR = \infty, \tau = 0^o$ (with respect to y direction); (c) $AR = \infty, \tau = 45^o$; (d) $AR = \infty, \tau = 90^o$; and (e) $AR = 1.5, \tau = 67.5^o$. Ans (a) 0.707; (b) 0; (c) 0.707; (d) 1; € 0.79.

Class work: Polarization matching factor. Find the polarization matching factor F for the following

cases: (a) wave VLP, antenna HLP; (b) wave SLP ($\tau = 60^0$), antenna HLP; (c) wave RCP antenna REP; (d) wave RCP antenna VLP; (e) wave RCP, antenna HLP; (f) wave RCP, antenna REP ($AR = -3, \tau = 0^0$) and (g) wave LEP $AR = 4, \tau = 0^0$), antenna REP $AR = -4, \tau = 45^0$). VLP = vertical linear polarization, HLP = horizontal linear polarization, SLP = slant linear polarization, RCP = right circular polarization, LCP = left circular polarization, REP = right elliptical polarization and LEP = left elliptical polarization. Ans (a) 0; (b) 0.25; (c) 0 (d) 0.5 (e) 0.5; (f) 0.8 (g) 0.39.

8.17 Oblique Incidence: Reflection and Refraction

consider a linearly polarized plane wave obliquely incident on a boundary between two media as shown in Fig. 8.21. The incident wave (from medium 1) makes an angle of θ_i with the y axis, the reflected wave (medium 1) makes an angle of θ_r with the y axis and the transmitted wave make an angle θ_r with the negative y axis.

Consider two cases: (1) the electric field perpendicular to the plane of incidence (the xy plane) and (2) the electric field parallel to the plane of incidence. These waves are said to be perpendicularly polarized and parallel polarized, respectively. The field vectors shown in Fig. 8.21 are for the case of perpendicular polarization. It is clear that any arbitrary plane wave can be resolved into perpendicular and parallel components.

Perpendicular Case (E)

From the boundary conditions.

$$\eta_1 \sin\theta_i = \eta_1 \sin\theta_r = \eta_2 \sin\theta_t \qquad 8.107$$

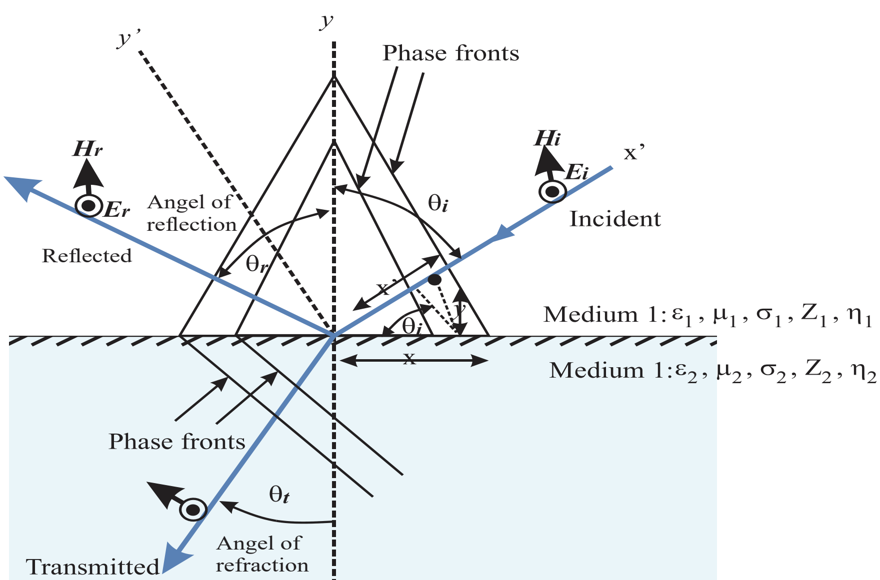


Figure 8.21 Geometry in the plane of incidence (x-y plane or plane of the page) for linearly polarized wave at oblique incidence and for perpendicular polarization. The z direction is outward from the page.

From the first equality

$$\theta_r = \theta_i \qquad 8.107.1$$

i.e., the angle of reflection is equal to the angle of incidence. From the second equality.

$$\boxed{\sin\theta_t = \frac{\eta_1}{\eta_2}\sin\theta_i \quad Snell's\ law} \qquad 8.108$$

Where η_1 and η_2 are the indices of refraction of medium 1 and medium 2, respectively. Equ (8.108) is known as Snell's law and is a relation of fundamentals importance in geometrical optics. For a lossless medium the index of refraction η can be written as equal to $\mu_r\varepsilon_r$ and Snell's law becomes.

$$\boxed{\sin\theta_t = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2} \qquad Snell's\ law}} \qquad 8.109$$

Example 8.13 Polystyrene-air interface. Polystyrene has a relative permittivity of 2.7. if a wave is incident at an angle of $\theta_i = 30^o$from air onto polystyrene, (a) calculate the angle of transmission θ_t and (b) interchange polystyrene and air and repeat the calculation.

Solution. From air onto polystyrene $\varepsilon_1 = \varepsilon_0, \mu_1 = \mu_0, \varepsilon_2 = 2.7\ \varepsilon_0\ \ and\ \mu_2 = \mu_0$. From (3)

$$\sin\theta_t = \sqrt{\frac{1}{2.7}}(0.5) = 0.304$$

$$\theta_t = 17.7^0\ \ Ans\ \ (a)$$

From Polystyrene onto air $\varepsilon_1 = 2.7\varepsilon_0, \mu_1 = \mu_0, \varepsilon_2 = \varepsilon_0\ \ and\ \mu_2 = \mu_0$

$$\sin\theta_t = \sqrt{2.7}(0.5) = 0.822$$

$$\theta_t = 52.2^0\ \ Ans\ \ (b)$$

We have also

$$-\cos\theta_i + \rho_\perp \cos\theta_i = -\tau \frac{Z_1}{Z_2}\cos\theta_t$$

8.110

And $$1 + \rho_\perp = \tau_\perp$$ 8.111

And on substituting Equs (8.111 into 8.110) and solving for the Fresnel reflection coefficient $\rho_\perp$, we have

$$\rho_\perp = \frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t}$$ 8.112

Where Z_1 and Z_2 are the impedance of medium 1 and medium 2, respectively. It is seen that the previously derived reflection coefficient for normal incidence, Equs (8.108-113-8.118). Is obtained as a special case; when $\theta_i = 0$.

If medium 2 is a perfect conductor $Z_2 = 0\ \ and\ \rho_\perp = 1$. If both media are lossless nonmagnetic dielectric (6) becomes.

$$\rho_\perp = \frac{\cos\theta_i - \sqrt{\left(\frac{\varepsilon_2}{\varepsilon_1}\right) - sin^2\theta_i}}{\cos\theta_i + \sqrt{\left(\frac{\varepsilon_2}{\varepsilon_1}\right) - sin^2\,\theta_i}} \qquad Reflection\ coefficient\ \perp$$ 8.113

Provided medium 2 is a denser dielectric than medium 1 ($\varepsilon_2 > \varepsilon_1$), the quantity under the square root will be positive and $\rho_\perp$ will be real. If, however, the wave is incident from the denser medium onto the less dense medium ($\varepsilon_1 > \varepsilon_2$), and if $sin^2\theta_i \geq\ \varepsilon_1/\varepsilon_2$ then, $\rho_\perp$ becomes complex and $|\rho_\perp| = 1$. Under these conditions, the incident wave is totally internally reflected back into the denser medium. The incident angle for which $\rho_\perp = 1 < 0^0$ is called the critical angle θ_{ic}. From 8.113 it is seen that this happens when the radical is zero, so that

$$\theta_{ic} = sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \qquad Critical\ angle$$ 8.114

Defines the critical angle. For all angles greater than the critical angle, $|\rho_\perp| = 1$. Using Snell's law, we see that when $\theta_i = \theta_{ic}\ then \sin\theta_t > 1,\ and \cos\theta_t$ must be imaginary, i.e.

$$\cos\theta_t = \sqrt{1 - sin^2\theta_t = jA} \qquad 8115$$

Where $A = \sqrt{(\varepsilon_1/\varepsilon_2)sin^2\theta_i - 1}$ is a real number

The electric field in the less dense medium can now be written

$$E_t = \hat{z}\tau_\perp E_0 \exp(-\alpha y)\exp(j\beta_2 x \sin\theta_t) \qquad 8.116$$

Where $$\alpha = \beta_2 A = \omega\sqrt{\mu_2\varepsilon_2}\sqrt{\frac{\varepsilon_1}{\varepsilon_2}\, sin^2\theta_i - 1} \qquad 8.117$$

Thus, $E_\perp$ in the less dense medium has a magnitude $\tau_\perp\ E_0$, decaying exponentially away from the surface (y direction) and propagating without loss in the x direction. Waves whose fields are of the form of Equa (8.116) are called surface waves. These results can be summarized by the principle of total internal reflection as follows. When a wave is incident from the denser onto the less dense medium at an angle equal to or exceeding the critical angle, the wave will be totally internally reflected and will also be accompanied by a surface wave in the less dense medium.

Example 8.14 Total internal reflection with surface wave. Referring to Fig. 8.22, a linearly polarized plane wave is incident from water onto the water-air interface at 45^0. Calculate the magnitude of the electric field in air (a) at the interface and (b) $\lambda/4$ above the surface if the incident electric field $E_i = 1\, Vm^{-1}$. Take the water constants to be those of distilled water: $\varepsilon_r = 81, \mu_r = 1, \sigma \approx 0$.

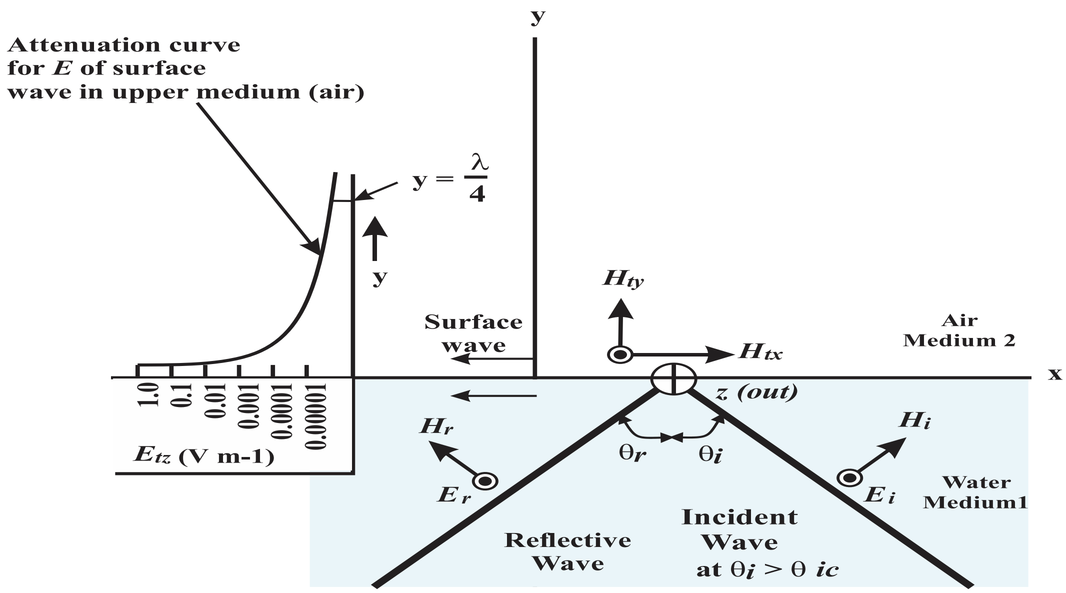


Figure 8.22 Total internal reflection of incident wave with accompanying surface wave which attenuates exponentially above surface (y direction) as shown by graph at left. No power is transmitted in y direction (up).

Solution. From Equ (8.101), the critical angle

$$\theta_{ic} = sin^{-1}\sqrt{\frac{1}{81}} = 6.38^o$$

Thus, the angle of incidence θ_{ic} $(= 45^0)$ exceeds the critical angle and the wave will be totally internally reflected (see Fig. 8.22 From 8.100

$$\sin\theta_t = \sqrt{81\ (0.707)} = 6.36$$

From Equ (8.115)

$$\cos\theta_t = jA = \sqrt{1 - 6.36^2} = j6.28$$

From Equ (8.117)

$$\alpha = \beta_2 A = \frac{2\pi}{\lambda_0} 6.28 = \frac{39.49}{\lambda_0}\ Np\ m^{-1}$$

From Equs (8.102 and 8.104),

$$\tau_{\perp} = 1 + \rho_{\perp} = 1 + \frac{0.707 - \sqrt{\frac{1}{81} - 0.5}}{0.707 + \sqrt{\frac{1}{81} - 0.5}} = 1.42 < -44.64^{o}$$

Therefore, the magnitude of the field strength is

(a) At the interface:

$$|E_t| = 1.42\ V\ m^{-1}$$

(b) $\lambda/4$ away from the interface:

$$|E_{\tau}| = 1.42 \exp\left(-\frac{39.49\, \lambda_0}{\lambda_0}\frac{\lambda_0}{4}\right) = 73.2\, \mu\, Vm^{-1} \quad Ans \quad (a)$$

Thus, the field $\lambda/4$ above the surface is

$$20 \log \frac{73.2 \times 10^{-6}}{1.42} = -85.8dB \qquad Ans \quad (b)$$

Less than the field at the surface. Recalling that a power ratio of 1 billion equals 90dB, it is evident that the field attenuates very rapidly above the surface (in the y direction) (see Fig. 8.22), meaning that the surface is very tightly bound to the water surface. Note that $\sin\theta_t$ is greater than 1 but real, while $\cos\theta_t$ is imaginary, from Equ (8.116)

$$E_t = \hat{z}\tau_{\perp}E_0 e^{-(\beta_2 A)y} e^{j\beta_2 x \sin\theta_t} \qquad 8.118$$

And

$$H_t = \left(-\frac{\hat{x}}{jA} + \hat{y}\sin\theta_t\right)\tau_{\perp}\frac{E_0}{Z_2}e^{-(\beta_2 A)y}\, e^{j\beta_2 x \sin\theta_t} \qquad 8.119$$

Where $A = -\sqrt{sin^2\theta_t - 1}$.

From Equs (8.118 and 8.119), the average Poynting vector of the wave in the y direction in air (above the water surface) is

$$S_{y(av)} = \frac{1}{2}\ ReE \times H^* = \hat{y}\frac{1}{2}E_{tz}H_{tx}\sin\phi\cos\phi$$

Where ϕ =space angle between E and H $(= 90^0)$ $and\ \theta$ =time phase angle between E and H.

The exponentials in Equs (8.118 and 8.119) are identical. However, H_{tx} has a factor where E_{tz} does not, indicating a 90^0 time phase difference between E and H. hence, $\theta = 90^0$, and since $\sin\theta = \sin 90^0 = 1$,

$$S_{y(av)} = \tfrac{1}{2}E_{tz}H_{tx}\cos 90^0 = 0 \qquad 8.120$$

Thus, no power is transmitting in the y direction (wave reactive). Both $E_t\ \ and\ H_t$ decay exponentially with y. similar waves, called evanescent waves, exist in hollow conducting wave guides at wavelength too long to propagate through the guide

The wave in medium 2 (air) involving E_{tz} and H_{ty} propagate without attenuation as a surface wave in the x direction with a velocity V_x equal to the wave velocity in the water (medium 1) as observed parallel to the x axis $(v_x = v_{water}/\sin\theta_t)$. The traveling wave is simply the matching field at the boundary. Total internal reflection with a surface wave can also occur for $E_{||}$, but the details differ.

8.17.1 Parallel case ($\boldsymbol{E}_{||}$)

Consider now the case of parallel (||) polarization. The geometry is the same as in Fig. 8.21 but with $E_i, E_r\ \ and\ E_t$ parallel to the plane of incidence as would be obtained by replacing $H_t\ by\ E_i, H_r\ by\ E_r\ \ and\ H_t\ by\ E_t$. By matching boundary conditions, as before, it is found that the angle of incidence equals the angle of reflection and that Snell's law (2) holds. It can also be shown that

$$1 + \rho_{||} = \frac{\cos\theta_t}{\cos\theta_i}\ \tau_{||} \qquad 8.121$$

The Fresnel reflection coefficient is found to be

$$\rho_{||} = \frac{Z_2\cos\theta_t - Z_1\cos\theta_i}{Z_1\cos\theta_i - Z_2\cos\theta_t} \qquad 8.122$$

Which for lossless nonmagnetic dielectrics becomes

$$\rho_{||} = \frac{-\left(\frac{\varepsilon_2}{\varepsilon_1}\right)\cos\theta_i + \sqrt{\left(\frac{\varepsilon_2}{\varepsilon_1}\right) - \sin^2\theta_i}}{\left(\frac{\varepsilon_2}{\varepsilon_1}\right)\cos\theta_i + \sqrt{\left(\frac{\varepsilon_2}{\varepsilon_1}\right) - \sin^2\theta_i}} \qquad Reflection\ coefficient\ || \qquad 8.123$$

And reduces to $\rho_{\|} = -1$ if medium 2 is a perfect conductor.

It is of especial interest that, for parallel polarization, it is possible to find an incident angle so that $\rho_{\|} = 0$ and the wave is totally transmitted into medium 2. This angle, called the Brewster angle θ_{iB}, can be found by setting the numerator of Equ (8.123) to zero, giving.

$$\theta_{iB} = \sin^{-1}\sqrt{\frac{\varepsilon_2/\varepsilon_1}{1+(\varepsilon_2/\varepsilon_1)}} = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \qquad Brewster\ angle \qquad 8.124$$

The Brewster angle is also sometimes called the polarizing angle since a wave composed of both perpendicular and parallel components and incident at the Brewster angle produces a reflected wave with only a perpendicular component. Thus, circular polarized wave incident at the Brewster angle becomes linearly polarized on reflection.

Example 8.15 Brewster angle. A parallel polarized wave is incident from air onto (a) distilled water ($\varepsilon_r = 81$), (b) flint glass ($\varepsilon_r = 10$) and (c) paraffin ($\varepsilon_r = 2$). Find the Brewster angle for each of these cases.

Solution

$$\theta_{iB} = \tan^{-1}\sqrt{81} = 83.7^o \ \ Ans \ \ (a)$$

$$\theta_{iB} = \tan^{-1}\sqrt{10} = 72.4^o \ \ Ans \ \ (b)$$

$$\theta_{iB} = \tan^{-1}\sqrt{2} = 54.7^o \ \ Ans \ \ (c)$$

Example 8.16 Effect of ground reflection on antenna patter. A linear in-phase antenna in free space radiates equally in all directions perpendicular to its length. Above ta perfectly conducting ground the field may double or go to zero depending on the relative phase of the direct and ground-bounce waves. If the antenna height $h = \lambda$, what is the field patter?

Solution. Referring to Fig. 8.23a the antenna is horizontal and perpendicular to the page. The pattern is the same as that of the antenna and its image as given by

$$E(\theta) = 2E_0 \sin\left(2\pi \frac{h}{\lambda} \sin\theta\right)$$

Where E_0 = free space field, V m^{-1} and h =height above ground, m. no mutual coupling antenna and image is assumed. In practice this is small when h is large ($> \lambda/2$).

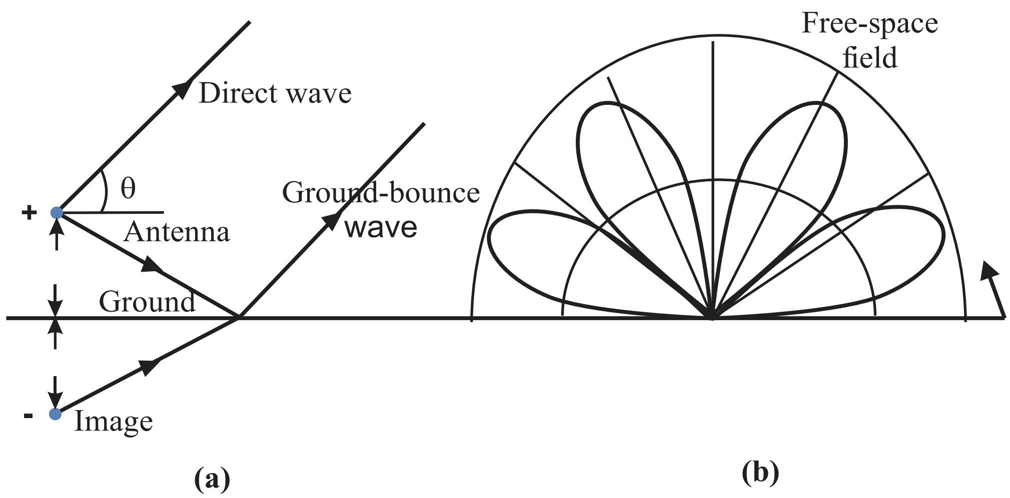


Figure 8.23(a) The pattern is shown in Fig 8.23 (b)

Class work: angle of maximum field. Find the angles for which the field in Fig. 8.23b is maximum.

Ans 14.5^0 and 48.6^0.

Class work: Angle of maximum field. Find the angles for which the field is maximum if $h = 2\lambda$. Ans. 7.2^0, 22.0^0, 38.7^0 and 61.0^0.

8.18 Exercise

1. For a lossy dielectric material having $\mu_r = 1, \varepsilon_r = 48, \sigma = 20\text{S/m}$, calculate the attenuation constant, phase constant and intrinsic impedance at a frequency of 16GHz.

2. A uniform plane wave in a medium having $\sigma = 10^{-3}$ S/m, $\varepsilon = 80\,\varepsilon_0$ and $\mu = \mu_0$ is having a frequency of 10KHz. Calculate the different parameters of wave.
3. For a non-magnetic material having $\varepsilon_r = 2.20, \sigma = 10^{-4}$ s/m. fine (i) loss tangent (ii) attenuation constant (iii) phase constant and (iv) intrinsic impedance for a wave having a frequency of 5MHz. assume material to be good dielectric.
4. Find skin depth S at frequency of 1.8 MHz is aluminum, where $\sigma = 34$ MS/m and $\mu_r = 1$. Also find propagation constant and wave velocity.
5. A travelling electric field $\vec{E}$ in free space with an amplitude of 200V/m strikes a sheet of silver of thickness 6μm as shown in Fig 8.14. by taking conductivity of silver $\sigma = 61.5$ MS/m and a frequency 250 MHz find the magnitude of E_2, E_3 and E_4.

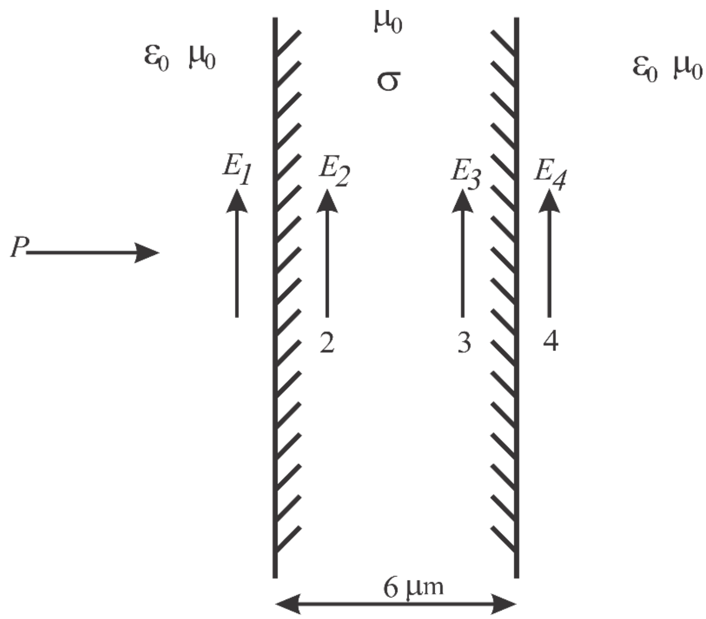


Fig 8.12

6. The electric field intensity in the radiation fields of an antenna located at origin of a spherical coordinates system is given by

$$E\,(r, \theta, \emptyset) = E_0 \frac{\sin\theta}{r} \cos(\omega t - \beta z)\hat{\theta}$$

Where E_0, ω_2 and $\beta\left(= \omega\sqrt{\mu_0 \varepsilon_0}\right)$ are constants. Find

i. Magnetic field associated with E
ii. Poynting vector
iii. Total power radiated over a spherical surface of radius ‘r’ centered at origin

7. A travelling E field in free space of amplitude 100V/m strikes a perfect dielectric as shown in Fig 8.15. find the electric field strength in medium 3

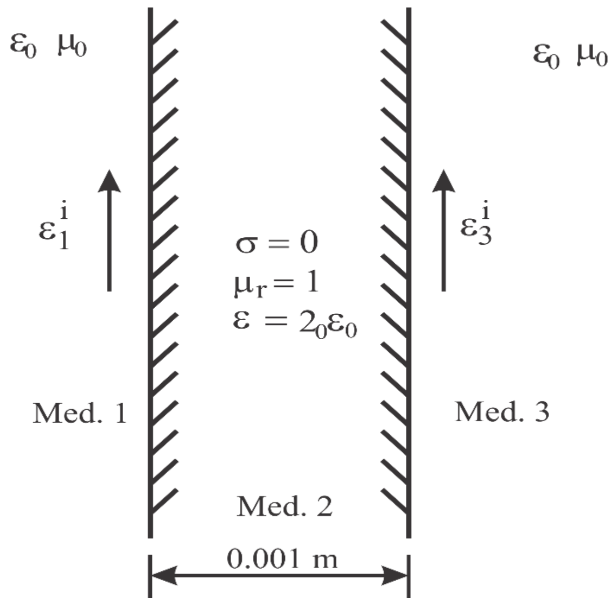


Fig 8.13

8. A medium has following characteristic: Propagation constant $r = 528 + j\ 244m^{-1}$ and intrinsic impedance $\eta = 50 < L12^o \Omega$ at a wave frequency of $f = 350\ MHz$. The electric field is given by $E_x = 200e^{-az}\cos(2\pi\ \times 10^8 t - \beta z)\hat{x}$. Calculate the expression for magnetic field intensity and the average power/m^2 at $z = 1\text{mm}$, given by electromagnetic wave.
9. Explain wave propagation in lossless medium
10. In lossless medium, derive expression for α, β.
11. Explain wave propagation in conducting medium.
12. In conducting medium, derive expression for α, β
13. Explain significance of term $\left(\frac{\sigma}{\omega\varepsilon}\right)$.
14. What are standing wave?
15. Explain wave propagation in good dielectric
16. Define depth of penetration
17. Define depth of propagation
18. What are direction cosines? What is their significance?
19. Define equiphasic surface.

20. Derive the expression for reflection of uniform plane waves by perfect dielectric for
    a. Normal incidence
    b. Oblique incidence
21. What is Snell's law?
22. Derive condition for horizontal and vertical polarization.
23. Derive the expression for reflection by perfect conductor of uniform plane wave.
24. Define surface impedance of conductor
25. Derive expression for surface impedance of conductor
26. What do you understand by Poynting Vector?
27. Derive the expression for Poynting's theorem
28. Explain significance of term $(\vec{E}\,.\vec{J})$.
29. What do you understand by electrostatic and magnetostatics' energy in Poynting's equations?
30. Write note on interpretation of $\vec{E} \times \vec{H}$.
31. Explain why there is negative sign in terms on R.H.S. of Poynting equation.
32. Attenuation in lossy medium. A medium has constant $\sigma = 1.112 \times 10^2$ ℧/ $m, \mu_r = 5 - j4$, and $\varepsilon_r = 5 - j2$. At 100MHz find (a) impedance of the medium and (b) distance required to attenuate a wave by 20dB after entering the medium.
33. **Index of refraction**. The measured phase velocity of a dielectric medium is 186 Mm/s at f_1 and 223Mm/s at f_2 find: index of refraction at the two frequencies
34. **Field intensity**. Find the magnetic field intensity for a TEM wave with electric field intensity of $1\ \mu V/m$ in (a) air (b) lossless dielectric with $\varepsilon_r = 5$, and (c) a lossless dielectric with $\varepsilon_r = 14$.
35. **medium impedance**. What is the impedance of a medium with $\sigma = 10^{-2}$℧$/m$, $\varepsilon_r = 3$, and $\mu_r = 1$ (a) at 1MHz (b) at 50MHz and (c) at 1GHz?
36. **Medium impedance**. Find the impedance of a conducting medium with $\sigma = 10^{-4}\frac{℧}{m}$,
    $\mu_r = 1 + j0.5$, and $\varepsilon_r = 12 - j4$ at a frequency of 800MHz.
37. **Poynting vector**. A plane wave is traveling in a medium for which $\sigma = 0$, $\mu_r = 1$, and $\varepsilon_r = 3$. If E(peak) = 5V/m find (a) peak Poynting vector, (b) average Poynting vector, (c) peak value of H, (d) the phase velocity and (e)the impedance Z of the medium.

38. **Poynting vector**. A plane 200MHz wave is traveling in a medium for which $\sigma = 0$, $\mu_r = 2$, and $\varepsilon_r = 4$. If the average Poynting vector is 5W/m^2 find (a) rms E; (b) rms H; (c)phase velocity; and (d) the impedance of the medium.
39. **Phase velocity**. What is the relative permittivity of a nonferrous medium for which the phase velocity is 150Mm/sec?
40. **Poynting vector.** A plane traveling wave has a peak electric field E_0 = 15V m^{-1} if the medium is lossless with $\mu_r = 1$, and $\varepsilon_r = 12$. find (a) velocity of the wave, (b) peak Poynting vector, (c) the impedance of medium.
41. **Poynting vector**. A plane traveling 800MHz wave has an average Poynting vector of 8m W/m^2. If the medium is lossless with $\mu_r = 1.5$, and $\varepsilon_r = 6$. find (a) velocity of wave, (b) wavelength, (c) impedance of medium (d) rms electric field E and (e)rms magnetic field H.
42. **Poynting vector**. A plane wave propagating in free space has a peak electric field of 750m V/m. find the average power through a square area 120cm on a side perpendicular to the direction of propagation.
43. **Energy density**. Find the energy density in a plane traveling wave with electric field intensity E = 5 V/m in a nonmagnetic medium with impedance Z = 100Ω.
44. **Angles of reflection and transmission**. A plane wave is incident from air onto a medium with $\varepsilon_r = 5$ at an angle of 30^0. Find (a) the angle of reflection and (b) the angle of transmission (c) repeats with the materials interchanged.
45. **Reflection coefficient, perpendicular polarization**. Find the reflection coefficient for a plane wave with polarization perpendicular to the plane of incidence from air onto a medium with permittivity $\varepsilon_r = 5$ at an angle of 30^0.
46. **Critical angle**. If the media Exercise 45 were interchanged, find the critical angle at which total internal reflection occurs.
47. **Reflection coefficient, parallel polarization**. Repeat Exercise 45 for parallel polarization.
48. **Brewster angle**. Find the Brewster angle for the conditions of Exercise 45

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